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MDCCCXCI.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume, the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*, which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them, without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society, the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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" " B " " " " Series B, and Proceedings
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" " p " " Proceedings only

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America (North). (See United States)

America (South).

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Brunn

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Gratz

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Hermannstadt

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AB Az Erdélyi Muzeum Das Siebenburgische Museum

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Vienna

p Anthropologische Gesellschaft

AB Kaiserliche Akademie der Wissenschaften

p K K Geographische Gesellschaft

AB K K Geologische Reichsanstalt

B K K Zoologisch-Botanische Gesellschaft

B Naturhistorisches Hof Museum

p Œsterreichische Gesellschaft fur Meteorologie

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Liége

AB Société des Sciences

p Société Géologique de Belgique.

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AB Royal Society of Canada
Toronto
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B. Royal Gardens
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B British Museum (Nat Hist).
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p "Electrician," Editor of the
B Entomological Society.
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AB Geological Survey of Great Britain.
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p Royal Asiatic Society
A Royal Astronomical Society.
B Royal College of Physicians.
B Royal College of Surgeons.
p Royal Engineers (for Libraries abroad, six copies)
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p Royal Institute of British Architects
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p Royal Microscopical Society
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AB Royal United Service Institution
AB Society of Arts
p Society of Biblical Archæology.
p. Society of Chemical Industry (London Section).

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- AB The War Office
- AB University College
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- AB Literary and Philosophical Society
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- AB Royal Institution

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- *p* Faculté des Sciences
- *p* Société de Médecine et de Chirurgie
- *p* Société des Sciences Physiques et Naturelles

France (continued)

Cherbourg

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- *p* Académie des Sciences

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- *p* Faculté des Sciences

Lyons

- AB Académie des Sciences, Belles-Lettres et Arts

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- AB Académie des Sciences de l'Institut
- *p* Association Française pour l'Avancement des Sciences.
- *p*. Bureau des Longitudes
- *p* Bureau International des Poids et Mesures
- *p* Commission des Annales des Ponts et Chaussées
- *p* Conservatoire des Arts et Métiers
- *p* Cosmos (M L'ABBÉ VALETTE)
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- AB École des Mines
- AB École Normale Supérieure
- AB École Polytechnique
- AB Faculté des Sciences de la Sorbonne
- AB Jardin des Plantes
- A L'Observatoire
- *p* Revue Internationale de l'Électricité
- *p*. Revue Scientifique (Mons H DE VARIGNY)
- *p* Société de Biologie
- AB Société d'Encouragement pour l'Industrie Nationale
- AB Société de Géographie
- *p* Société de Physique
- B Société Entomologique
- AB Société Géologique.
- *p* Société Mathématique
- *p* Société Météorologique de France

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- A Faculté des Sciences

Germany.

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- A Die Sternwarte
- *p* Gesellschaft fur Erdkunde
- AB Konigliche Preussische Akademie der Wissenschaften
- A Physikalische Gesellschaft

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- AB Universitat.

Germany (continued)

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Breslau
- *p.* Schlesische Gesellschaft fur Vaterländische Kultur.

Brunswick.
- *p* Verein fur Naturwissenschaft

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Emden
- *p* Naturforschende Gesellschaft

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- *p* Zoologische Gesellschaft.

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- AB Naturwissenschaftlicher Verein

Heidelberg.
- *p.* Naturhistorisch-Medizinische Gesellschaft.
- AB Universitat

Jena.
- AB Medicinisch-Naturwissenschaftliche Gesellschaft.

Karlsruhe
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Kiel.
- A. Sternwarte
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- *p* Naturwissenschaftlicher Verein.

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- AB. Universität

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- *p.* Kónigl. Ungarische Geologische Anstalt.
- AB. Á Magyar Tudós Társaság. Die Ungarische Akademie der Wissenschaften.

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Bombay.
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Calcutta.
- AB. Asiatic Society of Bengal.
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- *p.* Great Trigonometrical Survey of India
- AB Indian Museum
- *p.* The Meteorological Reporter to the Government of India.

Madras
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- *p.* Roorkee College.

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Armagh.
- A. Observatory.

Belfast.
- AB. Queen's College.

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p Philosophical Society
AB Queen's College

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AB Royal Dublin Society
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AB Queen's College

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AB Reale Istituto Lombardo di Scienze, Lettere ed Arti
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p Società di Naturalisti
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p University

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p Asiatic Society of Japan

Java

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AB Bataviaasch Genootschap van Kunsten en Wetenschappen

Buitenzorg

p Jardin Botanique

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Luxembourg

p Société des Sciences Naturelles

Malta

p Public Library

Mauritius.

p Royal Society of Arts and Sciences

Netherlands

Amsterdam

AB Koninklijke Akademie van Wetenschappen
p K Zoologisch Genootschap 'Natura Artis Magistra'

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Haarlem

AB Hollandsche Maatschappij der Wetenschappen
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Wellington

AB New Zealand Institute

Norway.

Bergen

AB Bergenske Museum

Christiania

AB Kongelige Norske Frederiks Universitet

Tromsoe

p Museum

Trondhjem

AB Kongelige Norske Videnskabers Selskab

Nova Scotia.

Halifax

p Institute of Natural Science

Windsor

p King's College Library

Portugal.

Coimbra

AB Universidade.

Lisbon

AB Academia Real das Sciencias

Portugal (continued)
Lisbon (continued)
p Secção dos Trabalhos Geologicos de Portugal
Russia
Dorpat
AB Université
Kazan
AB Imperatorsky Kazansky Universitet
Kharkoff
p Section Médicale de la Société des Sciences Expérimentales, Université de Kharkow
Kieff
p Société des Naturalistes
Moscow
AB Le Musée Public
B Société Impériale des Naturalistes
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AB Comité Géologique
p Compass Observatory
A L'Observatoire Physique Central.
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Aberdeen.
AB University
Edinburgh
p Geological Society
p. Royal College of Physicians (Research Laboratory)
p Royal Medical Society
A Royal Observatory
p Royal Physical Society.
p Royal Scottish Society of Arts
AB Royal Society
Glasgow
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Sicily
Acireale
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Palermo
A Circolo Matematico.
AB Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).
A. Reale Osservatorio

Spain
Cadiz
A Observatorio de San Fernando
Madrid
p Comisión del Mapa Geológico de España
AB Real Academia de Ciencias
Sweden
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AB Kongl Vetenskaps och Vitterhets Samhalle
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A Technological Institute
Brooklyn
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AB Harvard University

B Museum of Comparative Zoology

Chapel Hill (N C)

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Charleston

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AB Academy of Sciences

Davenport (Iowa)

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p New York Medical Journal

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p United States Commission of Fish and Fisheries

AB United States Geological Survey

AB United States Naval Observatory

West Point (N Y)

AB United States Military Academy

Adjudication of the Medals of the Royal Society for the year 1890, by the President and Council.

The Copley Medal to Simon Newcomb, For Mem.R S , for his Contributions to the Progress of Gravitational Astronomy

The Rumford Medal to Heinrich Hertz, for his work in Electro-magnetic Radiation.

A Royal Medal to David Ferrier, F R S , for his Researches on the Localisation of Cerebral Functions

A Royal Medal to John Hopkinson, F R S , for his Researches in Magnetism and Electricity

The Davy Medal to Emil Fischer, for his Discoveries in Organic Chemistry, and especially for his Researches on the Carbo-hydrates.

The Darwin Medal to Alfred Russel Wallace, for his Independent Origination of the Theory of the Origin of Species by Natural Selection.

The Bakerian Lecture, "The Discharge of Electricity through Gases (Preliminary Communication)," was delivered by Professor A. Schuster, F R.S

The Croonian Lecture, "On some Relations between Host and Parasite in certain Epidemic Diseases of Plants," was delivered by Professor H. M. Ward, F.R.S.

CONTENTS.

(A.)

VOL. 181.

LIST OF ILLUSTRATIONS

ERRATA.

'PHIL TRANS,' 1890, A.

Bakerian Lecture

Page 260, *for* $N = H_c \cos \delta_c - H \cos \delta$ *read* $N = H \cos \delta - H_c \cos \delta_c$

A similar correction should be made in the two following equations

Page 290, in fig. 21 the direction of the horizontal disturbing force at St. Leonards should be as in Plate 13.

Plate 13. The angle made by the horizontal disturbing force at Campbelton with the geographical meridian is $+136°$ not $-136°$.

'PHIL. TRANS.,' 1889, A.

G. H. BRYAN *on a Rotating Liquid Spheroid.*

Pages 214, 216, equations (96), (103),

$$\textit{for}\quad K_n(\zeta) - \frac{4q_2(\zeta)}{n(n+1)}\quad \textit{read}\quad K_n(\zeta) + \frac{4q_2(\zeta)}{n(n+1)},$$

and in equation (97) remove the sign $-$ on the right hand.

PHILOSOPHICAL TRANSACTIONS.

I *On the Effect of Temperature on the Specific Inductive Capacity of a Dielectric*

By W CASSIE, *M A., Trinity College, Cambridge, Examiner in the Universities of Aberdeen and Durham*

Communicated by J J THOMSON, *M A., F R S , Cavendish Professor of Experimental Physics in the University of Cambridge.*

Received May 24,—Read June 20, 1889

THE object of the experiments described in this paper was to learn how the specific inductive capacity of a dielectric is affected by change of temperature

CAVENDISH observed an increase in the capacity of a glass condenser when it was heated, but gave no measure of the effect Dr. HOPKINSON observed in light flint glass an increase of $2\frac{1}{2}$ per cent in the capacity between 12° and 83° C And Messrs GIBSON and BARCLAY showed that there is no appreciable change in the case of paraffin between temperatures —12° and 24° C Except these, no measurements of the effect appear to have been hitherto published.

The present investigation shows an increase of specific inductive capacity with rise of temperature in all the solids* examined, and a decrease in all the liquids except one

As paraffin, which is a substance comparatively near its melting point at ordinary temperatures, shows no change, these results seem to indicate, as far as they go, that the specific inductive capacity of a substance has a maximum value about the melting point But it may be questioned whether the data are sufficient as yet to warrant so general an induction

The relation which CLERK MAXWELL's electromagnetic theory of light indicates between specific inductive capacity and refractive index makes it interesting to compare the effects of temperature on these two quantities. Four of the liquids

* The results given here for mica, ebonite, and the first specimen of glass are all less than those mentioned by Professor J J THOMSON, in his treatise on "Applications of Dynamics to Physics and Chemistry", because the results quoted there were obtained from an earlier set of observations, in which the precautions for insulation were inferior to those described here

investigated here are amongst those for which DALE and GLADSTONE have observed the refractive indices at several temperatures For one of these four the temperature effects show no similarity whatever, for another the relation is fairly close to that indicated by the theory, and for the other two, though not agreeing exactly with the theory, the relation does not differ from it very greatly, not more, perhaps, than might be explained by differences in the specimens used

The experiments were done in the Cavendish Laboratory, Cambridge, and I am glad of this opportunity of thanking Professor J. J THOMSON, F R S, at whose suggestion they were undertaken, for placing the resources of the Laboratory at my disposal, and for valuable advice during the course of the work

1 SOLIDS

The method adopted in the case of solids was to make a condenser with the substance to be experimented upon and observe its capacity at different temperatures The condenser consisted of a pile of alternate thin sheets of the dielectric and discs of lead foil, with flat plates of iron above and below, and a weight on the top to keep them together. The condenser could not be fixed together in any more permanent way, because the unequal expansion with heat of the different materials would have altered the degree of compression of the pile, and so produced a change of capacity greater than the effect under investigation.

Special precautions were required to secure that the insulation between the two sides of the condenser be as far as possible independent of the temperature To secure this the condenser was suspended in a stirrup from a bracket about four feet above the air bath in which it was heated. The suspending wires, which formed the connection for the side of the condenser in contact with the stirrup, passed without touching through holes in the top of the air bath, and over a short glass tube varnished with shellac, which was fixed across the bracket above. The connection for the other side of the condenser consisted of a stiff wire, which also passed without touching through a hole in the top of the air bath, and was carried through the glass tube on the bracket Thus the condenser had no contact with anything about the air bath, and was supported solely by the glass tube on the bracket, which was always cold and perfectly insulating. The perfect freedom of the condenser and its supports was easily tested before each observation by tapping the air bath and seeing that no motion was produced in the suspending wires.

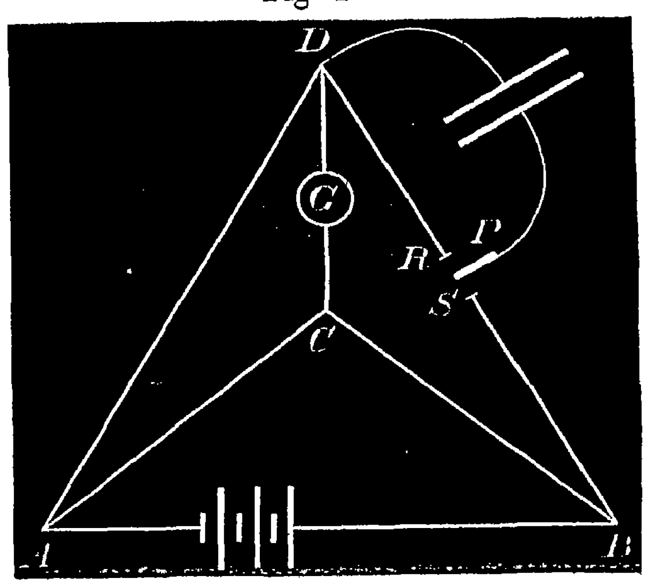
The condenser was heated by raising the bath quickly to a temperature considerably above that aimed at, and then leaving under it a small flame adjusted by experience to give the right temperature The temperature was read by a thermometer passing through a cork in the top of the bath. About three hours was the shortest time required to ensure that the condenser was uniformly heated throughout, it was usually left a good deal longer.

The condensers needed several heatings before settling down to a steady capacity in the cold state The temperatures in these preliminary heatings were always above the highest used in the actual observations

The capacities were measured by the method used by Professor J. J THOMSON in his determination of the ratio of the electrostatic and electromagnetic units ('Phil Trans,' 1883) It is thus described in his paper —

" In a Wheatstone's bridge, A B C D, with the galvanometer at G, and the battery between A and B, the circuit B D is not closed, but the points B and D are connected with the two poles, R and S, of a commutator, between which a travelling piece, P, moves backwards and forwards; P is connected with one plate of a condenser, the other plate of which is connected with D Thus when P is in contact with S, the condenser will be charged, and until it is fully charged, electricity will flow into it from the battery, this will produce a momentary current through the various arms of the bridge When the moving piece P is in contact with R, the two plates of the condenser are connected, and the condenser will discharge itself through D R, and as the resistance of D R is infinitesimal in comparison with the resistance of

Fig 1



any other circuit, the discharge of the condenser will not send an appreciable amount of electricity through the galvanometer. Thus, if we make the moving piece P oscillate quickly from R to S, there will, owing to the flow of electricity to the condenser, be a succession of momentary currents through the galvanometer The resistances are so adjusted that the deflection of the galvanometer produced by these momentary currents is balanced by the deflection due to the steady current through the galvanometer, and the resultant deflection is zero. When this is the case there is a relation between the capacity of the condenser, the number of times the condenser is charged and discharged per second, and the resistances in the various arms of the bridge."

This relation, which is worked out in the paper, is expressed by taking a, b, c, d, g to represent the resistances of A C, A B, A D, B C, D C respectively. The resistances

of D R and S B are so small as to be negligible. Then if the condenser has a capacity C, and is charged and discharged n times per second

$$nC = \frac{a\left\{1 - \dfrac{a^2}{(a+c+g)(a+b+d)}\right\}}{cd\left\{1 + \dfrac{ab}{c(a+b+d)}\right\}\left\{1 + \dfrac{ag}{d(a+c+g)}\right\}}$$

The resistances used were about

$a = 10,$
$b = 8,$
c between 3000 and 6000,
d „ 6000 „ 2000,
$g = 4060$ ohms.

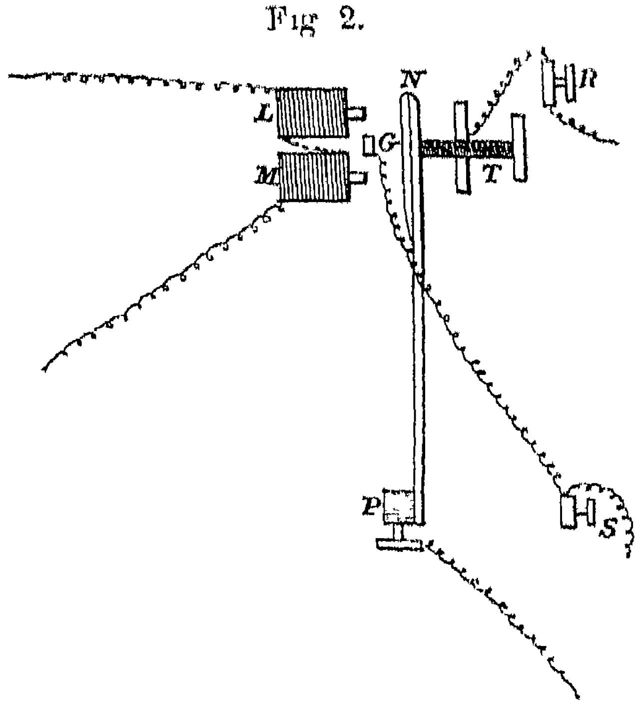
With these values

$$nC = \frac{a}{cd}$$

is correct to within ·2 per cent., and is the formula used here.

The commutator P R S (fig. 2) was that used by Professor THOMSON and is thus described in his paper :-

Fig 2.



"The current from two GROVE's cells passes first through a tuning-fork interruptor, and then through the coils L M of an electromagnet. P N is a strip of brass with a piece of iron attached to it. When there is no current passing through the electromagnet, the elasticity of the rod P N makes it press against a screw T, which is electrically connected with a binding screw R: when the current passes through the electromagnet, the magnet attracts the iron attached to the rod P N and brings it into contact with the stop G, which is electrically connected with the binding screw S. The letters P, R, S indicate the same points in this figure as in fig. 1. All the places where contact is made by the vibrating piece P N are covered with platinum, and the

whole arrangement is fastened down to an ebonite board. As the current passes intermittently through the coils L M of the electromagnet, the vibrating piece strikes alternately against the parts G and T, when it strikes against G the opposite plates of the condenser are connected with the two poles of the battery, when it strikes against T the condenser is discharged (see fig 1)"

To be able to allow for conduction or absorption in the dielectric by means of observations with forks of different rates, we must know the comparative times during which the vibrating piece P is in contact with G, in contact with T, and travelling from one to the other. I therefore fixed a stiff arm to the end of P and took tracings with it on smoked paper, moved across while the interruptor was vibrating. These tracings showed that when the interruptor is going steadily (this is easily known by the sound) the vibrator is in contact with G and T for equal times and the time occupied in passing from one to the other is negligible compared to the time of contact.

The battery used was six LECLANCHÉ cells, and the variable resistances were taken from a box by ELLIOTT Tuning forks were used, making 99, 64, and 49 complete vibrations per second. The forks were worked by a current from the storage cells in the Laboratory.

The coefficients of expansion with temperature of lead, ebonite, and glass are ·000029, 000077, and 000009 respectively. That of mica does not seem to have been determined, but it is probably less than that of lead So that the combined effects of expansion with rise of temperature do not affect the capacity before the fifth decimal place; and, as the results do not profess to be correct to more than four decimal places, the expansion of the materials may be neglected.

Some observations were also taken with ebonite by an electrostatic method based upon that described in MAXWELL'S "Electricity and Magnetism," § 229 (second edition); the only difference being that there was no guard ring. The condenser was compared with a variable condenser by dividing a charge between them, separating them, and connecting opposite poles together and to the electrometer, then, if they were unequal, the difference deflected the electrometer For this method a special key was required to make the connections quickly and to keep the electrometer to earth until everything except the charge to be measured had been discharged. If the electrometer were not kept to earth it would be deflected in spite of ordinary screens by induction inside the key on account of the high potentials required

When neither of the condensers being compared has any absorption, there is no difficulty with this method, and any ordinary quadrant electrometer may be used. But when this is not the case the residual discharge will begin to come out as soon as the condensers are connected, and this makes the needle always go ultimately to the same side. The only way to get a balance is to make the variable condenser too large, and gradually diminish it until the initial motion of the needle due to it disappears This requires a needle with a short time of swing, because with a slow

needle the residual charge comes out and overpowers the first effect before the needle has time to show it With a small ebonite condenser it was just possible to make an ELLIOTT'S electrometer sufficient by making the connection as short as possible, but with a large condenser it was useless Consequently, this experiment would require an electrometer with a needle of very short time of swing One with small cylindrical needle and quadrants seems most suitable

Mica

The mica condenser was made of twenty-two sheets of brown Muscovite with equilateral triangular markings. The sheets were about 3½ inches diameter, and the thicknesses varied from about a quarter to about a tenth of a millimetre Some difficulty was found in getting rid of surface conduction on the mica. No cleaning would make it insulate, but in the end perfect insulation was secured by putting a border of shellac varnish round the edges of the sheets The shellac did not come between the plates of the condenser, and so did not affect the capacity. The highest temperatures in the preliminary heating were less than 140° C

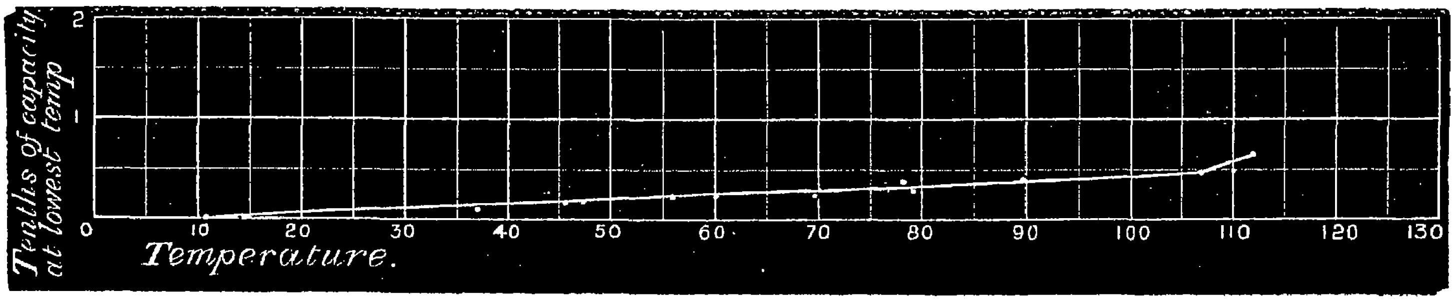
The insulation was constantly tested by an electrometer, and was found perfect throughout for temperatures below 110° C Above this it began to give way

The results are given in the following table and diagram (fig. 3).

Temperature	Variable resistance	Change of capacity	Rate of increase per degree.
Fork making 99 complete vibrations per second			
11° C	4890		
37	4850	0082	00032
46	4830	0122	·00035
56	4810	·0163	00035
70	4790	0204	·00035
79	4770	0245	00036
89	4750	0286	00035
107	4700	0387	00040
112	4600	0591	00059
Fork making 64 complete vibrations per second			
14	5530		
48	5460	0127	00037
60	5440	0163	00035
70	5425	0190	·00034
78	5390	·0253	·00039
89	5340	·0343	00045
110	5270	·0470	00049

The fact that both periods give the same value shows that, as was to be expected from its crystalline structure, there is no absorption in mica for those short times of charging

Fig 3



Ebonite

The ebonite condenser was made of twenty discs of ebonite, about a third of a millimetre thick, and three and a quarter inches in diameter. The ebonite sheets were carefully cleaned with paraffin dissolved in benzol, and this gave perfect insulation without any shellac border At temperatures above 70° C the capacity began to increase rapidly owing to softening The highest temperature reached in the preliminary heating was a little under 80° C

The insulation was constantly tested as in the case of mica, and was found perfect up to about 70° C

The results are given in the following table and diagram (fig 4)

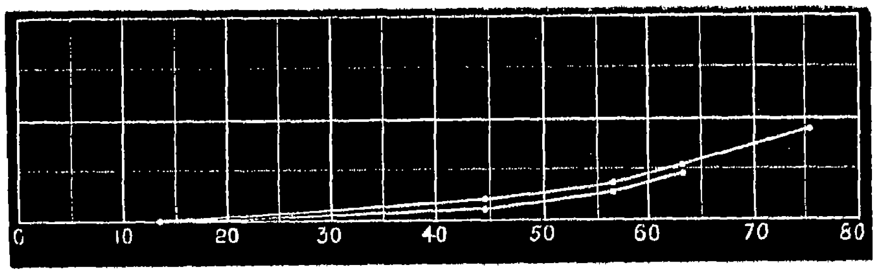
As the insulation was perfect, the greater capacity shown by the longer time of charging must be due to absorption—an effect which the structure of ebonite would lead us to expect The column of values for instantaneous charge have been calculated on the assumption that the absorption goes on at the same rate from the beginning of the charge, as during the interval observed.

Temperature	Variable resistance	Change of capacity	Rate of increase per degree
Fork making 99 complete vibrations per second			
13° C	3520		
44	3460	017	00055
57	3415	030	·00068
63	3355	044	00088
Fork making 64 complete vibrations per second.			
13	5250		
44	5143	020	00065
57	5060	036	00081
63	4960	055	00110
75	4800	085	00170

VALUES of rate of change corrected for instantaneous charge

44° C	00037
57	00043
63	00048

Fig 4



Electrostatic Observations—In the observations on ebonite by the electrostatic method the condenser consisted of two sheets of ebonite between the iron discs, and with a copper disc between them. The condenser rested upon a glass tripod inside the air bath, and the leading wires entered the air bath through glass tubes. The capacity of this was adjusted so as to be within the range of a fine sliding condenser in the Laboratory against which it was balanced; and the readings were taken on an Elliott electrometer. It required great care and patience to make so slow an electrometer suffice, for the reasons already stated. The insulation of all parts of the apparatus also required much attention.

The cold temperature, 8°, was obtained by running iced water round the bath containing the condenser Subsequent observations showed that this had affected the insulation and so increased the apparent diminution of capacity

The greater rate of change shown by the electrostatic results is probably due to the time of charging by the key worked by hand being greater than with the tuning fork.

Any leak would produce an apparent decrease of capacity by this method; so that, although the electrostatic results are of less weight than the others, the agreement of the results of two methods in which the main source of error tends in opposite directions is a confirmation of their accuracy

The electrostatic results are given in the following table .—

Temperature	Rate of change of capacity per degree
17° C	
40	·0008
48	0010
50	0016
57	0015
(8	0021)

Glass.

The glass condenser was made with seventeen sheets of thin microscope-slide cover-glass, about three inches diameter They consisted of a soda glass of high conductivity, so that, although the discs were carefully cleaned and bordered with shellac as in the case of mica, the insulation of the condenser, as tested by the electrometer, was never perfect. Since it is only the charging of the condenser that affects the galvanometer, the discharge passing sensibly all through D R (fig 1), the defective insulation introduces simply a steady current during each time of charging Each time the condenser is charged, the quantity of electricity passing D and B consists of (1) the charge of the condenser, and (2) the current through the condenser, which lasts during the time the vibrator P is in contact with S For our present purpose, the current may be considered as immediately established at full strength when the circuit is closed So that, neglecting absorption, the apparent capacity exceeds the true capacity by the conductivity of the condenser multiplied by the duration of the contact between P and S And from observations on the apparent capacity with two forks of known speed, the true capacity can easily be found

The highest temperature in the preliminary heating was less than 110° C

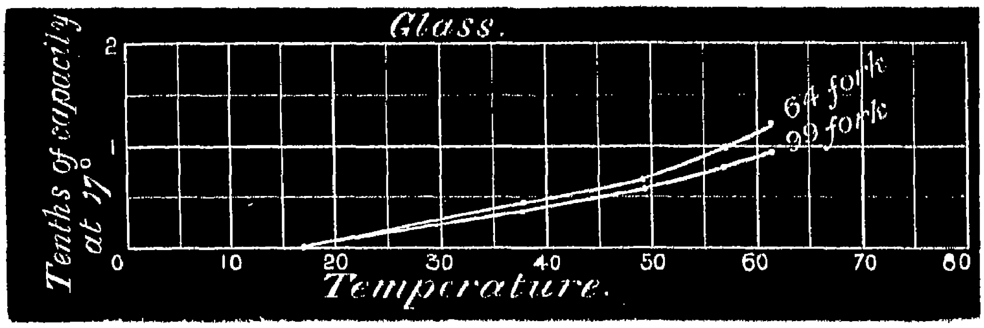
The results are given in the following table and diagram (fig 5) —

Temperature	Variable resistance	Change of capacity	Rate of change per degree
Fork making 99 complete vibrations per second			
17° C	4795		
38	4630	034	0016
49	4540	053	0017
57	4425	077	0019
62	4320	099	0022
Fork making 64 complete vibrations per second			
17	6070		
38	5830	039	0018
49	5670	066	0021
57	5470	098	0024
62	5360	117	0026

RATE of change corrected for instantaneous charge

17° to 38°	0013
49	0010
57	0010
62	0015

Fig 5



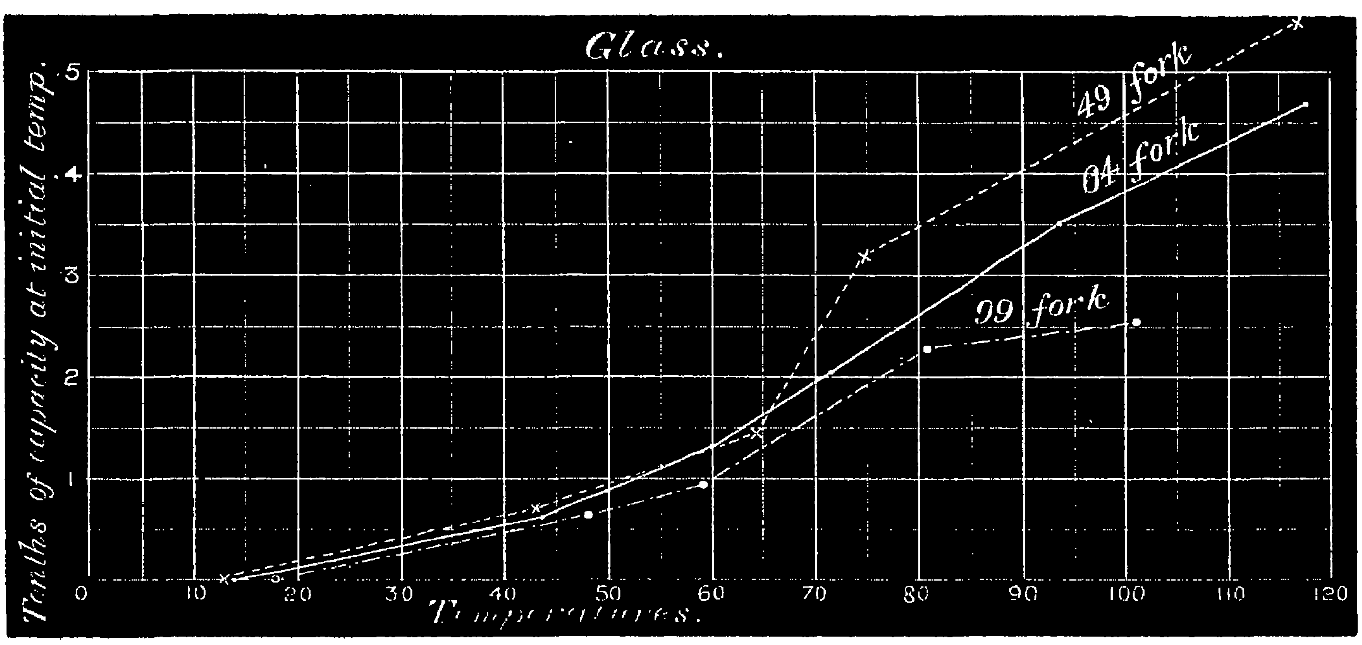
The next three tables and diagram give the results of experiments upon another condenser of microscope-slide cover-glass. This specimen of the glass had less conductivity than the previous, but the precautions for general insulation were not equal to those in the other experiments described. The condenser rested on a glass tripod inside the air bath, and the leading wires passed into the bath through glass tubes. The condenser consisted of twenty-two discs of glass cleaned and bordered with shellac in the usual way, with discs of lead foil between them

The curves are almost exactly parallel up to between 40° and 50°, and diverge at higher temperatures in consequence of conduction, so that we may take the temperature change of specific inductive capacity for this specimen of glass to be about 2 per cent. up to 50°.

An attempt was made to observe this temperature effect for shellac. The condenser was made by dipping the lead discs in shellac varnish, and carefully and thoroughly evaporating the alcohol. But after the condenser was made it was found that at a temperature below 50° the shellac softened, so that the plates were pressed together by the weights on the top of the condenser. To diminish the weight would have been useless, because, in any case, it would have been impossible to say how far a change of capacity was due to softening, and without a weight at all the results are quite unreliable

Temperature	Variable resistance	Change of capacity in terms of capacity at lowest temperature	Rate of change per degree
	With 99 fork		
18° C	4010		
48	3750	0645	00215
59	3630	0941	00230
81	3070	2311	00369
101	2890	2541	00251
	With "64 fork" (between 64 and 65).		
14	6110		
44	5700	0671	00224
50	5560	0900	00250
60	5270	1351	00293
71 5	4850	2062	00362
94	3840	3551	00444
117	3230	4712	00506
	With 49 fork		
13	8100		
43	7480	0765	00255
64 5	7020	1538	00300
75	5850	3201	00516
115	4250	5484	00537

Fig 6.

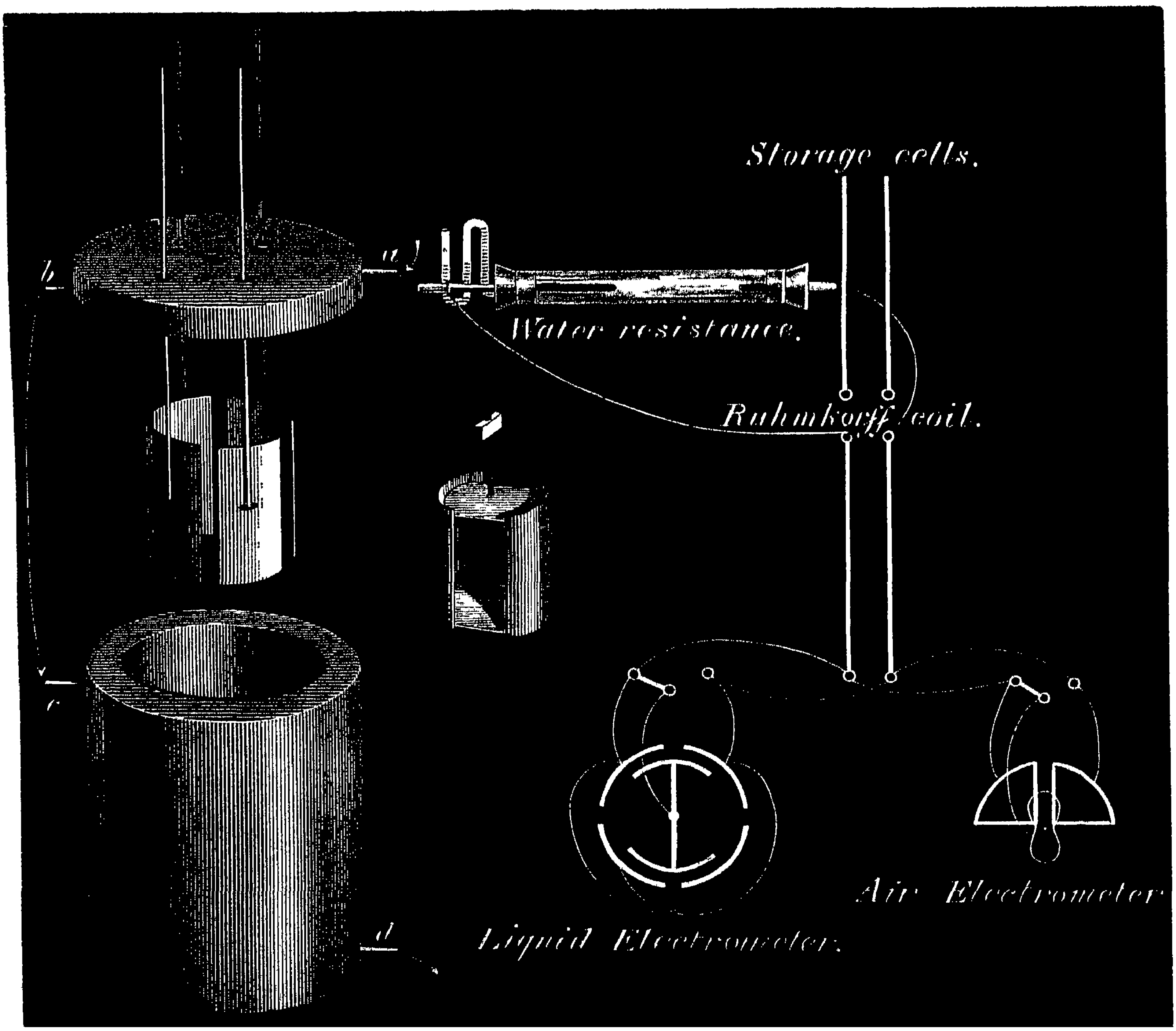


II. Liquids.

The apparatus for liquids consisted essentially of a quadrant electrometer immersed in the liquid.

The heating was done by a water bath, and on account of the highly inflammable character of some of the liquids experimented upon, the water was heated in a separate vessel at a distance, from the top and bottom of which pipes went to *a* and *d* in fig 7. The hot water was pumped through in the direction *d c b a*.

Fig 7



The provision for the insulation was on a similar principle to that adopted in the case of solids. Each quadrant was supported by a long stiff rod, the upper part of which was fixed in a varnished glass tube. These four glass tubes were separately

clamped to a support about 18 inches above the surface of the liquid, so that the quadrants touched nothing except the liquid and this perfectly insulating support The needle was suspended by a fine wire from the same height so as to oscillate under torsion

The hollow top of the water bath had seven holes through it as shown in the figure Through the outside four passed, without touching, the rods supporting the quadrants, through the centre one hung the needle, and through the two next the centre thermometers could be inserted This top was permanently fixed to the support of the needle and quadrants, and the lower part of the bath containing the liquid was moved up on a smooth sliding arrangement into contact with the top so as to immerse the needle and quadrants Above the fixed top of the bath was a box with a window on one side, through which the movements of the needle and mirror were read by a scale and telescope The deflections of the needle were observed when connected first to one pair of quadrants and then to the other pair

As the needle and quadrants were parts of cylinders between three and four inches diameter and about a quarter of an inch apart, a large electromotive force was required to produce a deflection The electromotive force had also to be rapidly reversed in direction to avoid as far as possible polarisation, convection, &c, in the liquid The electromotive force was obtained from a Ruhmkorff coil without a condenser, and with a high resistance between the terminals to prevent sparking This high resistance consisted of a wide glass tube, about 6 inches long, filled with distilled water, and having a thick copper wire sliding through a cork at each end. By altering the distance of the ends of the copper wires in the water the resistance could be adjusted and the deflection controlled as desired. The coil was worked by a current from the storage cells in the Laboratory.

As the electromotive force given by this arrangement was variable, and also the loss of electromotive force by conduction was different for each liquid and for each temperature, it was necessary to be independent of such changes Accordingly a second electrometer was placed between the terminals just outside the liquid one, which being always in the same state gave the comparative values of the electromotive force This second electrometer was an Elliott's lecture-room pattern with two quadrants removed and the needle connected to one of the remaining quadrants. The advantage of this form for the present purpose was that by moving the needle up from the quadrants the deflection could be diminished to any desired extent. Then the quotient of the deflection of the liquid electrometer by the deflection of the second electrometer was proportional to the specific inductive capacity of the liquid. For every deflection of the liquid electrometer a pair of deflections of the second electrometer were taken with the needle connected first to one pair of quadrants and then to the other. The general arrangement of the apparatus is shown in the figure.

The liquids experimented upon were turpentine, carbon bisulphide, glycerine, benzene, benzylene, olive oil, and paraffin oil Methylated spirit was also tried; but

its conductivity was so great that no deflection could be obtained. All except paraffin oil showed a decrease of specific inductive capacity with rise of temperature The paraffin oil was that used in the lamps in the Laboratory, and its exceptional behaviour may have been due to some secondary action arising from impurity.

The results are given in the following tables and diagram And to show the way in which the observations were made, the readings are given for turpentine at two temperatures

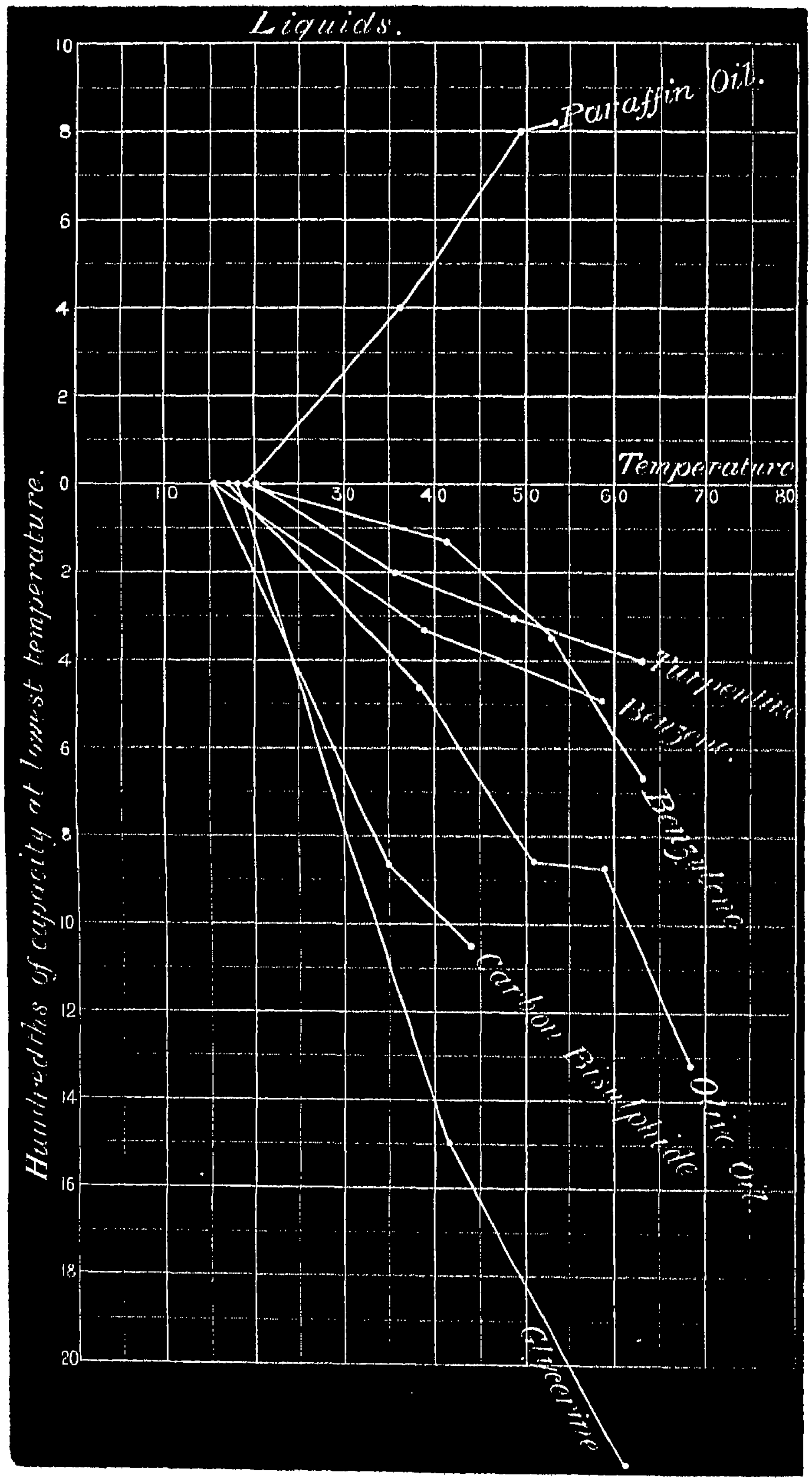
	Temperature	Ratio of specific inductive capacity to that at the lowest temperature	Rate of decrease
Turpentine			
	20° C	1 0000	
	36	·9800	·0012
	49	9700	0011
	62	·9600	·0009
Carbon bisulphide			
	15	1·0000	
	35	9130	·0040
	43	·8940	·0040
Glycerine			
	18	1·0000	
	41	8500	0060
	61	·7760	·0053
Benzylene			
	19	1 0000	
	41	·9870	0006
	52	·9640	0011
	63	9350	0015
Benzene			
	15	1 0000	
	39	·9665	·0014
	58 5	9507	0012
Olive oil ·			
	17	1·0000	
	38	9530	·0021
	51	9140	·0025
	59	·9130	0021
	68	8670	0026

	Temperature	Ratio of specific inductive capacity to that at the lowest temperature	Rate of increase
Paraffin oil	18° C	1 000	
	36 5	1 040	0022
	49 5	1 080	0025
	54	1 081	0022

READINGS for Turpentine.

Temperature	Liquid Electrometer	Air electrometer	Ratio of electrometer readings
20°			
Zero	16 5		
Needle connected to one pair	6 2	85	098
” ”	6 2	— 20	
” other ”	26 6	— 21	102
” ”	26 0	75	
Zero	16 8		
			Mean 100
36°			
Zero	17 3		
Needle connected to one pair	26 3	83	098
” ”	26 3	— 15	
” other ”	7 4	— 17	098
” ”	7 4	79	
			Mean 098

Fig 8



If the relation indicated by CLERK MAXWELL'S electromagnetic theory of light held good, and the specific inductive capacity were equal to the square of the refractive index, then the rate of change of specific inductive capacity with temperature ought to be twice that of the refractive index. Amongst the liquids, for which DALE and GLADSTONE have observed the refractive indices at different temperatures, are four of those dealt with here The mean rates of change, with temperature of the refractive index for the A line in the spectrum, deduced from their observations, are

Turpentine	00035	for temperature	range	from	10°	to	17°
Benzene	00040	,,	,,	,,	10	,,	39
Benzylene	00037	,,	,,	,,	25	,,	39
Glycerine .	00018	,,	,,	,,	20	,,	48

Thus, it appears that although the two rates of change for Glycerine present no similarity whatever, those of the rest of the four are in a ratio not very far from 1 to 2, the approach being nearest in the case of Benzylene

Appendix

(Received October 18, 1889)

A suggestion having been made that, as the opposite effects of rise of temperature upon solids and liquids were observed by different methods, it would be well that both should be tested by the same method, a qualitative experiment was made on a solid by the method used for liquids, to see whether the result would agree with that already obtained.

A cylinder of glass was placed between the quadrants and needle of the liquid electrometer, leaving the needle free to oscillate, and observations were taken at different temperatures, exactly as already described for the case of a liquid dielectric The result was always an increase of specific inductive capacity with rise of temperature All the precautions for insulation, &c, were observed as in the experiments already described But the heating was not maintained long enough to secure that the glass had acquired the full temperature of the air in the bath, so that the results obtained are only qualitative, the change being less than that corresponding to the temperature indicated by the air inside the bath In one case, where the heating had lasted several hours, the rate of change rose as high as 0024 per degree for a range of 50° C., a result sufficiently close to that obtained for glass by the other method.

II. On the Interchange of the Variables in Certain Linear Differential Operators.

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Communicated by Professor SYLVESTER, F.R.S.

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CONTENTS.

1. The operators to be considered, include or involve all those which have presented themselves as annihilators and generators in recent theories of functional differential invariants, reciprocants, cyclicants, &c. The general form of the binary operators, operators whose arguments are the derivatives of one dependent with regard to one independent variable, which I propose first to consider, is adopted in accordance with that used in two remarkable papers by Major MACMAHON.* They are his operators in four elements. The analogous ternary operators to which I subsequently devote attention, are distinct from his operators of six elements. Their arguments are the partial derivatives of one of three variables, supposed connected by a single relation, with regard to the two others.

* "The Theory of a Multilinear Partial Differential Operator with applications to the Theories of Invariants and Reciprocants," 'London Math. Soc. Proc.,' vol. 18, 1887, pp. 61–88. "The Algebra of Multilinear Partial Differential Operators," 'London Math. Soc. Proc.,' vol. 19, pp. 112–128.

The only previous contribution, of which I am aware, to the subject of the reversion of MacMahon operators, is a paper by Professor L J Rogers,* in which he obtains the operator reciprocal to $\{\mu, \nu, 1, 1\}$, and alludes to the self reciprocal property of V which is expressed with more precision in (38) below.

I Binary Operators

2 Let x and y be two variables connected by any relation. Let y_r denote $\frac{1}{r!}\frac{d^r y}{dx^r}$, and x_r denote $\frac{1}{r!}\frac{d^r x}{dy^r}$.

Let ξ and η be corresponding increments of x and y, so that by Taylor's theorem

$$\eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots \qquad (1)$$

and

$$\xi = x_1\eta + x_2\eta^2 + x_3\eta^3 + \ldots, \qquad (2)$$

the one expansion being a reversion of the other.

Let $Y_s^{(m)}$ denote the coefficient of ξ^s in the expansion of η^m, i e, of $(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$ in ascending integral powers of ξ; and $X_s^{(m)}$ the coefficient of η^s in the expansion of ξ^m, i e, of $(x_1\eta + x_2\eta^2 + x_3\eta^3 + \ldots)^m$ in ascending integral powers of η It is supposed that m is not fractional It need not, however, be positive Nor is it necessary to exclude the value zero, which, though somewhat special, will be seen to be of importance later

Let n be a positive or negative integer or zero, and let μ and ν be any numerical quantities

Denote

$$\frac{1}{m}\Sigma\left\{(\mu + \nu s)\, X_s^{(m)} \frac{d}{dx_{n+s}}\right\} \text{ by } \{\mu, \nu, m, n\}_x, \qquad (3)$$

and

$$\frac{1}{m}\Sigma\left\{(\mu + \nu s)\, Y_s^{(m)} \frac{d}{dy_{n+s}}\right\} \text{ by } \{\mu, \nu, m, n\}_y, \qquad (4)$$

the summations being, with regard to s, which assumes in turn all integral values not less than the greater of the two m and $-n+1$, so that, if $m+n > 1$, only symbols of differentiation with regard to all derivatives from y_{m+n} onwards may occur, while, if $m + n \overline{\gtrless} 1$, symbols of differentiation with regard to all derivatives may be present

It is the operators $\{\mu, \nu, m, n\}_x$ and $\{\mu, \nu; m, n\}_y$ of which I propose to speak as MacMahon operators in x and y, respectively, dependent. It will be seen upon reference to the first paper referred to above that they are the results of substitution in Major MacMahon's operator (μ, ν, m, n) for

* "Note on Conjugate Annihilators of Homogeneous and Isobaric Differential Equations," 'Messenger of Mathematics,' vol 18, pp 153–158.

$$a_0, a_1, a_2, a_3,$$

of

$$0, x_1, x_2, x_3, \qquad \text{in the one case,}$$

and of

$$0, y_1, y_2, y_3, \qquad \text{in the other}$$

MACMAHON himself generally takes them as meaning

$$y_2, y_3, y_4, y_5 \qquad ,$$

a fact which must not be forgotten in connecting his results with those to be here obtained

The essential difference between the cases of $m + n \nless 1$ and $m + n < 1$ should be noticed at the outset In the former case, the complete set of coefficients $X_s^{(m)}$ appears in the operator $\{\mu, \nu, m, n\}_x$ In the latter, one or more of those coefficients (a number of them equal to the excess of $-n + 1$ over m) is wanting at the beginning

3. The aim in view is to express any MacMahon operator $\{\mu, \nu, m, n\}_x$ in x dependent as an operator or sum of operators of like form $\{\mu', \nu', m', n'\}_y$ in y dependent We need the linear expressions in $d/dy_1, d/dy_2, d/dy_3, \ldots$ which, when operating on any function of $y_1, y_2, y_3,$ are equivalent to $d/dx_1, d/dx_2, d/dx_3, \ldots$ operating on the equal function of $x_1, x_2, x_3,$ The expressions in question I have obtained in the second* of a series of papers on Cyclicants, &c The best form for present purposes is hardly there given to the conclusions It will therefore result in a gain of clearness and no loss of brevity if in the present article the proof is given rather than the result quoted The same remark will apply to Article 17 below.

We may look upon $x_1, x_2, x_3, \ldots$ as a number of independent quantities, upon $y_1, y_2, y_3, \ldots$ as determinate functions of these quantities, and upon ξ and η as two other quantities connected with one another and with $x_1, x_2, x_3, \ldots$ by (1) or its equivalent (2)

Give x_r alone of all the quantites $x_1, x_2, x_3, \ldots$ an infinitesimal variation Keep η constant In virtue of (2) or its equivalent (1) ξ will vary in consequence of the variation of x_r. Also, as some or all of $y_1, y_2, y_3, \ldots$ are functions of x_r, some or all of those quantities will vary. Thus, from (2) we shall obtain

$$\delta\xi = \eta^r \, \delta x_r,$$

and from (1)

$$0 = \{y_1 + 2y_2\xi + 3y_3\xi^2 + \ldots\} \delta\xi + \left\{\frac{dy_1}{dx_r}\xi + \frac{dy_2}{dx_r}\xi^2 + \frac{dy_3}{dx_r}\xi^3 + \ldots\right\} \delta x_r = 0.$$

* "On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants," 'London Math Soc Proc,' vol 18, 1887, pp 142–164

Accordingly it follows that

$$\frac{dy_1}{dx_r}\xi + \frac{dy_2}{dx_r}\xi^2 + \frac{dy_3}{dx_r}\xi^3 + \quad = -\eta^r\{y_1 + 2y_2\xi + 3y_3\xi^2 + \ \cdot\ \}, \tag{5}$$

and consequently, this being true for all values of ξ, that if by aid of (1) this right hand member be expanded in ascending powers of ξ, the coefficients of ξ, ξ^2, ξ^3, . are exactly the expressions for dy_1/dx_r, dy_2/dx_r, dy_3/dx_r,

Now

$$\frac{d}{dx_r} = \frac{dy_1}{dx_r}\frac{d}{dy_1} + \frac{dy_2}{dx_r}\frac{d}{dy_2} + \frac{dy_3}{dx_r}\frac{d}{dy_3} + \quad ,$$

and is therefore the result of replacing each power ξ^s of ξ on the left, and therefore on the right, of (5) by the corresponding symbol d/dy_s. It follows that the expression on the right of (5) may be taken as a symbolical representation of the equivalent operator to d/dx_r, i e , that

$$\frac{d}{dx_r} = -(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ \cdot\ )^r(y_1 + 2y_2\xi + 3y_3\xi^2 + \ \cdot\cdot)$$

$$= -\eta^r\frac{d\eta}{d\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{6}$$

where the meaning of the symbolisation on the right is that η is to be replaced by its equivalent in terms of ξ from (1), that the differentiation with regard to ξ is partial, that the product on the right is to be expanded as a sum of multiples of powers of ξ, and that then each power ξ^s is to be replaced by the corresponding symbol of differentiation d/dy_s

4 The proof of (6) is the same *for all positive integral values (including unity) of r*. Thus the means of transforming any differential operator whatever is obtained.

The rule according to which any linear operator is transformed may be very simply expressed.

Exactly companion to the symbolical notation ξ^s for d/dy_s in an operator linear in d/dy_1, d/dy_2, d/dy_3, . . is the notation η^s for d/dx_s in an operator linear in d/dx_1, d/dx_2, d/dx_3, . . Now, writing a linear operator

$$\mathrm{A}\frac{d}{dx_\alpha} + \mathrm{B}\frac{d}{dx_\beta} + \mathrm{C}\frac{d}{dx_\gamma} + \ldots$$

in the symbolical form

$$\mathrm{A}\eta^\alpha + \mathrm{B}\eta^\beta + \mathrm{C}\eta^\gamma + \ldots,$$

we learn by (6) that its equivalent in d/dy_1, d/dy_2, d/y_3, . . . is obtained by multiplication by $-d\eta/d\xi$, expansion in terms of ξ by (1), and substitution for each power ξ^s in the expanded result, of the corresponding d/dy_s.

5 Now the symbolical form of any MacMahon operator for which $m + n \nless 1$ is very simple By (4) that of,

$$\{\mu, \nu, m, n\}_y$$

is

$$\frac{1}{m} \Sigma_{s \nless m} (\mu + \nu s) \mathrm{Y}_s^{(m)} \xi^{n+s}$$

i e

$$\frac{\mu}{m} \xi^n (y_1 \xi + y_2 \xi^2 + y_3 \xi^3 + \quad)^m + \frac{\nu}{m} \xi^{n+1} \frac{d}{d\xi} (y_1 \xi + y_2 \xi^2 + y_3 \xi^3 + \quad)^m$$

i e

$$\frac{\mu}{m} \xi^n \eta^m + \nu \xi^{n+1} \eta^{m-1} \frac{d\eta}{d\xi} \tag{7}$$

Thus in particular

$$\{1, 0, m, n,\}_y = \frac{1}{m} \xi^n \eta^m, \tag{8}$$

and

$$\{0, 1, m, n\}_y = \xi^{n+1} \eta^{m-1} \frac{d\eta}{d\xi}, \tag{9}$$

the right hand members being supposed to be expanded in terms of ξ by aid of (1), and then to have each power ξ^s of ξ which occurs replaced by the corresponding d/dy_s

We may of course write (7)

$$\{\mu, \nu, m, n\}_y = \mu\{1, 0, m, n\}_y + \nu\{0, 1, m, n\}_y, \tag{10}$$

so that in (8) and (9) we have involved all MacMahon operators in y for which $m + n$ is not less than unity A reservation must for the moment be made of the case $m = 0$

Exactly corresponding to (8) and (9) we have the symbolical forms

$$\{1, 0, m, n\}_x = \frac{1}{m} \eta^n \xi^m, \tag{11}$$

$$\{0, 1, m, n\}_x = \eta^{n+1} \xi^{m-1} \frac{d\xi}{d\eta}, \tag{12}$$

where the expansions on the right are to be in ascending powers of η by (2), and where in an expansion each η^s is to be replaced by the corresponding d/dx_s.

6 The transformation of $\{\mu, \nu; m, n\}_x$ for the cases at present under consideration of $m + n$ not less than unity is now immediate. By Art 4 the transformed form of the expansion in terms of η of

$$\eta^n \xi^m,$$

considered as the symbolical form of an operator in x dependent, is the expansion in terms of ξ of

$$-\xi^m\eta^n\frac{d\eta}{d\xi},$$

considered as an operator in y dependent.

In other words, by (11) and (9),

$$m\{1,0,m,n\}_x = -\{0,1,n+1,m-1\}_y \quad . \tag{13}$$

Again, the transformed form of the symbolical expansion in powers of η of

$$\eta^{n+1}\xi^{m-1}\frac{d\xi}{d\eta}$$

is, by Art 4, the symbolical expansion in powers of ξ of

$$-\xi^{m-1}\eta^{n+1}\frac{d\xi}{d\eta}\cdot\frac{d\eta}{d\xi},$$

i e, of

$$-\xi^{m-1}\eta^{n+1},$$

since in $d\xi/d\eta$ and $d\eta/d\xi$ the derivatives x_1 x_2, and $y_1, y_2, \ldots$ are not regarded as variables In other words, by (12) and (8),

$$\{0,1;m,n\}_x = -(n+1)\{1,0;n+1,m-1\}_y \tag{14}$$

It is to be remarked that (13) and (14) are entirely in accord Either of them is produced from the other by the interchange of x and y and of m and $n+1$

From (13) and (14) by aid of (10) we produce the more general equality of operators

$$\{\mu,\nu,m,n\}_x = -\left\{\nu(n+1),\frac{\mu}{m},n+1,m-1\right\}_y \quad . \tag{15}$$

which may be given the more symmetrical form

$$\{m\mu,\mu',m,m'-1\}_x = -\{m'\mu',\mu,m',m-1\}_y, \tag{16}$$

in which $m+m'$ has to be positive

In (16) are included two interesting classes of particular cases, viz :—

$$\{-m,1,m,m-1\}_x = \{-m,1;m,m-1\}_y, \quad . \quad . \quad . \tag{17}$$

and

$$\{m,1,m,m-1\}_x = -\{m,1,m,m-1\}_y \quad . \quad . \tag{18}$$

Corresponding to each positive degree m there are then two self-reciprocal operators.* The first is of positive character, being entirely unaltered in form by

* Self reciprocal operators, of course, generate from absolute reciprocants other absolute reciprocants

interchange of x and y, and the second of negative character, persisting in form but for a change of sign. (A complex self reciprocal operator can of course be found by taking the sum or difference of any two correlative operators; e g,

$$m\{1,\ 0,\ m,\ n\}_x \mp \{0,\ 1,\ n+1,\ m-1\}_x)$$

At greater length (17) and (18) are

$$\overset{(m)}{\mathrm{X}}_{m+1}\frac{d}{dx_{2m}} + 2\overset{(m)}{\mathrm{X}}_{m+2}\frac{d}{dx_{2m+1}} + 3\overset{(m)}{\mathrm{X}}_{m+3}\frac{d}{dx_{2m+2}} +$$

$$= \overset{(m)}{\mathrm{Y}}_{m+1}\frac{d}{dy_{2m}} + 2\overset{(m)}{\mathrm{Y}}_{m+2}\frac{d}{dy_{m+1}} + 3\overset{(m)}{\mathrm{Y}}_{m+3}\frac{d}{dy_{m+2}} + \quad , \qquad . \ . \ . \quad (17\text{A})$$

and

$$2m\,\overset{(m)}{\mathrm{X}}_{m}\frac{d}{dx_{2m-1}} + (2m+1)\,\overset{(m)}{\mathrm{X}}_{m+1}\frac{d}{dx_{2m}} + (2m+2)\,\overset{(m)}{\mathrm{X}}_{m+2}\frac{d}{dx_{2m+1}} + \quad .$$

$$= -\left\{2m\,\overset{(m)}{\mathrm{Y}}_{m}\frac{d}{dy_{2m-1}} + (2m+1)\,\overset{(m)}{\mathrm{Y}}_{m+1}\frac{d}{dy_{2m}} + (2m+2)\,\overset{(m)}{\mathrm{Y}}_{m+2}\frac{d}{dy_{2m+1}} + \quad \right\} \quad (18\text{A})$$

In particular for $m = 1$ we have

$$x_2\frac{d}{dx_2} + 2x_3\frac{d}{dx_3} + 3x_4\frac{d}{dx_4} + \quad . \quad = y_2\frac{d}{dy_2} + 2y_3\frac{d}{dy_3} + 3y_4\frac{d}{dy_4} + \quad . \qquad (19)$$

and

$$2x_1\frac{d}{dx_1} + 3x_2\frac{d}{dx_2} + 4x_3\frac{d}{dx_3} + \quad = -\left\{2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + 4y_3\frac{d}{dy_3} + \quad \right\} \quad (20)$$

Again $m = 2$ gives us that

$$2x_1x_2\frac{d}{dx_4} + 2\,(2x_1x_3 + x_2^2)\frac{d}{dx_5} + 3\,(2x_1x_4 + 2x_2x_3)\frac{d}{dx_6}$$

$$+ 4\,(2x_1x_5 + 2x_2x_4 + x_3^2)\frac{d}{dx_7} + \ . \ \ . \ , \qquad . \qquad (21)$$

and

$$4x_1^2\frac{d}{dx_3} + 5\ \ 2x_1x_2\frac{d}{dx_4} + 6\,(2x_1x_3 + x_2^2)\frac{d}{dx_5} + 7\,(2x_1x_4 + 2x_2x_3)\frac{d}{dx_6} + \ \ . \ \ , \qquad (22)$$

are self reciprocal operators of positive and negative characters respectively.

7 As other examples of the important formulæ of transformation (13) and (14), let us write down cases corresponding to $m = 1$, $n \nless 0$

For $m = 1$, $n = 0$ we obtain

$$x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3} + \ .\ . \quad = -\left\{y_1\frac{d}{dy_1} + 2y_2\frac{d}{dy_2} + 3y_3\frac{d}{dy_3} + \ .\ .\ .\ \right\}, \qquad (23)$$

$$x_1\frac{d}{dx_1} + 2x_2\frac{d}{dx_2} + 3x_3\frac{d}{dx_3} + \ . \quad . = -\left\{y_1\frac{d}{dy_1} + y_2\frac{d}{dy_2} + y_3\frac{d}{dy_3} + \ . \quad \right\}, \qquad (24)$$

which together are equivalent to (19) and (20) together. From (23) it follows that a homogeneous function of $x_1\, x_2\, x_3, \ldots$ transforms into an isobaric function of $y_1\, y_2\, y_3, \ldots$, and that, i and w meaning degree and weight respectively,

$$i_x = -w_y,$$

while from (24) follows the equivalent fact that an isobaric function transforms into a homogeneous one, and that

$$w_x = -i_y.$$

From (19) follows the especially interesting fact that, if a function of $x_1, x_2\, x_3, \ldots$ is isobaric in $x_2, x_3, x_4, \ldots$ upon considering the weight of x_r to be $r-1$, so also is the transformed function of $y_1, y_2, y_3, \ldots$ isobaric in the same sense and of the same weight in $y_2, y_3, y_4, \ldots$

Again the substitution $m = 1$, $n = 1$ in (13) and (14) produces for us

$$x_1 \frac{d}{dx_2} + x_2 \frac{d}{dx_3} + x_3 \frac{d}{dx_4} + \ldots = -\tfrac{1}{2}\left\{2\mathrm{Y}_2^{(2)} \frac{d}{dy_2} + 3\mathrm{Y}_3^{(2)} \frac{d}{dy_3} + 4\mathrm{Y}_4^{(2)} \frac{d}{dy_4} + \ldots\right\}, \qquad (25)$$

and

$$x_1 \frac{d}{dx_2} + 2x_2 \frac{d}{dx_3} + 3x_3 \frac{d}{dx_4} + \ldots = -\left\{\mathrm{Y}_2^{(2)} \frac{d}{dy_2} + \mathrm{Y}_3^{(2)} \frac{d}{dy_3} + \mathrm{Y}_4^{(2)} \frac{d}{dy_4} + \ldots\right\}, \qquad (26)$$

where

$$\mathrm{Y}_2^{(2)} = y_1^2,\ \mathrm{Y}_3^{(2)} = 2y_1y_2,\ \mathrm{Y}_4^{(2)} = 2y_1y_3 + y_2^2,\ \mathrm{Y}_5^{(2)} = 2y_1y_4 + 2y_2y_3, \ldots$$

These two transformations have been obtained by Professor ROGERS (see note to Art 1). The second tells us that what he calls primary invariants in $x_1, x_2, x_3, \ldots$ have for their transforms what he calls secondary invariants in $y_1, y_2, y_3, \ldots$

We might now consider the results of putting $m = 1$ and $n = 2, 3, \ldots$ in (13) and (14). By this means the transformation of lineo-linear operators of two, three, &c., steps is effected For the general case $m = 1$, $n = n$ the results are

$$\begin{aligned} x_1 \frac{d}{dx_{n+1}} &+ x_2 \frac{d}{dx_{n+2}} + x_3 \frac{d}{dx_{n+3}} + \ldots \\ &= -\frac{1}{n+1}\left\{(n+1)\, \mathrm{Y}_{(n+1)}^{(n+1)} \frac{d}{dy_{n+1}} + (n+2)\, \mathrm{Y}_{(n+2)}^{(n+1)} \frac{d}{dy_{n+2}} + \ldots\right\} . \end{aligned} \qquad (27)$$

$$x_1 \frac{d}{dx_{n+1}} + 2x_2 \frac{d}{dx_{n+2}} + 3x_3 \frac{d}{dx_{n+3}} + \ldots = -\left\{\mathrm{Y}_{(n+1)}^{(n+1)} \frac{d}{dy_{n+1}} + \mathrm{Y}_{(n+2)}^{(n+1)} \frac{d}{dy_{n+2}} + \ldots\right\} \qquad (28)$$

Perhaps the most interesting fact to be deduced from (25) and (26) is the transformation of $\{-1, 1; 1, 1\}$, the second annihilator Ω of projective reciprocants. By subtraction of (25) from (26), or directly from (15)

$$\{-1, 1; 1, 1\}_x = \{-2, 1; 2, 0\}_y,$$

i e,

$$x_2 \frac{d}{dx_3} + 2x_3 \frac{d}{dx_4} + 3x_4 \frac{d}{dx_5} + $$

$$= \tfrac{1}{2} \left\{ Y_3^{(2)} \frac{d}{dy_3} + 2Y_4^{(2)} \frac{d}{dy_4} + 3Y_5^{(2)} \frac{d}{dy_5} + \qquad \right\}$$

$$= \tfrac{1}{2} \left\{ 2y_1 y_2 \frac{d}{dy_3} + 2\,(2y_1 y_3 + y_2^2) \frac{d}{dy_4} + 3\,(2y_1 y_4 + 2y_2 y_3) \frac{d}{dy_5} + . \qquad \right\}$$

$$= y_1 \left\{ y_2 \frac{d}{dy_3} + 2y_3 \frac{d}{dy_4} + 3y_4 \frac{d}{dy_5} + \quad . \right\} + 2\,\frac{y_2^2}{2} \frac{d}{dy_4} + 3y_2 y_3 \frac{d}{dy_5}$$

$$+ 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_6} + \qquad ,$$

or

$$\Omega\,(x, y) = y_1\, \Omega\,(y, x) + 2\,\frac{y_2^2}{2} \frac{d}{dy_4} + 3y_2 y_3 \frac{d}{dy_5} + 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_6} + . \tag{29}$$

Since $y_1 x_1 = 1$ we infer from this conclusion and its correlative that

$$x_1^{-\frac{1}{2}} \left\{ 2\,\frac{x_2^2}{2} \frac{d}{dx_4} + 3x_2 x_3 \frac{d}{dx_5} + 4\left(x_2 x_4 + \frac{x_3^2}{2}\right) \frac{d}{dx_6} + \quad . \right\} = y_1^{\frac{1}{2}}\, \Omega\,(y, x) - x_1^{\frac{1}{2}}\, \Omega\,(x, y)$$

$$= -\, y_1^{-\frac{1}{2}} \left\{ 2\,\frac{y_2^2}{2} \frac{d}{dy_3} + 3y_2 y_3 \frac{d}{dy_4} + 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_5} + \qquad \right\} . \tag{30}$$

is a self reciprocal operator of negative character The operator is one of considerable interest in connection with the theories of invariants and reciprocants. (See MACMAHON, 'London Math Soc Proc,' vol 18, p 75)

It also follows that the sum of $2x_1^{\frac{1}{2}}\, \Omega\,(x, y)$ and the operator on the left of (30) is a self reciprocal operator of positive character

8 To complete our theory of the reversion of MacMahon operators, for which $m + n$ is not less than unity, we must consider the somewhat special and exactly correlative cases $m = 0$, $n \nless 1$, and $n = -1$, $m \nless 2$

The operator $0\,\{1, 0\,,\, 0, n\}_x$ is d/dx_n in accordance with the general definition of $m\,\{\mu, \nu\,;\; m, n\}_x$ in (3) Thus the identity (6) may be written, by aid of (9),

$$0\,\{1,\, 0\,,\, 0,\, r\}_x = -\,\{0,\, 1\,;\; r + 1, -1\}_y, \qquad . \quad . \quad . \quad . \quad . \quad (31)$$

for any positive integral value, not excluding unity, of r which is strictly in agreement with the general formula of transformation (13)

On the other hand the general definition (3) gives to $0\,\{0, 1\,;\; 0, n\}_x$ no other meaning than zero. So far then the operator $\{0, 1\,;\; 0, n\}_x$ is indeterminate in form. An interpretation of it is now sought which shall make the case not exceptional to the general formula of transformation (14).

To discover this interpretation let us reverse the order of investigation and seek the operator in x, which is equivalent to the operator in y obtained by putting $m = 0$ in the right hand member of (14).

By (8) the symbolical form of

$$-(n+1)\{1, 0, n+1, -1\}_y$$

is

$$-\xi^{-1}\eta^{n+1}$$

The equivalent operator in x dependent has then for its symbolical form, as in Article 4,

$$\eta^{n+1}\xi^{-1}\frac{d\xi}{d\eta},$$

i e,

$$\eta^n + \eta^{n+1}\frac{d}{d\eta}\log\frac{\xi}{\eta}$$

Now

$$\begin{aligned}\frac{d}{d\eta}\log\frac{\xi}{\eta} &= \frac{d}{d\eta}\log(x_1 + x_2\eta + x_3\eta^2 + \quad .) \\ &= \frac{d}{d\eta}e^{\eta(x_2 d/dx_1 + 2x_3 d/dx_2 + 3x_4 d/dx_3 + \quad)}\cdot\log x_1 \\ &= e^{\eta(x_2 d/dx_1 + 2x_3 d/dx_2 + 3x_4 d/dx_3 + \,.\,)}\cdot\frac{x_2}{x_1}, \\ &= \frac{x_2}{x_1} + \frac{2x_1x_3 - x_2^2}{x_1^2}\eta + \frac{3x_1^2x_4 - 3x_1x_2x_3 + x_2^3}{x_1^3}\eta^2 + \\ &= \frac{x_2}{x_1} + \frac{Gx_2}{x_1^2}\eta + \frac{G^2x_2}{x_1^3}\cdot\frac{\eta^2}{1.2} + \frac{G^3x_2}{x_1^4}\cdot\frac{\eta^3}{1\ 2.3} + \cdots \end{aligned} \qquad (32)$$

$$\begin{aligned}\text{where } G &= x_1\left(x_2\frac{d}{dx_1} + 2x_3\frac{d}{dx_2} + 3x_4\frac{d}{dx_3} + \ldots\right) - x_2\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3} + \ldots\right) \\ &= (2x_1x_3 - x_2^2)\frac{d}{dx_2} + (3x_1x_4 - x_2x_3)\frac{d}{dx_3} + (4x_1x_5 - x_2x_4)\frac{d}{dx_4} + \ldots, \end{aligned} \qquad (32\text{A})$$

so that the numerators Gx_2, G^2x_2, G^3x_2, ..., are a set of seminvariant protomorphs in x_1, x_2, $2!\,x_3$, $3!\,x_4$, ...

Consequently the transformation in x dependent of $-(n+1)\{1, 0;\ n+1, -1\}_y$ is

$$\frac{d}{dx_n} + \frac{x_2}{x_1}\frac{d}{dx_{n+1}} + \frac{Gx_2}{x_1^2}\frac{d}{dx_{n+2}} + \frac{G^2x_2}{2!\,x_1^3}\frac{d}{dx_{n+3}} + \frac{G^3x_2}{3!\,x_1^4}\frac{d}{dx_{n+4}} + \ldots;$$

and it is accordingly this operator which has to be defined as

$$\{0, 1;\ 0, n\}_x{}^* \qquad (33)$$

* Cf. HAMMOND, 'London Math. Soc Proc.,' vol 18, p. 64, note.

that (14) may be regarded as holding for the value $m = 0$ as well as for non-vanishing values of m

It affords an instructive verification to conduct the investigation of the same transformation in the order of Article 6

9 For the case $n = 1$ the two formulæ of transformation,

$$0\{1, 0;\ 0, n\}_x = -\{0, 1,\ n+1, -1\}_y,$$
$$\{0, 1;\ 0, n\}_x = -(n+1)\{1, 0,\ n+1, -1\}_y,$$

of the last article become respectively

$$\begin{aligned}\frac{d}{dx_1} &= -\tfrac{1}{2}\left\{2\mathrm{Y}_2^{(2)}\frac{d}{dy_1} + 3\mathrm{Y}_3^{(2)}\frac{d}{dy_2} + 4\mathrm{Y}_4^{(2)}\frac{d}{dy_3} + \quad .\right\}\\ &= -\tfrac{1}{2}\left\{2y_1^2\frac{d}{dy_1} + 3.2y_1y_2\frac{d}{dy_2} + 4(2y_1y_3 + y_2^2)\frac{d}{dy_3} + \quad\right\}, \quad . \quad (34)\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dx_1} + \frac{x_2}{x_1}\frac{d}{dx_2} + \frac{2x_1x_3 - x_2^2}{x_1^2}\frac{d}{dx_3} + \frac{3x_1^2x_4 - 3x_1x_2x_3 + x_2^3}{x_1^3}\frac{d}{dx_4} + \\ &= -\left\{y_1^2\frac{d}{dy_1} + 2y_1y_2\frac{d}{dy_2} + (2y_1y_3 + y_2^2)\frac{d}{dy_3} + \quad ..\right\} \quad . \quad (35)\end{aligned}$$

By combination of these we have the equivalence, free from d/dx_1 and d/dy_1,

$$\begin{aligned}&\frac{x_2}{x_1}\frac{d}{dx_2} + \frac{2x_1x_3 - x_2^2}{x_1^2}\frac{d}{dx_3} + \frac{3x_1^2x_4 - 3x_1x_2x_3 + x_2^3}{x_1^3}\frac{d}{dx_4} + \dots\\ &= y_1y_2\frac{d}{dy_2} + 2\left(y_1y_3 + \frac{y_2^2}{2}\right)\frac{d}{dy_3} + 3(y_1y_4 + y_2y_3)\frac{d}{dy_4} + \quad .\\ &= y_1\left(y_2\frac{d}{dy_2} + 2y_3\frac{d}{dy_3} + 3y_4\frac{d}{dy_4} + \dots\right) + 2\frac{y_2^2}{2}\frac{d}{dy_3} + 3y_2y_3\frac{d}{dy_4}\\ &\qquad + 4\left(y_2y_4 + \frac{y_3^2}{2}\right)\frac{d}{dy_5} + \dots \qquad . \quad (36)\end{aligned}$$

In like manner, for any positive integral value of n,

$$\begin{aligned}&\{0, 1,\ 0, n\}_x - 0\{1, 0;\ 0, n\}_x = -(n+1)\{1, 0;\ n+1, -1\}_y\\ &\qquad - \{0, 1;\ n+1, -1\}_y \qquad . \quad (37)\end{aligned}$$

is an equivalence of operators which do not involve any lower symbols of operation than d/dx_{n+1} and d/dy_{n+1} respectively

10. From (34) is easily derived in exact form the known fact that the annihilator V

of pure reciprocants is, when affected with a simple multiplier, self reciprocal. We may write (34)

$$\frac{d}{dx_1} = y_1^2 \frac{d}{dy_1} - y_1 \left\{2y_1 \frac{d}{dy_1} + 3y_2 \frac{d}{dy_2} + 4y_3 \frac{d}{dy_3} + \quad . \right\}$$
$$- \left\{4 \frac{y_2^2}{2} \frac{d}{dy_3} + 5y_2y_3 \frac{d}{dy_4} + 6 \left(y_2y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_5} + \ldots \right\}$$
$$= y_1^2 \frac{d}{dy_1} - y_1 \{1, 1, 1, 0\}_y - \mathrm{V}(y, x)$$
$$x_1 \frac{d}{dx_1} - y_1 \frac{d}{dy_1} = - \{1, 1, 1, 0\}_y - y_1^{-1}\mathrm{V}(y, x)$$

So too, correlatively

$$y_1 \frac{d}{dy_1} - x_1 \frac{d}{dx_1} = - \{1, 1; 1, 0\}_x - x_1^{-1}\mathrm{V}(x, y)$$

But

$$\{1, 1, 1, 0\}_x = - \{1, 1, 1, 0\}_y \text{ by (18) or (20)}$$
$$\therefore \quad x_1^{-1}\mathrm{V}(x, y) = - y_1^{-1}\mathrm{V}(y, x), \qquad \ldots \quad (38)$$

so that, to use a familiar notation, $t^{-1}\mathrm{V}$ is a self reciprocal operator of negative character

11 It remains to consider operators $\{\mu, \nu; m, n\}$ in cases when $m + n < 1$. In such cases the formulæ of Arts 5 and 6 have to be replaced by others. The essential difference between them and the cases already considered lies in the fact that the lower limit of s in (3) and (4), and, therefore, in what replaces (7), is now $-n+1$ instead of m, i.e, is greater than m, so that the coefficients in $\{\mu, \nu; m, n\}_y$ are no longer multiples of the complete set of coefficients in the expansion of $(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$, but of those coefficients with the exception of one or more at the beginning

In the present article attention is confined to the case of $m + n = 0$, i.e., $n = -m$.

Proceeding to write down the symbolical form of $\{\mu, \nu; m, -m\}_y$ as in Art. (5) we see that the whole expansion from which we there started is present except the first term,

$$\frac{1}{m}(\mu + \nu m)\, \mathrm{Y}_m^{(m)} \xi^{n+m}.$$

Thus the symbolical form of

$$\{1, 0; m, -m\}_y \text{ is } \frac{1}{m} \xi^{-m} (\eta^m - y_1^m \xi^m), \qquad \ldots \quad (39)$$

and that of

$$\{0, 1, m, -m\}_y \text{ is } \xi^{-m+1}\left(\eta^{m-1} \frac{d\eta}{d\xi} - y_1^m \xi^{m-1}\right), \qquad \ldots \quad (40)$$

the right-hand members standing for their expansions in powers of ξ

In like manner

$$\frac{1}{m}\eta^{-m}(\xi^m - x_1^m\eta^m) \quad \text{and} \quad \eta^{-m+1}\left(\xi^{m-1}\frac{d\xi}{d\eta} - x_1^m\eta^{m-1}\right), \qquad (41, 42)$$

standing for their expansions in terms of η, are the symbolical forms of

$$\{1, 0, m, -m\}_x \quad \text{and} \quad \{0, 1, m, -m\}_x$$

As in Art (6) the result of transforming $\{1, 0, m, -m\}_x$ to its form in y dependent is, then, symbolically,

$$-\frac{1}{m}\eta^{-m}(\xi^m - x_1^m\eta^m)\frac{d\eta}{d\xi},$$

i e,

$$-\frac{1}{m}\xi^m\eta^{-m}\frac{d\eta}{d\xi} + \frac{1}{m}y_1^{-m}\frac{d\eta}{d\xi},$$

i e,

$$-\frac{1}{m}\xi^m\left(\eta^{-m}\frac{d\eta}{d\xi} - y_1^{-m+1}\xi^{-m}\right) + \frac{1}{m}y_1^{-m}\left(\frac{d\eta}{d\xi} - y_1\right),$$

whence, by aid of (40),

$$m\{1, 0, m, -m\}_x = -\{0, 1, 1-m, m-1\}_y + y_1^{-m}\{0, 1, 1, -1\}_y \qquad (43)$$

Once more the result of transforming $\{0, 1, m, -m\}_x$ is, in like manner,

$$-\eta^{-m+1}\left(\xi^{m-1}\frac{d\xi}{d\eta} - x_1^m\eta^{m-1}\right)\frac{d\eta}{d\xi},$$

i e.,

$$-\xi^{m-1}\eta^{-m+1} + y_1^{-m}\frac{d\eta}{d\xi},$$

i e.,

$$-\xi^{m-1}(\eta^{-m+1} - y_1^{-m+1}\xi^{-m+1}) + y_1^{-m}\left(\frac{d\eta}{d\xi} - y_1\right),$$

so that, by (39) and (40),

$$\{0, 1, m, -m\}_x = -(1-m)\{1, 0; 1-m, m-1\}_y + y_1^{-m}\{0\ 1, 1, -1\}_y \qquad (44)$$

From (43) and (44) follows the more general identity,

$$m\{\mu, \nu; m, -m\}_x = -\{\nu m(1-m), \mu; 1-m, m-1\}_y$$
$$+ (\mu + m\nu)y_1^{-m}\{0, 1, 1, -1\}_y;$$

or, replacing νm by ν,

$$m\left\{\mu, \frac{\nu}{m}, m, -m\right\}_x = -(1-m)\left\{\nu, \frac{\mu}{1-m}, 1-m, m-1\right\}_y + (\mu+\nu)\, y_1^{-m}\{0, 1, 1, -1\}_y \qquad (45)$$

In particular,

$$m\left\{\mu, -\frac{\mu}{m}, m, -m\right\}_x = (1-m)\left\{\mu, -\frac{\mu}{1-m}, 1-m, m-1\right\}_y \qquad (46)$$

12 The value zero of m is somewhat special in these cases of $n = -m$, just as in the more general cases already discussed So too, of course, is the conjugate value $m = 1$.

For (43) and (44) to hold for these special values of m we must have,

$$\left.\begin{aligned} 0\{1, 0; 0, 0\}_x &= -\{0, 1, 1, -1\}_y + \{0, 1, 1, -1\}_y, \\ \{1, 0, 1, -1\}_x &= -\{0, 1, 0, 0\}_y + y_1^{-1}\{0, 1, 1, -1\}_y, \\ \{0, 1, 0, 0\}_x &= -\{1, 0, 1, -1\}_y + \{0, 1, 1, -1\}_y, \\ \{0, 1, 1, -1\}_x &= -0\{1, 0; 0, 0\}_y + y_1^{-1}\{0, 1, 1, -1\}_y. \end{aligned}\right\} \qquad (47)$$

Of these four equalities the first is a mere identity of two zero operators In fact to $0\{1, 0, 0, 0\}$ no other meaning than zero could be attached consistently with the general definition Thus the form of $\{1, 0, 0, 0\}$ is left indeterminate.

The fourth of (47) becomes

$$\{0, 1, 1, -1\}_x = y_1^{-1}\{0, 1, 1, -1\}_y,$$

i.e.,

$$x_1^{-\frac{1}{2}}\left\{2x_2\frac{d}{dx_1} + 3x_3\frac{d}{dx_2} + 4x_4\frac{d}{dx_3} + \ldots\right\} = y_1^{-\frac{1}{2}}\left\{2y_2\frac{d}{dy_1} + 3y_3\frac{d}{dy_2} + 4y_4\frac{d}{dy_3} + \ldots\right\}, \qquad (48)$$

of which the left-hand member is merely $x_1^{-\frac{1}{2}}\frac{d}{dy}$, and the right $y_1^{-\frac{1}{2}}\frac{d}{dx}$, the symbols of differentiation being total

The remaining equalities, the second and third of (47), now become the same but for an interchange of x and y. Consequently there will be complete consistency if we define the at present undetermined operator $\{0, 1, 0, 0\}$ as that which obeys the equation of transformation

$$\begin{aligned} \{0, 1; 0, 0\}_x &= -\{1, 0; 1, -1\}_y + \{0, 1, 1, -1\}_y \\ &= \{-1, 1; 1, -1\}_y \\ &= y_2\frac{d}{dy_1} + 2y_3\frac{d}{dy_2} + 3y_4\frac{d}{dy_3} + \ldots \end{aligned} \qquad (49)$$

Now, proceeding exactly as in Art. 8, it is seen that the symbolical form of an x operator equal to this is $\eta\, d/d\eta \log(\xi/\eta)$, and that its expanded form is obtained by omitting the first term, and then putting $n = 0$ in the general value (33). Thus, the operator which has to be defined as $\{0, 1, 0, 0\}_x$ is

$$\frac{x_2}{x_1}\frac{d}{dx_1} + \frac{Gx_2}{x_1^2}\frac{d}{dx_2} + \frac{G^2x_2}{2!\,x_1^3}\frac{d}{dx_3} + \ldots, \tag{50}$$

where G is the generator defined in (32A).

13. Examples of important operators which occur among those transformed in the last two articles are before us in (48) and in the equality of (49) and (50). It is unnecessary to multiply particular instances as they can be deduced without number by giving m particular integral values in (43) . . (46). It is to be noticed that, excluding the special cases of $m = 0$ and $m = 1$, one or other of the two equivalent operators will involve as coefficients those in a multinomial expansion of negative index. Thus, for instance,

$$2\{1, 0;\ 2, -2\}_x = -\{0, 1, -1, 1\}_y + y_1^{-2}\{0, 1;\ 1, -1\}_y,$$

a particular case of (43), is at more length

$$\begin{aligned} &2x_1x_2\frac{d}{dx_1} + (2x_1x_3 + x_2^2)\frac{d}{dx_2} + (2x_1x_4 + 2x_2x_3)\frac{d}{dx_3} + \\ &= -\left\{\frac{y_2^2 - y_1y_3}{y_1^3}\frac{d}{dy_2} - 2\frac{(y_2^3 - 2y_1y_2y_3 + y_1^2y_4)}{y_1^4}\frac{d}{dy_3} + \ldots\right\} \\ &\quad + \frac{1}{y_1^2}\left\{2y_2\frac{d}{dy_1} + 3y_3\frac{d}{dy_2} + 4y_4\frac{d}{dy_3} + \ldots\right\} \end{aligned} \quad . \ . \ . \ . \tag{51}$$

The transformation of the operator G of (32A) is an application of (49). Thus

$$\begin{aligned} G(x, y) &= x_1\left(x_2\frac{d}{dx_1} + 2x_3\frac{d}{dx_2} + 3x_4\frac{d}{dx_3} + \ldots\right) - x_2\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3} + \ldots\right) \\ &= x_1\{-1, 1, 1, -1\}_x - x_2\{1, 0;\ 1, 0\}_x \\ &= y_1^{-1}\{0, 1;\ 0, 0\}_y - y_1^{-3}y_2\{0, 1;\ 1, 0\}_y, \end{aligned}$$

by (49) and (23),

$$\begin{aligned} &= y_1^{-1}\left\{\frac{y_2}{y_1}\frac{d}{dy_1} + \frac{Gy_2}{y_1^2}\frac{d}{dy_2} + \frac{G^2y_2}{2!\,y_1^3}\frac{d}{dy_3} + \ldots\right\} \\ &\qquad - y_1^{-3}y_2\left\{y_1\frac{d}{dy_1} + 2y_2\frac{d}{dy_2} + 3y_3\frac{d}{dy_3} + \ldots\right\} \\ &= \frac{2y_1y_3 - 3y_2^2}{y_1^3}\frac{d}{dy_2} + \frac{3y_1^2y_4 - 6y_1y_2y_3 + y_2^3}{y_1^4}\frac{d}{dy_3} + \ldots \end{aligned} \tag{52}$$

14. Operators for which $m+n$ is negative still remain to be considered In particular, those of the type $\{0, 1, 0, -n'\}$ have still to be defined About the right definition of them there can, however, after articles 8 and 12, be no doubt

The general principle by means of which if $\{\mu, \nu, m, n\}_y$ is known, the form of $\{\mu, \nu, m, n-r\}_y$ is deduced is expressed by the rule—"Write $\{\mu, \nu, m, n\}_y$ symbolically, by putting ξ^p for each d/dy_p, divide through by ξ^r, reject all terms, if any now occur, which do not contain a positive power of ξ as factor, and then for each ξ^p write d/dy_p."

Thus to accord with (33) and (50) the right operator to be defined as

$$\{0, 1, 0, -n'\}_y,$$

where $-n'$ is a negative integer, is

$$\frac{G^{n'}y_2}{n'!\,y_1^{n'+1}}\frac{d}{dy_1}+\frac{G^{n'+1}y_2}{(n'+1)!\,y_1^{n'+2}}\frac{d}{dy_2}+\frac{G^{n'+2}y_2}{(n'+2)!\,y_1^{n'+3}}\frac{d}{dy_3}+\qquad(53)$$

We now proceed to the transformation of $\{\mu, \nu, m, -m-r\}_x$ where r is a positive integer.

The symbolical form of $m\{1, 0, m, -m-r\}_x$ is as in Arts. 6 and 11

$$\eta^{-m-r}\{\xi^m-X_m^{(m)}\eta^m-X_{m+1}^{(m)}\eta^{m+1}-\quad\cdot\quad-X_{m+r}^{(m)}\eta^{m+r}\}\qquad(54)$$

The symbolical form of its transformation is, therefore,

$$-\xi^m\eta^{-m-r}\frac{d\eta}{d\xi}+X_m^{(m)}\eta^{-r}\frac{d\eta}{d\xi}+X_{m+1}^{(m)}\eta^{-r+1}\frac{d\eta}{d\xi}+\qquad+X_{m+r}^{(m)}\frac{d\eta}{d\xi}\qquad(55)$$

This when expanded in terms of ξ can involve no zero or negative powers For it is a sum of multiples of $\eta\, d\eta/d\xi$, $\eta^2\, d\eta/d\xi$, . only, since $\{1, 0, m, -m-r\}_x$ is a sum of multiples of η, η^2, . only, and these when expressed in terms of ξ are all free from ξ^0, ξ^{-1} ξ^{-2} . . Thus the coefficients of ξ^{-r}, ξ^{-r+1}, . . 1, which would appear to occur in the above symbolical form of the transformation of $\{1, 0; m, -m-r\}$ are in reality absent, and, consequently,

$$m\{1, 0, m, -m-r\}_x=-\{0, 1, 1-m-r, m-1\}_y+X_m^{(m)}\{0, 1, 1-r, -1\}_y$$
$$+X_{m+1}^{(m)}\{0,1,2-r,-1\}_y+\ \cdot\ +X_{m+r-1}^{(m)}\{0,1;0,-1\}_y+X_{m+r}^{(m)}\{0,1,1,-1\}_y,\qquad(56)$$

the various terms on the right consisting of the parts with positive indices of ξ from the corresponding terms of (55).

In like manner $m\{0, 1, m, -m-r\}_x$ whose symbolical form is

$$\eta^{1-m-r}\frac{d}{d\eta}\{\xi^m - X^{(m)}_m\eta^m - X^{(m)}_{m+1}\eta^{m+1} - \dots - X^{(m)}_{m+r}\eta^{m+r}\} \tag{57}$$

a form proceeding by positive integral powers of η, and therefore of ξ, transforms into

$$\begin{aligned}-m\xi^{m-1}\eta^{1-m-r} + mX^{(m)}_m\eta^{-r}\frac{d\eta}{d\xi} + (m+1)X^{(m)}_{m+1}\eta^{1-r}\frac{d\eta}{d\xi} + \\ + (m+r-1)X^{(m)}_{m+r-1}\eta^{-1}\frac{d\eta}{d\xi} + (m+r)X^{(m)}_{m+r}\frac{d\eta}{d\xi},\end{aligned} \tag{58}$$

of which the terms in zero and negative powers of ξ must, as before, disappear, leaving as the result of transformation

$$\begin{aligned}m\{0, 1, m, -m-r\}_x = -m(1-m-r)\{1, 0, 1-m-r, m-1\}_y \\ + mX^{(m)}_m\{0, 1, 1-r, -1\}_y + (m+1)X^{(m)}_{m+1}\{0, 1, 2-r, -1\}_y + \\ + (m+r-1)X^{(m)}_{m+r-1}\{0, 1, 0, -1\}_y + (m+r)X^{(m)}_{m+r}\{0, 1, 1, -1\}_y\end{aligned} \tag{59}$$

By addition of μ times (56) to ν times (59) the transformation of the more general $\{\mu, \nu, m, -m-r\}_x$ is at once deduced

15 The forms taken by (56) and (59) for the case $r = 1$, *i e*, $m + n = -1$, since

$$X^{(m)}_m = x_1^m = y_1^{-m}$$

and

$$X^{(m)}_{m+1} = mx_1^{m-1}x_2 = -my_1^{-m-2}y_2,$$

may be written

$$\begin{aligned}m\{1, 0, m, -m-1\}_x = -\{0, 1, -m, m-1\}_y + y_1^{-m}\{0, 1, 0, -1\}_y \\ - my_1^{-m-2}y_2\{0, 1, 1, -1\}_y \,.\end{aligned} \tag{60}$$

and

$$\begin{aligned}\{0, 1; m, -m-1\}_x = m\{1, 0, -m, m-1\}_y + y_1^{-m}\{0, 1, 0, -1\}_y \\ -(m+1)y_1^{-m-2}y_2\{0, 1, 1, -1\}_y\end{aligned} \tag{61}$$

One or two particular cases of these formulæ deserve mention. The value zero of m makes (60) an identity. In (61) the substitution of the same value produces

$$\{0, 1; 0, -1\}_x = \{0, 1, 0, -1\}_y - y_1^{-2}y_2\{0, 1; 1, -1\}_y,$$

in verification of which we may notice that it is unaltered by interchange of x and y, in virtue of (48). Another way of writing the result is to say that

$$2\{0, 1, 0, -1\}_x - x_1^{-2} x_2 \{0, 1; 1, -1\}_x \\ = 2\{0, 1, 0, -1\}_y - y_1^{-2} y_2 \{0, 1; 1, -1\}_y \quad . \quad (62)$$

is a self reciprocal operator of positive character

We are now enabled to write (61)

$$m\{0, 1, m, -m-1\}_x \\ = m\{1, 0; -m, m-1\}_y + y_1^{-m}\{0, 1; 0, -1\}_x - m y_1^{-m-2} y_2 \{0, 1; 1, -1\}_y,$$

which becomes (60) upon interchanging x and y, replacing m by $-m$, and using (48) and the values for x_1 and x_2, in terms of y_1 and y_2. Thus we have another verification of the consistency of our results.

II *Ternary Operators.*

16. Let x, y, z be three variables connected by a relation of any form known or unknown. Let x_{rs}, y_{rs}, z_{rs} denote respectively

$$\frac{1}{r!s!}\frac{d^{r+s}x}{dy^r dz^s}, \quad \frac{1}{r!s!}\frac{d^{r+s}y}{dz^r dx^s}, \quad \frac{1}{r!s!}\frac{d^{r+s}z}{dx^r dy^s}$$

Let $\xi, \eta; \zeta$ be any set of corresponding increments of x, y, z They are connected by a single relation, which may be written in either of the forms

$$\xi = (x_{10}\eta + x_{01}\zeta) + (x_{20}\eta^2 + x_{11}\eta\zeta + x_{02}\zeta^2) \\ + (x_{30}\eta^3 + x_{21}\eta^2\zeta + x_{12}\eta\zeta^2 + x_{03}\zeta^3) + \ldots, \quad (63)$$

$$\eta = (y_{10}\zeta + y_{01}\xi) + (y_{20}\zeta^2 + y_{11}\zeta\xi + y_{02}\xi^2) \\ + (y_{30}\zeta^3 + y_{21}\zeta^2\xi + y_{12}\zeta\xi^2 + y_{03}\xi^3) + \ldots \quad (64)$$

$$\zeta = (z_{10}\xi + z_{01}\eta) + (z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2) \\ + (z_{30}\xi^3 + z_{21}\xi^2\eta + z_{12}\xi\eta^2 + z_{03}\eta^3) + \ldots \quad (65)$$

Let m be a positive integer, and let $X_{rs}^{(m)}$ denote the coefficient of $\eta^r\zeta^s$ in ξ^m when expanded in ascending products of positive integral powers of η and ζ, so that

$$\xi^m = \{\Sigma x_{pq}\,\eta^p\,\zeta^q\}^m = \sum_{r+s \nless m} X_{rs}^{(m)}\,\eta^r\,\zeta^s\,; \tag{66}$$

and in like manner write

$$\eta^m = \{\Sigma y_{pq}\,\zeta^p\,\xi^q\}^m = \sum_{r+s \nless m} Y_{rs}^{(m)}\,\zeta^r\,\xi^s, \tag{67}$$

and

$$\zeta^m = \{\Sigma z_{pq}\,\xi^p\,\eta^q\}^m = \sum_{r+s \nless m} Z_{rs}^{(m)}\,\xi^r\,\eta^s \quad . \tag{68}$$

We may include, if we please, the value zero of m, but the expansions of ξ^o, η^o, ζ^o consist only of the single terms $\eta^o\,\zeta^o$, $\zeta^o\,\xi^o$, $\xi^o\,\eta^o$.

The operators to be considered and transformed are the following —

$$m\{\mu,\,\nu,\,\nu',\,m,\,n,\,n'\}_x = \Sigma\,(\mu + \nu r + \nu' s)\,X_{rs}^{(m)}\,\frac{d}{dx_{n+r,\,n'+s}}, \tag{69}$$

$$m\{\mu,\,\nu,\,\nu',\,m,\,n,\,n'\}_y = \Sigma\,(\mu + \nu r + \nu' s)\,Y_{rs}^{(m)}\,\frac{d}{dy_{n+r,\,n'+s}}, \tag{70}$$

$$m\{\mu,\,\nu,\,\nu',\,m,\,n,\,n'\}_z = \Sigma\,(\mu + \nu r + \nu' s)\,Z_{rs}^{(m)}\,\frac{d}{dz_{n+r,\,n'+s}}; \quad . \tag{71}$$

where μ, ν, ν' are any numerical quantities,

m a positive integer,

n, n' positive integers or one or both zero, and

r, s quantities which take in succession all zero and positive integral values subject to $r + s \nless m$.

Cases of m negative, and of n, n' either or both less than -1, which have been dealt with in the analogous theory of binary operators, will not be here considered.

The cases of m zero, and of n or n' equal to -1, will not be entirely excluded, but will be only dealt with as far as their accordance with the results for m positive and n, n' not negative needs no elaboration to make it clear.

Thus our field of investigation *is narrower* than in that of the analogous theory hitherto considered Were negative values of n and n' admitted, the lower limit of r in the operators (69) (71) would be $-n+1$ instead of zero, and that of s would be in like manner $-n'+1$. Thus when we admit the value -1 of n we must exclude the value 0 of r, and when the value -1 of n' we must exclude the value 0 of s.

Let us now express (69), (70), (71) symbolically as follows .—

$$m\,\{\mu,\,\nu,\,\nu'\,;\,m,\,n,\,n'\}_x = \Sigma\,(\mu + \nu r + \nu' s)\,X_{rs}^{(m)}\,\eta^{n+r}\,\zeta^{n'+s}, \quad . \quad . \tag{72}$$

$$m\,\{\mu,\,\nu,\,\nu'\,;\,m,\,n,\,n'\}_y = \Sigma\,(\mu + \nu r + \nu' s)\,Y_{rs}^{(m)}\,\zeta^{n+r}\,\xi^{n'+s}, \quad . \quad . \quad . \tag{73}$$

$$m\,\{\mu,\,\nu,\,\nu'\,;\,m,\,n,\,n'\}_z = \Sigma\,(\mu + \nu r + \nu' s)\,Z_{rs}^{(m)}\,\xi^{n+r}\,\eta^{n'+s}, \quad . \quad . \quad . \tag{74}$$

i e, let us in any x-operator symbolize d/dx_{pq} by $\eta^p \zeta^q$, in any y-operator d/dy_{pq} by $\zeta^p \xi^q$, and in any z-operator d/dz_{pq} by $\xi^p \eta^q$.

We may in this way write (71) or (74)

$$m\{\mu, \nu, \nu', m, n, n'\}_z = \mu\xi^n \eta^{n'} \{z_{10}\xi + z_{01}\eta + z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2 + \quad \}^m$$
$$+ \nu\zeta^{n+1}\eta^{n'} \frac{d}{d\xi}\{z_{10}\xi + z_{01}\eta + z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2 + \quad \}^m$$
$$+ \nu'\xi^n \eta^{n'+1} \frac{d}{d\eta}\{z_{10}\xi + z_{01}\eta + z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2 + \quad \}^m$$
$$= \mu\xi^n \eta^{n'} \zeta^m + \nu\xi^{n+1}\eta^{n'} \frac{d}{d\xi}(\zeta^m) + \nu'\xi^n \eta^{n'+1} \frac{d}{d\eta}(\zeta^m), \tag{75}$$

where ζ means the expansion in terms of ξ and η given in (65), and where the symbolization denotes that the right-hand member is to be expanded in terms of ξ and η, and to have each product $\xi^p \eta^q$ in its expansion replaced by the corresponding d/dz_{pq}, in order to produce the operator in z dependent which is represented by the notation on the left

Thus in particular, assigning to different pairs in succession of the three parameters, μ, ν, ν', zero values,

$$m\{1, 0, 0; m, n, n'\}_z = \xi^n \eta^{n'} \zeta^m, \tag{76}$$

$$m\{0, 1, 0; m, n, n'\}_z = \xi^{n+1}\eta^{n'} \frac{d}{d\xi}(\zeta^m) = m\xi^{n+1}\eta^{n'}\zeta^{m-1}\frac{d\zeta}{d\xi}, \tag{77}$$

$$m\{0, 0, 1; m, n, n'\}_z = \xi^n \eta^{n'+1} \frac{d}{d\eta}(\zeta^m) = m\xi^n \eta^{n'+1}\zeta^{m-1}\frac{d\zeta}{d\eta}, \tag{78}$$

while

$$\{\mu, \nu, \nu'; m, n, n'\}_z = \mu\{1, 0, 0, m, n, n'\}_z + \nu\{0, 1, 0, m, n, n'\}_z + \nu'\{0, 0, 1, m, n, n'\}_z \tag{79}$$

Precisely similar symbolical expressions to (75) . (78) are, of course, assigned to the corresponding operators in x and in y dependent We have only cyclically to interchange ξ, η, ζ once and twice respectively, and to regard the expressions on the right thus obtained as short ways of writing their expansions by aid of (63) and (64) in terms of η, ζ and ζ, ξ respectively.

17. In the present article the expression of each operative symbol d/dx_{rs}, on a function of the derivatives of x with regard to y and z, in terms of the operative symbols d/dz_{pq} on the equivalent function of the derivatives of z with regard to x and y, is investigated.

If, as in the earlier part of the last article, ξ, η, ζ are simultaneous increments of x, y, z, we may look upon

$$x_{10},\ x_{01},\ x_{20},\ x_{11},\ x_{02},$$

as a number of independent quantities, upon

$$y_{10},\ y_{01},\ y_{20},\ y_{11},\ y_{02},$$

and

$$z_{10},\ z_{01},\ z_{20},\ z_{11},\ z_{02},$$

as determinate functions of these quantities, upon ξ, η, ζ as three quantities connected with one another, and with x_{10}, x_{01}, x_{20}, . . by a relation of which (63), (64), and (65) are equivalent forms.

Of the quantities x_{10}, x_{01}, x_{20}, let one x_{rs} alone receive an infinitesimal variation also of ξ η, ζ, let η and ζ be kept constant so that ξ receives a consequent variation Some or all of y_{10}, y_{01}, y_{20}, and some or all of z_{10}, z_{01}, z_{20}, . will also receive consequent variations From (63) we thus obtain

$$\delta\xi = \eta^r\zeta^s\,\delta x_{rs},$$

from (64)

$$0 = \{y_{01} + y_{11}\xi + 2y_{02}\eta + y_{21}\xi^2 + 2y_{12}\xi\eta + 3y_{03}\eta^2 + \quad\}\,\delta\xi$$
$$+ \left\{\frac{dy_{10}}{dx_{rs}}\zeta + \frac{dy_{01}}{dx_{rs}}\xi + \frac{dy_{20}}{dx_{rs}}\zeta^2 + \frac{dy_{11}}{dx_{rs}}\zeta\xi + \frac{dy_{02}}{dx_{rs}}\xi^2 + \quad\right\}\delta x_{rs},$$

and from (65)

$$0 = \{z_{10} + 2z_{20}\xi + z_{11}\eta + 3z_{30}\xi^2 + 2z_{21}\xi\eta + z_{12}\eta^2 \quad\}\,\delta\xi$$
$$+ \left\{\frac{dz_{10}}{dx_{rs}}\xi + \frac{dz_{01}}{dx_{rs}}\eta + \frac{dz_{20}}{dx_{rs}}\xi^2 + \frac{dz_{11}}{dx_{rs}}\xi\eta + \frac{dz_{02}}{dx_{rs}}\eta^2 + \ldots\right\}\delta x_{rs}.$$

The three relations are identical Let us study the identity of the first and third We obtain from them that

$$\frac{dz_{10}}{dx_{rs}}\xi + \frac{dz_{01}}{dx_{rs}}\eta + \frac{dz_{20}}{dx_{rs}}\xi^2 + \frac{dz_{11}}{dx_{rs}}\xi\eta + \frac{dz_{02}}{dx_{rs}}\eta^2 + \quad .$$
$$= -\eta^r\zeta^s\{z_{10} + 2z_{20}\xi + z_{11}\eta + 3z_{30}\xi^2 + 2z_{21}\xi\eta + z_{12}\eta^2 + \ \ldots\}, \tag{80}$$

for all values of ξ and η, and, consequently, that if by aid of (65) the right hand member be like the left, expanded in powers and products of powers of ξ and η, the coefficients of corresponding terms on the two sides will be equal. In other words, each dz_{pq}/dx_{rs} is the coefficient of the corresponding $\xi^p\eta^q$.

Now, in the equivalence of operators,

$$\frac{d}{dx_{rs}} = \frac{dz_{10}}{dx_{rs}} \cdot \frac{d}{dz_{10}} + \frac{dz_{01}}{dx_{rs}} \cdot \frac{d}{dz_{01}} + \frac{dz_{20}}{dx_{rs}} \cdot \frac{d}{dz_{20}} + \frac{dz_{11}}{dx_{rs}} \cdot \frac{d}{dz_{11}} + \frac{dz_{02}}{dx_{rs}} \cdot \frac{d}{dz_{02}} + \ldots$$

each dz_{pq}/dx_{rs} is the coefficient of d/dz_{pq}.

It follows that in the expansion in terms of ξ and η of the right hand member of (80) the substitution for each $\xi^p\eta^q$ of the corresponding d/dz_{pq} exactly produces the expression for d/dx_{rs}. In other words, for each r and s, the

$$z \text{ transform of } \frac{d}{dx_{rs}} = -\eta^r\zeta^s \frac{d\zeta}{d\xi}, \qquad . \qquad . \qquad (81)$$

where ζ and its partial differential coefficient are to be replaced by their equivalents in terms of ξ and η by (65), where the product is to be expanded in terms of ξ and η, and where in the expanded result each product $\xi^p\eta^q$ is to be replaced by the corresponding operative symbol d/dz_{pq}.

By (77) we see that d/dx_{rs} is thus replaced by a linear z-operator of the form under consideration; in fact that

$$\frac{d}{dx_{rs}} = -\{0,\ 1,\ 0\ ,\ s+1,\ -1,\ r\}_z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Since the μ and the ν' of this operator are zero, the fact that n is -1 gives no difficulty as to the presence or absence of coefficients on the right like $Z_{0q}^{(s+1)}$.

In precisely the same way, by giving variations to ξ and x_{rs} in (63) and (64) instead of (63) and (65), we might have obtained

$$\frac{d}{dx_{rs}} = -\{0,\ 0,\ 1\ ;\ r+1,\ s,\ -1\}_y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

of which y-operator the symbolical form is

$$-\zeta^s\eta^r \frac{d\eta}{d\xi}.$$

18 The rules for transforming any linear x-operator to its equivalent forms in y and in z dependent, are now very simply expressed just as was the analogous rule in in Art. 4. Since the x-operator

$$\frac{d}{dx_{rs}} \quad \text{or} \quad \eta^r\zeta^s$$

has for its equivalent y-operator

$$-\eta^r\zeta^s \frac{d\eta}{d\xi},$$

and for its equivalent z-operator

$$-\eta^r\zeta^s \frac{d\zeta}{d\xi},$$

these rules are merely—To find the equivalent y-operator to a given linear x-operator, multiply its symbolical form by $-d\eta/d\xi$, and to find its equivalent z-operator, multiply that symbolical form by $-d\zeta/d\xi$ The y-operator thus obtained has of course to be expanded in terms of ζ and ξ by (64), and the z-operator in terms of ξ and η by (65), before being intelligible except by means of (76) to (78)

In verification let it be noticed that, since by first principles of the theory of partial differentiation the three sets of ratios

$$\begin{array}{ccc} \dfrac{d\xi}{d\eta} & \dfrac{d\xi}{d\zeta} & -1, \\[2ex] -1 & \dfrac{d\eta}{d\zeta} & \dfrac{d\eta}{d\xi}, \\[2ex] \dfrac{d\zeta}{d\eta} & -1 & \dfrac{d\zeta}{d\xi}, \end{array}$$

are equal, precisely the same results are obtained by cyclical interchange of x, y, z and ξ, η, ζ

19. Now as in (76)

$$m\{1, 0, 0, m, n, n'\}_x = \eta^n\zeta^{n'}\xi^m.$$

Its forms in y and z respectively are then

$$-\zeta^{n'}\xi^m\eta^n\frac{d\eta}{d\xi}, \quad \text{and} \quad -\xi^m\eta^n\zeta^{n'}\frac{d\zeta}{d\xi}$$

Of these by two cyclic interchanges in (78), and by (77) itself, respectively, the expressions are

$$-\{0, 0, 1; n+1, n', m-1\}_y, \quad \text{and} \quad -\{0, 1, 0, n'+1, m-1, n\}_z$$

consequently

$$\begin{aligned} m\{1, 0, 0, m, n, n'\}_x &= -\{0, 0, 1, n+1, n', m-1\}_y \\ &= -\{0, 1, 0, n'+1, m-1, n\}_z \quad . \quad (84) \end{aligned}$$

In the same way the y and z transforms of

$$\{0, 1, 0, m, n, n'\}_x, \quad i\,e, \quad \eta^{n+1}\zeta^{n'}\xi^{m-1}\frac{d\xi}{d\eta},$$

are

$$-\zeta^{n'}\xi^{m-1}\eta^{n+1}\frac{d\xi}{d\eta}\cdot\frac{d\eta}{d\xi}, \quad \text{and} \quad -\xi^{m-1}\eta^{n+1}\zeta^{n'}\frac{d\xi}{d\eta}\;\frac{d\zeta}{d\xi},$$

i.e.,

$$-\zeta^{n'}\xi^{m-1}\eta^{n+1}, \quad \text{and} \quad +\xi^{m-1}\eta^{n+1}\zeta^{n'}\frac{d\zeta}{d\eta},$$

by the equalities of ratios at the end of the last article. Accordingly

$$\{0,\ 1,\ 0\ ;\ m,\ n,\ n'\}_x = -(n+1)\{1,\ 0,\ 0\ ,\ n+1,\ n',\ m-1\}_y$$
$$= \{0,\ 0,\ 1\ ;\ n'+1,\ m-1,\ n\}_z \quad . \quad (85)$$

And once more, precisely in the same way,

$$\eta^n \zeta^{n'+1} \xi^{m-1} \frac{d\xi}{d\zeta}, \quad \zeta^{n'+1} \xi^{m-1} \eta^n \frac{d\eta}{d\zeta}, \quad \text{and} \quad -\xi^{m-1} \eta^n \zeta^{n'+1}$$

are equivalent operators in x, y, z respectively dependent, so that also

$$\{0,\ 0,\ 1\ ;\ m,\ n,\ n'\}_x = \{0,\ 1,\ 0\ ;\ n+1,\ n',\ m-1\}_y$$
$$= -(n'+1)\{1,\ 0,\ 0\ ,\ n'+1,\ m-1,\ n\}_z \qquad (86)$$

Of these sets of equalities (85) and (86) may in reality be deduced from (84) by cyclical interchanges of the variables and alteration of parameters. The independent investigation above is justified by the verification it affords.

The general formulæ of transformation, including (84), (85), (86), follow from them by (79), and are

$$\{\mu,\ \nu,\ \nu'\ ;\ m,\ n,\ n'\}_x = \{-\nu(n+1),\ \nu',\ -\frac{\mu}{m}\ ;\ n+1,\ n',\ m-1\}_y$$

$$\{-\nu'(n'+1),\ -\frac{\mu}{m},\ \nu\ ;\ n'+1,\ m-1,\ n\}_z . \quad . \quad (87)$$

Included, it is interesting to notice that we have three distinct classes of self reproductive or cyclically persistent operators, of characters corresponding each to one of the cube roots of unity, viz..

$$\{-m,\ 1,\ 1\ ;\ m,\ m-1,\ m-1\}_x = \{-m,\ 1,\ 1\ ;\ m,\ m-1,\ m-1\}_y$$
$$= \{-m,\ 1,\ 1\ ;\ m,\ m-1,\ m-1\}_z \quad . \quad . \quad (88)$$

$$\{-m,\ \omega,\ \omega^2,\ m,\ m-1,\ m-1\}_x = \omega\{-m,\ \omega,\ \omega^2\ ;\ m,\ m-1,\ m-1\}_y$$
$$= \omega^2\{-m,\ \omega,\ \omega^2\ ;\ m,\ m-1,\ m-1\}_z \ . \quad (89)$$

$$\{-m,\ \omega^2,\ \omega\ ,\ m,\ m-1,\ m-1\}_x = \omega^2\{-m,\ \omega^2,\ \omega\ ;\ m,\ m-1,\ m-1\}_y$$
$$= \omega\{-m,\ \omega^2,\ \omega\ ;\ m,\ m-1,\ m-1\}_z \ . \quad (90)$$

20. Some of the simplest, and most important so far as actual experience goes, examples of the formulæ now proved will be considered in what follows

The only lineo-linear operators, of the classes with which we are dealing, both of whose cyclical transformations are also lineo-linear, are found by putting m, $n+1$ and $n'+1$ all equal to unity in the results of the last article. Thus

$$\{1, 0, 0;\ 1, 0, 0\}_x = -\{0, 0, 1,\ 1, 0, 0\}_y = -\{0, 1, 0,\ 1, 0, 0\}_z, \quad . \quad (91)$$

with the correlative equalities obtained by writing y, z, x and z, x, y respectively for x, y, z, involve the aggregate of all such operators At length the equalities (91) are

$$\begin{aligned} x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} + x_{20}\frac{d}{dx_{20}} + x_{11}\frac{d}{dx_{11}} + x_{02}\frac{d}{dx_{02}} + x_{30}\frac{d}{dx_{30}} + x_{21}\frac{d}{dx_{21}} \\ + x_{12}\frac{d}{dx_{12}} + x_{03}\frac{d}{dx_{03}} + \quad . \\ = -\left\{y_{01}\frac{d}{dy_{01}} + y_{11}\frac{d}{dy_{11}} + 2y_{02}\frac{d}{dy_{02}} + y_{21}\frac{d}{dy_{21}} + 2y_{12}\frac{d}{dy_{12}} + 3y_{03}\frac{d}{dy_{03}} + \quad\right\} \\ = -\left\{z_{10}\frac{d}{dz_{10}} + 2z_{20}\frac{d}{dz_{20}} + z_{11}\frac{d}{dz_{11}} + 3z_{30}\frac{d}{dz_{30}} + 2z_{21}\frac{d}{dz_{21}} + z_{12}\frac{d}{dz_{12}} + \quad\right\} \end{aligned} \quad (92)$$

We thus learn that, if a function of the derivatives of x with regard to y and z is homogeneous, the equivalent function of the derivatives of y with regard to z and x is isobaric in second suffixes, while the equivalent function of the derivatives of z with regard to x and y is isobaric in first suffixes, and that

$$\imath(x, yz) = -w_2(y, zx) = -w_1(z, xy), \quad . \quad . \quad . \quad . \quad (93)$$

where the notation explains itself The correlative facts are

$$-w_1(x, yz) = \imath(y, zx) = -w_2(z, xy), \quad . \quad . \quad . \quad . \quad (94)$$

and

$$-w_2(x, yz) = -w_1(y, zx) = \imath(z, xy) \quad . \quad . \quad . \quad . \quad (95)$$

The same aggregate as is involved in (91) and its correlatives is also expressed by the facts that

$$\{-1, 1, 1,\ 1, 0, 0\}, \quad . \quad . \quad . \quad (96)$$

$$\{-1, \omega, \omega^2,\ 1, 0, 0\}, \quad . \quad . \quad . \quad . \quad (97)$$

$$\{-1, \omega^2, \omega,\ 1, 0, 0\}, \quad . \quad . \quad . \quad . \quad (98)$$

obtained by giving m the value 1 in (88) to (90), are cyclically persistent lineo-linear operators of characters 1, ω, ω^2 respectively.

If the operation be on a homogeneous and doubly isobaric function we are thus told that

$$-i+w_1+w_2 \qquad -i+\omega w_1+\omega^2 w_2, \qquad -i+\omega^2 w_1+\omega w_2 \quad . \tag{99}$$

are characteristics which persist after one cyclical transformation but for the multipliers 1, ω, ω^2 respectively, and after a second but for 1, ω^2, ω

21 The quadro-linear operators (linear operators with coefficients quadratic in the derivatives) both whose cyclic transformations are also quadro-linear, are obtained by giving to every one of m, $n+1$, $n'+1$ the value 2 in the formulæ of Art 19. Their aggregate is involved in

$$2\{1, 0, 0, 2, 1, 1\}_x = -\{0, 0, 1, 2, 1, 1\}_y = -\{0, 1, 0, 2, 1, 1\}_z, \tag{100}$$

and its two correlatives in y, z, x and z, x, y

The same system is expressed by the three cyclically persistent quadro-linear operators

$$\{-2, 1, 1; 2, 1, 1\}, \text{ of character } 1, \quad . \quad . \quad . \quad . \tag{101}$$

$$\{-2, \omega, \omega^2; 2, 1, 1\}, \quad ,, \quad \omega, \quad . \tag{102}$$

$$\{-2, \omega^2, \omega; 2, 1, 1\}, \quad ,, \quad \omega^2 \quad . \quad . \quad . \quad . \tag{103}$$

Of these the first expanded to a few terms is

$$X_{30}^{(2)}\frac{d}{dx_{41}}+X_{21}^{(2)}\frac{d}{dx_{32}}+X_{12}^{(2)}\frac{d}{dx_{23}}+X_{03}^{(2)}\frac{d}{dx_{14}}+2\left(X_{40}^{(2)}\frac{d}{dx_{51}}+\dots\right)$$
$$+3\left(X_{50}^{(2)}\frac{d}{dx_{61}}+\dots\right)+\dots, \tag{104}$$

where

$$X_{20}^{(2)}=x_{10}{}^2, X_{11}^{(2)}=2x_{10}x_{01}, X_{02}^{(2)}=x_{01}{}^2,$$

$$X_{30}^{(2)}=2x_{10}x_{20}, X_{21}^{(2)}=2x_{10}x_{11}+2x_{01}x_{20}, X_{12}^{(2)}=2x_{10}x_{02}+2x_{01}x_{11}, X_{03}^{(2)}=2x_{01}x_{02},$$

$$X_{40}^{(2)}=2x_{10}x_{30}+x_{20}{}^2, X_{31}^{(2)}=2x_{10}x_{21}+2x_{01}x_{12}+2x_{20}x_{11}, X_{22}^{(2)}=2x_{10}x_{12}+2x_{01}x_{21}+x_{11}{}^2$$
$$+2x_{20}x_{02}, X_{13}^{(2)}=2x_{10}x_{03}+2x_{01}x_{12}+2x_{11}x_{02}, X_{04}^{(2)}=2x_{01}x_{03}+x_{02}{}^2,$$

$$X_{50}^{(2)}=2x_{10}x_{40}+2x_{20}x_{30}, \dots,$$

and generally

$$X_{mn}^{(2)}=\sum_{r+s \nless 1}^{r+s \ngtr m+n-1}(x_{rs}x_{m-r,\,n-s}).$$

The two imaginary cyclically persistent quadro-linear operators (102) and (103) are easily written out in like manner They commence with terms in d/dx_{31}, d/dx_{22}, d/dx_{13}, which it is to be observed are wanting from the above

Once more by giving to each of $m, n+1, n'+1$ the value 3 in Art 19, an aggregate is obtained of linear operators with coefficients of the third degree, whose transforms have both of them coefficients of the third degree also. The aggregate may, as before, be considered involved in three cyclically persistent operators of the type, one of each character Similarly as to operators with coefficients of any higher degree

22 Some of the most important linear operators which have been used in recent theories of functional invariants, cyclicants, &c, have the property of persistence of degree in the derivatives after one cyclical transformation, but not after a second

Such operators occur among those obtained by putting $n+1=m$ in (87), viz,

$$\{\mu, \nu, \nu', m, m-1, n'\}_x = \left\{-\nu m, \nu', -\frac{\mu}{m}, m, n', m-1\right\}_y$$
$$= \left\{-\nu'(n'+1), -\frac{\mu}{m}, \nu, n'+1, m-1, m-1\right\}_z \qquad (105)$$

In particular, there are three classes of operators which have a property closely akin to that of persisting in form after a first cyclical transformation, being, in fact, only altered by the interchange of first and second suffixes they are

$$\{-m, 1, 1, m, m-1, n'\}_x = \{-m, 1, 1, m, n', m-1\}_y$$
$$= \{-(n'+1), 1, 1, n'+1, m-1, m-1\}_z \qquad . \quad (106)$$

$$\{-m, \omega, \omega^2, m, m-1, n'\}_x = \omega\{-m, \omega, \omega^2, m, n', m-1\}_y$$
$$= \omega^2\{-(n'+1), \omega, \omega^2; n'+1, m-1, m-1\}_z \qquad . \quad (107)$$

$$\{-m, \omega^2, \omega; m, m-1, n'\}_x = \omega^2\{-m, \omega^2, \omega, m, n', m-1\}_y$$
$$= \omega\{-(n'+1), \omega^2, \omega; n'+1, m-1, m-1\}_z \qquad . \quad (108)$$

It is to be noticed, in the case of the first of these, that the second cyclical transformation, which is of different degree from the first, is quite symmetrical in first and second suffixes

Among the operators comprised in (106) occur the two, which I have called ω_1 and ω_2,* two of the six form annihilators of projective cyclicants, viz,

$$\omega_1(x, yz) = \sum_{r+s \nless 1} \left\{(r+s-1)\, x_{rs} \frac{d}{dx_{r, s+1}}\right\},$$
$$\omega_2(x, yz) = \sum_{r+s \nless 1} \left\{(r+s-1)\, x_{rs} \frac{d}{dx_{r+1, s}}\right\},$$

* 'London Math Soc. Proc,' vol 20, pp 131–160.

or in present notation $\{-1, 1, 1; 1, 0, 1\}$ and $\{-1, 1, 1; 1, 1, 0\}$. For the transformation of these we have, by putting 1 for each of m and n' in (106),

$$\omega_1(x, yz) = \omega_2(y, zx) = \{-2, 1, 1, 2, 0, 0\}_z, \quad . \quad . \quad . \quad (109)$$

of which right hand operator the expansion is

$$Z^{(2)}_{30}\frac{d}{dz_{30}} + Z^{(2)}_{21}\frac{d}{dz_{21}} + Z^{(2)}_{12}\frac{d}{dz_{12}} + Z^{(2)}_{03}\frac{d}{dz_{03}} + 2\left(Z^{(2)}_{10}\frac{d}{dz_{10}} + . \quad \right) + 3\left(Z^{(2)}_{00}\frac{d}{dz_{00}} + . \quad \right) + \quad . \; .$$

where the coefficients have meanings, as in Art. 21.

Closely resembling, but distinct from ω_1 and ω_2, are Mr. FORSYTH's Δ_2 and Δ_1,* *i.e.*,

$$\Delta_2(x, yz) = \sum_{r+s \nless 1}\left\{(r+s)\, x_{rs}\frac{d}{dx_{r, s+1}}\right\} = \{0, 1, 1; 1, 0, 1\}_z,$$

$$\Delta_1(x, yz) = \sum_{r+s \nless 1}\left\{(r+s)\, x_{rs}\frac{d}{dx_{r+1, s}}\right\} = \{0, 1, 1; 1, 1, 0\}_z.$$

These are also transformed by means of the present article, but have not the property of companionship belonging to ω_1 and ω_2. In fact, by (105),

$$\Delta_2(x, yz) = \{-1, 1, 0; 1, 1, 0\}_y = \{-2, 0, 1; 2, 0, 0\}_z, \quad . \quad . \quad (110)$$

and, by (105), with z, x, y put for x, y, z,

$$\Delta_1(x, yz) = \{-2, 1, 0; 2, 0, 0\}_y = \{-1, 0, 1; 1, 0, 1\}_z. \quad . \quad . \quad (111)$$

23. The special importance of many operators in which the first derivatives do not occur is well known. The form of such operators (in z dependent) is symbolically

$$\xi^p\eta^q\left(a + b\xi\frac{d}{d\xi} + c\eta\frac{d}{d\eta}\right)(\zeta - z_{10}\xi - z_{01}\eta)^m.$$

As every such operator is a sum of multiples of complete operators $\{\mu, \nu, \nu'; m, n, n'\}_z$ so that their theory is implicitly involved in that above discussed, no attempt will be made here to develope it independently. In the present article, however, an interesting class of cyclically persistent operators will be obtained, and a method of procedure in a much wider class of cases will be thus exemplified.

It is required to prove that the result of replacing each first derivative by zero in

$$\{-m, 1, 1; m, 0, 0\}$$

* See his Memoir "A Class of Functional Invariants," 'Phil. Trans.,' A., vol. 180 (1889), pp. 71–118.

is, but for a first derivative factor, an operator which persists in form after one and two cyclical interchanges of the variables

Symbolically we have, *if square brackets indicate that in an operator first derivatives are thus omitted,*

$$[-m, 1, 1, m, 0, 0]_x = \frac{1}{m}\left(-m + \eta\frac{d}{d\eta} + \zeta\frac{d}{d\zeta}\right)(\xi - x_{10}\eta - x_{01}\zeta)^m,$$

$$= (\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{-\xi + x_{10}\eta + x_{01}\zeta + \eta\frac{d\xi}{d\eta} \right.$$

$$\left. - x_{10}\eta + \zeta\frac{d\xi}{d\zeta} - x_{01}\zeta\right\}$$

$$= (\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{\eta\frac{d\xi}{d\eta} + \zeta\frac{d\xi}{d\zeta} - \xi\right\}$$

The y transform of this operator is therefore, by Art. 18,

$$-(\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{\eta\frac{d\xi}{d\eta} + \zeta\frac{d\xi}{d\zeta} - \xi\right\}\frac{d\eta}{d\xi},$$

which, since

$$x_{10} \quad x_{01} \cdot -1 = -1 \quad y_{10} \quad y_{01} = z_{01} \quad -1 \quad z_{10},$$

and

$$\frac{d\xi}{d\eta} \quad \frac{d\xi}{d\zeta} \quad -1 = -1 \cdot \frac{d\eta}{d\zeta} \quad \frac{d\eta}{d\xi} = \frac{d\zeta}{d\eta} \quad -1 \cdot \frac{d\zeta}{d\xi},$$

may be written

$$(-1)^{m-1}x_{10}^{m-1}(\eta - y_{10}\zeta - y_{01}\xi)^{m-1}\left\{\zeta\frac{d\eta}{d\zeta} + \xi\frac{d\eta}{d\xi} - \eta\right\},$$

and is consequently

$$(-1)^{m-1}x_{10}^{m-1}[-m, 1, 1; m, 0, 0]_y.$$

In exactly the same way the z transform of the same operator is

$$(-1)^{m-1}x_{01}^{m-1}[-m, 1, 1; m, 0, 0]_z$$

Thus we have the formula of transformation

$$[-m, 1, 1; m, 0, 0]_x = \left(-\frac{1}{y_{01}}\right)^{m-1}[-m, 1, 1; m, 0, 0]_y = \left(-\frac{1}{z_{10}}\right)^{m-1}[-m, 1, 1, m, 0, 0]_z,$$

which may be written in a form even more clearly expressive of the cyclically persistent property, viz.,

$$\left(\frac{1}{x_{10}x_{01}}\right)^{(m-1)/3}[-m, 1, 1, m, 0, 0]_x = \left(\frac{1}{y_{10}y_{01}}\right)^{(m-1)/3}[-m, 1, 1, m, 0, 0]_y$$

$$= \left(\frac{1}{z_{10}z_{01}}\right)^{(m-1)/3}[-m, 1, 1, m, 0, 0]_z \quad . \quad (112)$$

A first case of this formula occurs in (96) A second is that the operator on the right of (109) is the sum of terms involving z_{10}, z_{01}, and of an operator free from those derivatives which, when affected with the factor $(1/z_{10}z_{01})^{\frac{1}{2}}$, has the persistent property. The fact is interesting in its bearing on the theory of the transformation of ω_1 and ω_2. In fact (109) may be written

$$\omega_1(x, yz) = \omega_2(y, zx) = z_{10}\omega_2(z, xy) + z_{01}\omega_1(z, xy) + [-2, 1, 1, 2, 0, 0]$$

From this and its two correlatives a verification of the formula (112) for the case of $m = 2$ is readily obtained. The first few terms of this cyclically persistent operator are

$$\begin{aligned}\left(\frac{1}{x_{10}x_{01}}\right)^{\frac{1}{2}}\Big\{&2\frac{x_{20}^2}{2}\frac{d}{dx_{40}} + 2x_{20}x_{11}\frac{d}{dx_{31}} + 2\left(x_{20}x_{02} + \frac{x_{11}^2}{2}\right)\frac{d}{dx_{22}} + 2x_{11}x_{02}\frac{d}{dx_{13}} + 2\frac{x_{02}^2}{2}\frac{d}{dx_{04}} \\ &+ 3x_{20}x_{30}\frac{d}{dx_{50}} + 3(x_{20}x_{21} + x_{11}x_{30})\frac{d}{dx_{11}} + 3(x_{20}x_{12} + x_{11}x_{21} + x_{02}x_{30})\frac{d}{dx_{32}} \\ &+ 3(x_{20}x_{03} + x_{11}x_{12} + x_{02}x_{21})\frac{d}{dx_{23}} + 3(x_{11}x_{03} + x_{02}x_{12})\frac{d}{dx_{14}} + 3x_{02}x_{03}\frac{d}{dx_{05}} \\ &+ \ldots\Big\} \quad \ldots \quad (113)\end{aligned}$$

With it compare the analogous operator (30) of the binary theory. That, like this, is, of course, one of a whole class of persistent operators.

24. The transformation of the four form annihilators Ω_1, Ω_2, V_1, V_2 of pure cyclicants* will be now considered. They are all operators free from first derivatives, and might as such be treated by the methods of the last article. It seems preferable, however, to take them in connection with the formulæ of transformation

$$\frac{d}{dx_{10}} = -\{0, 0, 1, 2, 0, -1\}_y = -\{0, 1, 0, 1, -1, 1\}_z, \quad . \quad . \quad (114)$$

$$\frac{d}{dx_{01}} = -\{0, 0, 1, 1, 1, -1\}_y = -\{0, 1, 0; 2, -1, 0\}_z, \quad . \quad (114\text{A})$$

which are particular cases of (82) and (83), or again of (87) with $m = 0$.

The forms of the four annihilators are

$$\begin{aligned}\Omega_1(z, xy) &= \sum_{m+n \not< 2}\left\{m z_{mn}\frac{d}{dz_{m-1,\,n+1}}\right\} \\ &= \eta\frac{d}{d\xi}(\zeta - z_{10}\xi - z_{01}\eta) \\ &= \eta\frac{d\zeta}{d\xi} - z_{10}\eta \\ &= \{0, 1, 0; 1, -1, 1\}_z - z_{10}\frac{d}{dz_{01}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)\end{aligned}$$

* See my papers in the 'London Math. Soc. Proc.,' vol 18, pp 142, &c.; vol. 19, pp. 6, &c., and pp. 337, &c., vol. 20, pp. 131, &c.

$$\Omega_2(z, xy) = \sum_{m+n \nless 2} \left\{ n z_{mn} \frac{d}{dz_{m+1\ n-1}} \right\}$$

$$= \xi \frac{d}{d\eta}(\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \{0, 0, 1 ; 1, 1, -1\}_z - z_{01}\frac{d}{dz_{10}}, \quad . \quad . \quad . \quad . \quad (116)$$

$$\mathrm{V}_1(z, xy) = \sum_{m+n \nless 4} \left\{ \sum_{r+s \nless 2}^{r+s \ngtr m+n-2} (r z_{rs} z_{m-r,\ n-s}) \frac{d}{dz_{m-1,\ n}} \right\}$$

$$= \tfrac{1}{2}\frac{d}{d\xi}\{(\zeta - z_{10}\xi - z_{01}\eta)^2\}$$

$$= \zeta\frac{d\zeta}{d\xi} - z_{10}\xi\frac{d\zeta}{d\xi} - z_{01}\eta\frac{d\zeta}{d\xi} - z_{10}(\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \{0, 1, 0 , 2, -1, 0\}_z - z_{10}\{0, 1, 0 , 1, 0, 0\}_z - z_{01}\{0, 1, 0 , 1, -1, 1\}$$
$$- z_{10}\left\{\{1, 0, 0 , 1, 0, 0\}_z - z_{10}\frac{d}{dz_{10}} - z_{01}\frac{d}{dz_{01}}\right\}$$

$$= \{0, 1, 0 , 2, -1, 0\}_z - z_{10}\{1, 1, 0 , 1, 0, 0\}_z$$
$$z_{01}\Omega_1(z, xy) + z_{10}{}^2\frac{d}{dz_{10}}, \quad . \quad (117)$$

$$\mathrm{V}_2(z, xy) = \sum_{m+n \nless 4} \left\{ \sum_{r+s \nless 2}^{r+s \ngtr m+n-2} (s z_{rs} z_{m-r,\ n-s}) \frac{d}{dz_{m,\ n-1}} \right\}$$

$$= \tfrac{1}{2}\frac{d}{d\eta}\{(\zeta - z_{10}\xi - z_{01}\eta)^2$$

$$= \{0, 0, 1; 2, 0, -1\}_z - z_{01}\{1, 0, 1; 1, 0, 0\}_z - z_{10}\Omega_2(z, xy) + z_{01}{}^2\frac{d}{dz_{01}}\cdot \quad (118)$$

Thus (114A) may be written

$$\frac{d}{dx_{01}} = -\Omega_2(y, zx) - y_{01}\frac{d}{dy_{10}} = -\{0, 1, 0 , 2, -1, 0\}_z,$$

and the result of one cyclical interchange in (114) may be written ,

$$-\Omega_1(x, yz) - x_{10}\frac{d}{dx_{01}} = \frac{d}{dy_{10}} = -\{0, 0, 1 , 2, 0, -1\}_z,$$

from which two sets of identities, by aid of the facts that

$$x_{10} \quad x_{01} \quad -1 = -1 \quad y_{10}\cdot y_{01} = z_{01}\cdot -1\cdot z_{10},$$

it follows at once that

$$\frac{1}{x_{01}}\,\Omega_1(x,\,yz) = -\frac{1}{y_{10}}\,\Omega_2(y,\,zx)$$
$$= -z_{01}\{0,1,0,\ 2,-1,0\}_z + z_{10}\{0,0,1,\ 2,0,-1\}_z, \qquad (119)$$

which, and its correlatives, obtained by one and two cyclical transformations effect the transformation of Ω_1 and Ω_2.

Again, we may write (114A)

$$\frac{d}{dx_{01}} = -\Omega_2(y,\,zx) - y_{01}\frac{d}{dy_{10}}$$
$$= -V_1(z,\,xy) - z_{01}\,\Omega_1(z,\,xy) - z_{10}\{1,1,0,\ 1,0,0\}_z + z_{10}^2\frac{d}{dz_{10}},$$

and (114) in like manner

$$\frac{d}{dx_{10}} = -V_2(y,\,zx) - y_{10}\Omega_2(y,\,zx) - y_{01}\{1,0,1,\ 1,0,0\}_y + y_{01}^2\frac{d}{dy_{01}}$$
$$= -\Omega_1(z,\,xy) - z_{10}\frac{d}{dz_{01}}.$$

From these it follows at once that

$$x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} = y_{10}\frac{d}{dy_{10}} + y_{01}\frac{d}{dy_{01}} - \frac{1}{y_{01}}V_2(y,\,zx) - \{1,0,1,\ 1,0,0\}_y$$
$$= z_{10}\frac{d}{dz_{10}} + z_{01}\frac{d}{dz_{01}} - \frac{1}{z_{10}}V_1(z,\,xy) - \{1,1,0;\ 1,0,0\}_z$$

By a cyclical interchange of the variables we have, also,

$$x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} - \frac{1}{x_{10}}V_1(x,\,yz) - \{1,1,0,\ 1,0,0\}_x = y_{10}\frac{d}{dy_{10}} + y_{01}\frac{d}{dy_{01}}$$
$$= z_{10}\frac{d}{dz_{10}} + z_{01}\frac{d}{dz_{01}} - \frac{1}{z_{01}}V_2(z,\,xy) - \{1,0,1;\ 1,0,0\}_z;$$

and by a second cyclical interchange a third set of such equalities is obtained. From the two sets that have been written out, upon subtraction we obtain

$$\frac{1}{x_{10}}V_1(x,\,yz) + \{1,1,0;\ 1,0,0\}_x = -\frac{1}{y_{01}}V_2(y,\,zx) - \{1,0,1,\ 1,0,0\}_y$$
$$= -\frac{1}{z_{10}}V_1(z,\,xy) + \frac{1}{z_{01}}V_2(z,\,xy) + \{0,-1,1;\ 1,0,0\}_z$$

Now, from (91) and its correlative obtained by a cyclical interchange, the second parts of these three equal operators are themselves equal. Consequently

$$\frac{1}{x_{10}}\mathrm{V}_1(x, yz) = -\frac{1}{y_{01}}\mathrm{V}_2(y, zx) = -\frac{1}{z_{10}}\mathrm{V}_1(z, xy) + \frac{1}{z_{01}}\mathrm{V}_2(z, xy), \qquad (120)$$

which, and its correlatives, are the formulæ for the transformation of V_1 and V_2.

It is easy and very instructive to prove (120) directly from the symbolical expressions in (117) and (118) by the method of Art. 19 or 23.

Of other important operators Mr. FORSYTH's Δ_4 and Δ_3 ('Phil. Trans.,' A., vol. 180, p. 74) should have their formulæ of transformation noted. They are the complete operators $\{0, 1, 0\,,\ 1, -1, 1\}$ and $\{0, 0, 1\,,\ 1, 1, -1\}$ of which Ω_1 and Ω_2 are all but the first terms. Thus their formulæ of transformation are merely (114) and (114A) themselves, *i.e.*, cyclically interchanging the variables once,

$$\Delta_4(x, yz) = -\frac{d}{dy_{10}} = \{0, 0, 1\,,\ 2, 0, -1\}_z, \qquad (121)$$

$$\Delta_3(x, yz) = \{0, 1, 0\,,\ 2, -1, 0\}_y = -\frac{d}{dz_{01}} \qquad (122)$$

III. THE BAKERIAN LECTURE.—*A Magnetic Survey of the British Isles for the Epoch January* 1, 1886.

By A. W. RUCKER, *M.A., F.R.S., and* T. E. THORPE, *B.Sc., Ph.D., F.R.S.*

Lecture delivered April 11,—MS. received June 6, 1889.

[PLATES 1–14.]

CONTENTS

Two Magnetic Surveys of the British Isles have been made previous to that of which an account is given in this paper The observations necessary for these were taken between the years 1834–38 and 1857–62, and the results were reduced to the epoch 1842–5 by Sir E SABINE, in a paper published in the 'Philosophical Transactions' for 1870 ('Phil. Trans.,' vol 160, 1870, p. 265) As a full account of both surveys is given in that paper, it is unnecessary to describe them in detail here. The first was made by five observers, viz , Sir E. SABINE, Captain J C ROSS, Mr R W. FOX, and Professors LLOYD and JOHN PHILLIPS In the second survey (1857–58), Mr WELSH, Superintendent of the Kew Observatory, made an admirable series of observations in Scotland, though, unfortunately, the exposure to which he was subjected brought on an illness which terminated in his death Sir EDWARD SABINE made observations on the Force and Dip at 24 stations in England, and some declinations determined by several naval officers between the years 1855 and 1861, were utilized. Altogether observations were made at 243 stations. ('Phil Trans,' vol 162, 1872, p 319)

It has, we believe, for some time been thought by those interested in terrestrial magnetism, that another survey of the United Kingdom should be undertaken, and we ourselves drew attention to the matter in a paper "On the Irregularities in Magnetic Inclination on the West Coast of Scotland" ('Roy Soc Proc,' vol 36, 1884, p 10) Not only was this desirable in order that the secular changes in the direction of the lines of equal Inclination, Force, and Declination might be re-determined, but also because the earlier surveys left much to be desired with regard to the distribution of the stations and the number of the Declination observations. Thus the Declination was determined about the epochs 1836 and 1857 at 84 stations only, of which 11 were common to the two surveys The maps given by Sir E SABINE in the paper already referred to show that different districts have received very different degrees of attention Stations where the Dip has been determined cluster thickly about the coast of Scotland to the south of Oban, about the English lakes, and the south coast of England, and are thinly distributed in the North of Scotland, in the eastern counties of England, and in the centre of Ireland In like manner, while (owing chiefly to the labours of Mr. WELSH) the Declination had been measured at 40 places in Scotland, it had been observed with adequate instruments at only 28 stations in England, and 16 in Ireland. We have therefore undertaken, and, in the course of the five years 1884–88, both inclusive, have completed a new magnetic survey of the British Isles

In our opinion, it would probably have been better if a larger number of observers had been engaged in the task, so that it could have been finished in a shorter time In fact, we originally made a proposal in this sense which was, we believe, brought before the Kew Committee of the Royal Society. It was considered that the objections to the employment of many instruments and observers were sufficient to prevent the acceptance of such a scheme. With proper organisation, we believe that the errors thus introduced would have been much less serious than those due to the uncertainty as to the true secular corrections at any particular station As, however, it was more important that the survey should be made than that it should be made under the best possible conditions, we have in undertaking the task ourselves devoted most of our vacations and spare time to it, so that there should be as little delay as possible, and have collected all the facts which throw light on the value of the secular corrections.

Although the re-determination of the lines of equal magnetic Declination, Force, and Dip has been the main object of our investigation, a number of other questions have naturally come under consideration. We have made some alterations in the Kew magnetometer which have been described to the Physical Society ('Phil. Mag,' August, 1888, p. 122); the method of presenting the results of the experiments has been modified so as to afford a greater test of their accuracy, the validity of the method of correcting for diurnal variation and disturbance, especially at places far distant from the base station, has been reconsidered, and finally, and perhaps chiefly, we have given more attention than our predecessors have done to the distribution and causes of "local magnetic attraction" In view of the difficulties caused by such disturbances, we have taken special pains to indicate the position of our stations as accurately as possible. This has been done, not merely by verbal description, but by taking out the latitudes and longitudes from the inch Ordnance maps or the Admiralty charts, with far greater accuracy than is necessary for the calculations in which these quantities are afterwards employed For similar reasons we have selected, when possible, public parks, open commons, or other situations which are likely to be still available when the survey is repeated. It must, however, be remarked that, at some places which were not included in our original programme, observations have been made mainly because a favourable opportunity presented itself, and that, in such cases, it was not always possible to exercise the same care in the selection of a site.

Epoch.

The epoch of the survey is January 1, 1886, to which date all the observations have been reduced.

Instruments

The survey of Scotland was mainly made with a set of instruments which belong to Professor RUCKER. They are a Kew Magnetometer by ELLIOTT BROS, No. 60, and a

Dip Circle by Dover, No 74 These instruments were also used by Professor RUCKER in his portion of the surveys of England, Wales, and Ireland The instruments employed by Dr THORPE in these countries are the property of the Science and Art Department They are of the same patterns and by the same makers, the Magnetometer and Dip Circle are numbered 61 and 83 respectively

Arrangements have been made for placing this latter set of instruments in the Collection of Scientific Apparatus at South Kensington They will be kept for surveying only, and will be available for comparison with instruments which may be used by future observers

Base Station

Our base station is the Kew Observatory Both sets of instruments were tested there before they came into our possession, and we have also made numerous comparisons with the Kew instruments during the progress of the survey We take this opportunity of expressing the great obligation we are under to Mr G M WHIPPLE, the Superintendent, and to Mr T BAKER, the First Assistant in the Observatory, who, with the approval of the Kew Committee, rendered us every assistance in their power Our frequent visits and requests for information as to diurnal variation and disturbance have imposed much additional labour on these gentlemen. The help we required has, however, invariably been given with a readiness and heartiness which merit our grateful acknowledgments

We tested our instruments at Kew in 1884, 1886, and 1887.

A very large magnetic disturbance set in on March 30, 1886, the 31st was also much disturbed and the effects were still to be distinguished on April 1. We have, therefore, neglected the observations made on that day, but, with this exception, the following tables contain all the determinations made by us in the Magnetic House.

The first three columns give the date, instrument, and observer, and that headed S the value of the element obtained

The numbers given are not corrected either for diurnal variation or disturbance

In the column headed K is the corresponding value read off from the curves of the continuously recording instruments at Kew, standardised by means of the monthly observations which are taken there

The differences are given in the next column. They are very nearly constant, but are rather larger than we should *a priori* have expected Our own instruments are evidently in accord.

Subtracting the mean difference from the Kew values, we should, if both sets of observations were perfect, reproduce our own numbers, so that a comparison between columns S and $K-\beta$ gives the error of experiment and comparison which is tabulated in the last column

OBSERVATIONS in the Kew Magnetic House.

Declination.

Date	Instrument	Observer	S (Survey Instrument.)	K (Kew Instrument.)	$K-S=\beta$	$K-\beta=K'$	$S-K'$
			° ′	° ′	′	° ′	′
1884 July 17	60	T	18 26·3	18 28·6	2·3	18 26·1	+ 0·2
„ 18	60	R	18 27·4	18 29·9	2·5	18 27·4	0·0
1886 April 2	61	T	18 12·1	18 14·4	2·3	18 11·9	+ 0·2
1887 Sept 30	60	R	18 8·8	18 13·1	4·3	18 10·6	− 1·8
			18 5·5	18 9·4	3·9	18 6·9	− 1·4
Oct 11	61	T	18 6·0	18 8·7	2·7	18 6·2	− 0·2
			18 6·9	18 10·4	3·5	18 7·9	− 1·0
„ 12	60	R	18 9·5	18 11·9	2·4	18 9·4	+ 0·1
			18 11·1	18 13·4	2·3	18 10·9	+ 0·2
„ 13	61	T	18 5·8	18 6·7	0·9	18 4·2	+ 1·6
			18 8·6	18 8·8	0·2	18 6·3	+ 2·3
„ 18	61	T	18 6·3	18 8·4	2·1	18 5·9	+ 0·4
			18 6·0	18 10·4	4·4	18 7·9	− 1·9
„ 19	60	R	18 8·1	18 10·3	2·2	18 7·8	+ 0·3
			18 6·9	18 8·7	1·8	18 6·2	+ 0·7
					$\beta=2·5$	Mean	$=\pm 0·82$

Horizontal Force.

Date	Instrument	Observer.	S (Survey Instrument.)	K (Kew Instrument.)	$K-S=\beta$	$K-\beta=K'$	$S-K'$ = 0·000
1884 July 17	60	T	1·8080	1·8055	− 0·0025	1·8084	− 4
1886 April 2	61	T	1·8091	1·8070	− 0·0021	1·8099	− 8
	60	R	1·8107	1·8072	− 0·0035	1·8101	+ 6
„ 22	60	R	1·8110	1·8082	− 0·0028	1·8111	− 1
1887 Sept 30	60	R	1·8117	1·8087	− 0·0030	1·8116	+ 1
Oct 11	61	T	1·8114	1·8084	− 0·0030	1·8113	+ 1
„ 12	60	R	1·8096	1·8070	− 0·0026	1·8099	− 3
„ 13	61	T	1·8092	1·8062	− 0·0030	1·8091	+ 1
„ 18	61	T	1·8114	1·8088	− 0·0026	1·8117	− 3
			1·8117	1·8088	− 0·0029	1·8117	0
„ 19	60	R	1·8113	1·8074	− 0·0039	1·8103	+ 10
					$\beta=-0·0029$	Mean	± 3·5

To obviate, as far as possible, the use of decimals, the Forces are throughout this paper expressed in terms of metric units (metre, gram, second), and all numerical values of Forces can be reduced to C G S units by dividing by 10

In the case of the Dips, of which no continuous record is kept, the following plan was adopted We give, under the column S, our results corrected for diurnal variation and disturbance as hereafter described, each being the mean of the two needles In column K are the values obtained at Kew at the nearest date to the time of our observations taken from the published record The dates in question are July 28 and 31, 1884, Sept 22, 24, and October 25, 26, 1887 The mean of the last two is taken as applying to the whole period of the 1887 observations

INCLINATION

Date	Instrument	Observer	S (Survey Instrument)	K (Kew Instrument)	K − S = β	K − β = K′	S − K′
July 17, 1884	74	R	67° 36′ 0	67° 38′ 8	2′ 8	67° 36′ 1	− 0′ 1
„ 18, „	„	T	67 35 8		3 0		− 0 3
„ 19, „	„	R	67 36 4		2 4		+ 0 3
Sept 30, 1887	„	„	67 34 9	67 37 5	2 6	67 34 8	+ 0 1
Oct 11, „	83	T	67 34 8		2 7		0 0
„ 13, „	„	„	67 35 0		2 5		+ 0 2
„ 18, „	„	„	67 34 2		3 3		− 0 6
„ 19, „	74	R	67 34 9		2 6		+ 0 1
					β = 2 7	Mean	± 0 2

The mirrors used to form an image of the Sun which can be viewed through the telescope were tested independently We were, however, unfortunate in that on most of the days when we observed at Kew the Sun was invisible The meridian mark also, which is used in the Kew measurements, cannot be seen from the lawn immediately in front of the Magnetic House, and we were, therefore, obliged to use a rod which was fixed at some distance, as nearly as might be in the meridian line in which the instrument was also placed On one occasion (October 11, 1887) the latter adjustment was inaccurate, and a correction was applied deduced from the angle subtended at the pillar in the magnetic house by the centre of the tripod and the rod, and from the distances of the tripod and pillar from the rod At Parsonstown the instrument was placed in the meridian line used for the Observatory of the Earl of ROSSE, and a check on the declinometer was thus obtained From what has been said it will have been seen that the determinations of the geographical meridian at Kew were not made under the most favourable conditions, but the following Table is sufficient to show that no important error attached to the indications of the instruments.

AZIMUTH of Kew Meridian Mark from Sun Observations.

Date	Instrument	Observer	Azimuth
			° ′
April 22, 1886	60	R.	2 48·4
Oct. 11, 1887	61	T.	2 49·9
,, 12, ,,	60	R.	2 48·0
,, 12, ,,	,,	,,	2 50·1
,, 19, ,,	,,	,,	2 48·2
Mean .	.		2 48·9
Value employed at Kew			2 48·7

At Parsonstown the difference between the reading for the fixed mark and the meridian, as calculated from a sun observation, was 1′·1.

We also thought that it would be desirable to compare the instruments under the normal conditions of use in the field. We therefore made three sets of observations at Ard Point, Stornoway; at Bunnahabhain, in the Sound of Islay; and at Stranraer. At the first station the results of the individual experiments made with each instrument were in good accord, but, as the declinations differed from each other by an amount very much greater than any possible error of experiment, viz., 18′, it was obvious that the station was affected by local magnetism. At Bunnahabhain we took the precaution to exchange positions; these observations were made on a patch of limestone, and were in agreement. At Stranraer the sky became overcast, and it was impossible to move the instruments after the first determination of the geographical meridian. The results show that the station was a good one, and we therefore give here the declinations obtained at Bunnahabhain and Stranraer. As we shall have to refer to them again, we reproduce only the data which bear upon the point under discussion. At Bunnahabhain the thread of instrument 61 was broken, and had to be replaced in the field, which accounts for the fact that the first observation made with it is not in very good accord with the others. The Greenwich mean time is reckoned throughout from midnight.

BUNNAHABHAIN.—Aug. 25, 1888.

G. M. T.	Station.	Instrument	Observer	Declination
h m				° ′
10 55	1	61	T.	22 44·0
11 36	2	,,	,,	22 46·0
10 29	2	60	R.	22 46·0
12 25	1	,,	,,	22 47·3
14 30	1	,,	,,	22 47·9
15 9	1	,,	,,	22 47·3

STRANRAER —Aug 28, 1888.

G M T	Station	Instrument	Observer	Declination
h m				° ′
11 35	1	61	T	21 13 7
10 54	2	60	R	21 12 5
13 6	2	,,	,,	21 13 0

During the survey we employed various chronometers, some of which were hired from Messrs DENT, and others were lent to us by the late Professor BALFOUR STEWART, F R S, and by Captain WHARTON, R N, F R S, Chief Hydrographer to the Admiralty, to whom our thanks are due.

We enjoyed an important advantage over our predecessors in that we were able to determine the rates of our chronometers frequently by comparison with Greenwich, by means of the 10 A.M and 1 P M telegraphic signals, of which the former is sent to all post-offices in the kingdom We have to thank Mr PREECE, F.R S, Chief Electrician to the General Post-office, and Mr J C LAMB, the Head of the Telegraph Department, for their kindness in giving or obtaining for us permission to receive the signals. Many of the local post-office officials not only afforded us every facility for correcting our chronometers in accordance with their instructions, but gave us additional help in the selection of suitable stations When possible, we received the signal every day, and rarely omitted more than two consecutive days At places where the signal had undergone one or more re-transmissions, an error on this account was inevitable By the kindness of the Earl of ROSSE, we were able to determine its magnitude on the occasion of our visit to Parsonstown The time signal was observed both by Professor RUCKER and Herr BODDEKKER, the Superintendent of Lord ROSSE's Observatory, who agreed exactly as to the apparent error of the chronometer The value thus obtained differed by 4 sec from that given by the observatory clock, the error of which was known from star observations We are inclined to think that this amount was rarely exceeded, or the rates would have varied more widely than was actually the case No special precautions against error had been taken at Parsonstown, whereas in many districts we had, when practicable, made arrangements at the central office that particular care should be taken on those mornings when we informed the authorities that we intended to receive the signal. We have on several occasions received at the same post-office signals sent by two different routes, involving re-transmission, and have never detected any appreciable difference.

When visiting outlying stations at sea we kept one chronometer on board in a fixed position. This served as the standard. Another instrument, which was frequently compared with it, was used for making the comparisons with Greenwich and for the work of the actual observations At these periods of our survey longer intervals necessarily elapsed between successive receptions of the time signal, during which we

depended on the standard With the exception of two or three cases, which are referred to in the detailed account of the observations at the stations affected, we do not think that any error exceeding a minute of arc can have been introduced into the declinations from uncertainty as to the true time.

Selection of Stations

In selecting stations we have aimed at uniformity of distribution over the whole area under investigation, and, subject to this condition, have chosen places at which observations have previously been made. We have also avoided situations where disturbance by so-called local attraction was known to be great, except in special cases to be discussed hereafter On many tours we have carried geological maps which have aided us in choosing sites

In all we have made observations at 54 stations in Scotland, 102 in England, Wales,* and the Channel Isles, and 44 in Ireland At many of these places which are counted as a single station our instruments have been set up in different localities in order to study or eliminate the local effects of magnetic rocks These sub-stations were in some instances only a few yards and in others several miles apart. The more important of them are indicated by letters in the lists given hereafter, and if these are added they bring the number of stations up to 213. In addition to these we observed on the island of Canna at 23 places. If we took minor changes of position into account this total would be considerably increased, and as we have been able to use the observations made at Greenwich and Stonyhurst, and the data for Cherbourg and Berck-sur-Mer, furnished by the survey of France, recently completed by M MOUREAUX ("Détermination des Eléments Magnétiques en France," par M TH MOUREAUX Paris GAUTHIER-VILLARS, 1886), our conclusions are based upon observations made at about 250 different places

The average distance apart in normal districts is about 30 miles. At most of the principal stations we made two Dip observations, one with each needle, and a complete set of measurements for the determination of the Declination and Horizontal Force At many places we determined the Declination twice, making two independent sets of observations on the Sun and the needle In some cases only the geographical meridian or only the direction of the magnetic axis of the needle was re-determined When time was short or the weather unfavourable the deflection experiment was omitted The following Table shows the total number of observations made By magnetic meridian we mean determination of the direction of the magnetic axis, which, when combined with the Sun observations or geographical meridian, gives the Declination. We do not include 23 Declinations taken on Canna by means of an azimuth compass and chart.

* Exclusive of five supplementary stations along the valley of the Wye, at which Dip observations only were made in 1889 (*vide* p 84)

	Stations	Dips	Deflections	Vibrations	Geographical meridian	Magnetic meridian
Scotland	54	120	57	66	89	93
England and Wales	102	213	94	135	171	185
Ireland	44	89	35	58	84	84
Total	200	422	186	259	344	362

Most of these observations were taken by one or other of us We have, however, to thank Mr A P. LAURIE, now Fellow of King's College, Cambridge, for observations of Dip at 8 stations in Scotland The stations on the West Coast of Scotland were, for the most part, visited in Dr THORPE's yacht "Coventina" During the first year (1884) we generally observed together Afterwards, in order to save time, we travelled separately

Method of Taking the Observations.

The conditions and time of the observations necessarily varied, but the greatest number were made as follows —Shortly before 10 A M we visited the post-office to correct our chronometers by the time signal from Greenwich We then drove to the station which had been selected the night before or earlier in the morning The first observation taken was the solar azimuth, which was finished about an hour before noon This was followed by the declination, vibration, and deflection in this order The dip was then determined, and, if time and weather allowed, we often repeated the azimuth and declination The Sun observations were hardly ever taken within an hour of noon

In a variable climate like that of the United Kingdom the weather often presents a serious difficulty We carried with us a small tent, in which the dip and vibration observations could be made, and we were also provided with waterproof covers for the Magnetometer and Dip Circle We were thus able to make the Dip observations, for example, during showers, while the Magnetometer, though outside the tent, was protected by its covering The case containing the dip needles was carried in a box filled with soda-lime to prevent the axles being injured by rust.

During sunshine we shaded the deflecting magnet in the deflection experiment by a cardboard case, or by throwing a light piece of cloth over it At 42 principal and subsidiary stations the force was determined by means of the vibration experiment only, the deflection being omitted and the moment of the magnet deduced from the values obtained at neighbouring stations

The only part of the observations which requires special comment is the use of the mirror employed in the sun observations The adjustment can be effected by means of the reflected image of the cross wires in the telescope We found, however, that

in practice in the field this method is troublesome It is difficult to see the image unless the mirror is very bright, and there is fairly strong sunshine. On days when the weather and sunshine are uncertain it is very annoying to have to spend time when the sun is visible in making preliminary adjustments From and after our visit to Stornoway in 1884, *i e*, after observations had been made at about a dozen stations, we adopted a different plan. The mirror was frequently adjusted either indoors in accordance with the directions of the Admiralty 'Manual,' or by means of some elevated object in the field In the latter case the error of parallelism was first got rid of by observing the object when the observer's back was turned towards it, and the mirror was nearly vertical The mirror was adjusted until the image appeared in the same position as before when it was reversed in its bearings. The object was then viewed by reflection when the instrument was turned through 180°, *i e*, when the mirror was nearly horizontal and the axis of the mirror was made perpendicular to the axis of the telescope By repeating these processes twice or thrice a perfect adjustment was obtained, and it was found that under ordinary circumstances the instrument could be carried about from place to place for some time without this adjustment being seriously affected To make certain, however, that all was right, we always (when possible) took observations of the sun both in the "front" and "back" positions, *i e*, with the mirror approximately horizontal and vertical and reversed the mirror in its bearings in both positions By the latter precaution the error of parallelism (if any existed) was eliminated, and the observation thus afforded data for the calculation of the error of collimation, and for the corrections which in consequence of this error must be applied to the "front" observations

By this method the actual adjustment of the mirror could be made at any time or place If the time at our disposal was short, or the weather uncertain, the instrument was regarded as ready for use as soon as it was set up in the field, and thus no time was lost, but the observations were so taken that any error of adjustment could be calculated and allowed for. Lastly the agreement of the measurements made in the front and back positions gives a valuable test both of the adjustments and observations.

In the case of the observations made in 1888 with Magnetometer No. 60, the method of correcting the mirror by the image of the cross wires was again resorted to. Some modifications of the instrument which we have introduced (*loc. cit*) now make this method practicable, and it was found to work well under the new conditions.

Diurnal Variation and Disturbance.

In the survey of 1857 corrections for diurnal variation and disturbance were applied to the declination observations only. These two quantities are intimately connected. The total deflections of the continuously recording instruments at any instant are the sums of the normal diurnal variations and the disturbances. If the latter are assumed to occur simultaneously and in equal intensity over an area such as that of the United

Kingdom, and if the G M T. at which the observations were made is known they may be determined from the curves obtained at Kew On the other hand the magnitude of the diurnal variation at a given instant depends on the local time, and in the case of stations so far apart as Kew and Valentia, the diurnal variation of the declination would, between 10 and 11 a m G M T , differ by several minutes of arc

We have therefore taken out from the Kew curves the deviations from the mean at the G M T of our observations, and subtracting the corresponding value of the diurnal variations at Kew, have called these differences the temporary disturbances at the time of the observations

We have assumed further, that the Kew mean curves of diurnal variations may be taken as applying to the whole of the United Kingdom, but we have, in all cases, used the correction corresponding to the local time Thus the total correction is the algebraic sum of the diurnal variation at the local time, and of the disturbance registered at Kew at the G M T at which the observations were taken

As proof of the validity of this method, we have picked out all the stations in Ireland at which two or more observations were made at times when the diurnal variations differed by more than four minutes

As full particulars are given later on, it is only necessary here to tabulate the corrections for diurnal variation and disturbance in addition to the final results It may, however, be remarked that in the majority of cases the determinations of the geographical, as well as of the magnetic meridian, are quite independent, and that at Athlone, Gort, and Westport, the results given were obtained on different days The letters v and Δ are used throughout this memoir to indicate the departure from the mean value of the element at the time of experiment, due to diurnal variation and to temporary disturbance respectively.

Station	v	Δ	Corrected Declination
	′	′	° ′
Athlone	+ 6 3	+ 1 5	22 16 5
	− 3 9	0 0	22 16 8
Cavan	− 3 7	+ 1 5	22 26 1
	+ 6 2	− 4 0	22 26 7
	+ 3 5	− 5 0	22 27 4
Charleville	+ 4 9	+ 1 0	22 19 4
	+ 0 7	+ 2 5	22 19 1
Cork	0 0	0 0	22 3 8
	+ 5 8	− 1 0	22 1 7
Gort	+ 3 9	− 1 0	22 38 2
	− 4 2	+ 1 0	22 39 1
Lismore	− 1 0	0 0	21 54 1
	+ 6 4	+ 3 5	21 54 0
Westport	+ 0 7	0 0	23 5 5
	+ 1 8	− 0 5	23 4 2
	+ 6 3	+ 2 0	23 5 4

The following are the best stations for the application of a similar test to Scotland. The two sets of observations at Scarnish and Portree were taken on two different days Those at Stornoway were on the same day, but at different stations. The observations at Bunnahabhain and Stranraer, which have been already discussed, are here arranged in order of time without reference to the station or instrument —

Station	v	Δ	Corrected Declination
	′	′	° ′
Scarnish	+ 2 4		24 49 8
	− 3 6	+ 3 0	24 51 8
Portree (2)	+ 3 3	+ 2 0	22 21 6
	+ 3 7	− 2 0	22 22 7
Stornoway, Ard Point	+ 2 0	− 2 0	23 50 7
(1)	+ 6 3	− 1 0	23 48 4
	+ 4 4	·	23 50 7
Ard Point (2)	+ 2 4	− 3 0	24 8 7
	+ 6 4	− 1 0	24 7 9
	+ 4 4		24 8 3
Bunnahabhain	− 0 1	·	22 46 0
	+ 1 7		22 44 0
	+ 4 3	− 2 0	22 46 0
	+ 6 3	− 1 0	22 47 3
	+ 5 7		22 47 9
	+ 4 6		22 47 3
Stranraer .	+ 1 9	− 2 0	21 12 5
	+ 4 6	− 3 0	21 13 7
	+ 6 4	− 1 0	21 13 0

In some cases where there is an appearance of a regular change, such as would be caused if the diurnal variation was not properly estimated, it can be otherwise accounted for. Thus the first four observations at Bunnahabhain were independent in every particular, but the last two depend upon the same Sun observation as the fourth This probably accounts for the somewhat closer agreement in these cases The magnitude of the errors which would be introduced if the disturbance corrections were omitted can be estimated by noting that at Cavan the first and last observations when uncorrected give 22° 27′ 6 and 22° 22′ 4, those at Portree 22° 23′ 6 and 22° 20′·7, and those at Scarnish 24° 49′ 8 and 24° 54′·8 respectively, instead of the much more accordant numbers entered in the Table.

These results when corrected are quite as good as those we obtained when observing at Kew, and furnish a complete *a posteriori* justification of the method of correction.

The values of the diurnal variation employed for the year 1884–5 were furnished to us by Mr WHIPPLE For 1885–6 and 1886–7 we used the Tables given in the reports of the Kew Committee for those years ('Roy Soc Proc,' vol 41, p. 415, and vol. 43, p 226).

These differed rather markedly as to the magnitude of the maximum variation from the curve employed for 1884–5, the difference amounting to about 1′·5 As the report

of the Kew Committee for 1887–8 was not published when the observations made in 1888 were reduced, we used the same Table as for the 1887 observations.

The question as to whether we should apply similar corrections to the other elements has been carefully considered, and we have on the whole decided to do so.

Though we have worked up the observations on the Horizontal Force so that each experiment gives two nearly independent results, the times at which these have been taken are practically identical, and they do not serve as a test of the advantage of applying corrections. In 18 cases we have repeated either the deflection or the vibration experiment (generally the latter) in the field on the same day. The differences between the results when uncorrected and corrected for diurnal variations and disturbance are as follows. They are expressed in terms of 0·0001 metric or 0·00001 C.G.S. unit.

TABLE I

Station	Uncorrected	Corrected
Loch Aylort	− 5	0
Ayr	− 4	− 4
Stornoway	− 3	− 5
	+ 3	+ 2
Cambridge	+ 1	+ 1
Northampton	+18	− 7
Stoke	− 5	− 6
Swansea	+ 3	− 1
Taunton	+17	+16
Thirsk	0	+ 1
Tunbridge Wells	+17	− 3
Worthing	+ 8	− 2
Ballywilliam	0	− 1
Bangor	+ 4	− 6
Coleraine	− 7	−11
Drogheda	+ 9	+ 3
Kildare	−10	+ 5
Londonderry	+ 6	+ 5
Probable difference: Uncorrected	5·9	
Probable difference: Corrected	4·0	

The values of H obtained by us at Kew in 1887 also afford a test of the method of correction. The diurnal variation and disturbance are expressed in the following Table in terms of 0·0001 metric unit.—

Date	Deflection		Vibration		H	
	v_1	Δ_1	v_2	Δ_2	Observed	Corrected
September 30	+ 7	0	+ 3	0	1·8117	1·8112
October 11	2	0	− 2	0	1·8114	1·8114
,, 12	3	− 20	− 1	− 20	1·8096	1·8115
,, 13	− 5	− 20	− 10	− 10	1·8092	1·8114
,, 18	− 5	0	− 13	0	1·8114	1·8123
,, 18	− 5	0	− 11	0	1·8117	1·8125
,, 19	− 4	− 10	− 10	0	1·8113	1·8125

As then the diurnal variation between the hours at which our observations were generally taken amounts to 0·0025 metric unit, and as the probable difference between the corrected semi-independent results at the stations referred to in Table I., p 67, is ·0004 (the probable error being half that quantity), and as the probable error of the seven Kew observations is less than ·0004, there seems no doubt that the correction ought to be applied

In the case of the Dip the diurnal range (which is more than 2′) exceeds the error of experiment, but the application of a correction for disturbance is more difficult and uncertain It has to be deduced from the Vertical and Horizontal Force magnetograms and thus the chances of error are increased The quantities to be dealt with are generally small, and the uncertainty of the readings taken from the curves unquestionably affects the results. We have not many data which furnish any *a posteriori* arguments of importance, as in most cases the times at which the two Dip observations were taken were separated by too short an interval for the diurnal variation to have altered much. The observations made at Kew in 1887 are, however, decidedly improved by correction.

Date.	Instrument.	Needle	Dip observed.	v	Δ	Dip corrected
			° ′	′	′	° ′
September 30	74	1	67 35·4	− 0·7	+ 0·7	67 35·4
	74	2	34·7	− 0·8	+ 1·3	34·2
October 11	83	1	67 33·8	− 0·4	0	67 34·2
	83	2	34·9	− 0·5	0	35·4
,, 13	83	1	67 36·0	− 0·4	+ 1·4	67 35·0
	83	2	36·1	− 0·5	+ 1·6	35·0
,, 18	83	1	67 34·2	− 0·3	+ 0·3	67 34·2
	83	2	34·2	− 0·4	+ 0·3	34·3
,, 19	74	1	67 35·0	− 0·3	0	67 35·3
	74	2	33·5	− 0·3	− 0·7	34·5

The greatest differences in the corrected and uncorrected results are 1′·2 and 2′·6 respectively.

On the whole, then, we think the application of the correction for diurnal range is advantageous, but the correction for disturbance appears to be somewhat uncertain in its effect as applied to the field observations, and we have retained it chiefly for the sake of uniformity with our treatment of the other elements

The formulæ used in the application of the corrections to the Horizontal Force and Dip were as follows —

If H′, m' and H, m be the values of the Horizontal Force and the moment of the magnet deduced from the uncorrected and corrected observations respectively, and if ϕ and ψ be the total increments of the Horizontal Force due to diurnal variation and disturbance at the times when the deflection and vibration experiments are made,

$$\mathrm{H} = \mathrm{H}' - (\phi + \psi)/2;$$
$$m = m' \{1 + (\phi - \psi)/2\mathrm{H}'\}$$

As the disturbances of the Dips have to be deduced from those of the Horizontal and Vertical Forces, the calculations involved are rather troublesome

If $d\theta$ be the change, expressed in minutes of arc, produced by increments $d\mathrm{V}$ and $d\mathrm{H}$ in the Vertical and Horizontal Forces respectively,

$$d\theta = a d\mathrm{V} - b d\mathrm{H},$$

where a and b are quantities which depend on the Dip and Horizontal Force at the station Tables were prepared in which the values of these were entered for each complete degree between 66° and 72°, and for each tenth of a unit of Horizontal Force between 1 4 and 1·8. As they vary slowly, the value corresponding to any given Dip or Force could be readily determined, and the value of $d\theta$ was thus deduced

Method of Tabulating the Observations.

We have ventured to modify considerably the methods ordinarily adopted of presenting the results of a survey such as our own. It is often very difficult for any one who studies the records of magnetic observations to learn anything as to whether the instruments were in good or bad adjustment, or whether the actual observations were careful or careless Sometimes superfluous data are given which supply no information on these points. Thus, the quantities log mX and log m/X, or mX and m/X, which are sometimes tabulated in records of Force observations, add but little to their value. In an observatory, indeed, they ought to be nearly constant from month to month; but as, in both of them, the changes in the value of the moment of the magnet and those in the Force are mixed up with the error of experiment, it is not easy to draw any definite conclusion from them To test the observations, we should compare quantities the difference between which ought theoretically to vanish. In the case of a survey, log mX and log m/X must vary largely from station to station, and they are, therefore, quite worthless as tests of the observations.

There are now five British magnetic observatories working with instruments of substantially the same patterns, and we think it would be very desirable if an agreement could be come to as to the form in which the results of the observations are published In our own case we have been anxious to check the numbers given by experiment in every way, as it is only thus that we are able to determine what reliance may be placed on observations at stations where time or weather prevented the repetition of one or more of the measurements

The following is an example of the method of tabulation adopted in the case of the Declinations —

Σ is the interval between the southing of the Sun and the mean of the times at which the "front" observations were made on the Sun's azimuth It is taken as positive if the observations were made after noon.

Alt. signifies the altitude of the Sun above the horizon at the same time

μ is the correction made by means of the "back" observations to the geographical meridian determined from the front observations alone This quantity serves to indicate the order of the error that may have been introduced by the omission of the back readings on some occasions.

G M is the reading on the horizontal circle which corresponds to the geographical meridian. In most cases two determinations of the geographical and magnetic meridians were made at an interval of some hours In this time the tripod may have shifted slightly, owing to unequal sinking of the ground, warping of the legs, &c Hence, if no fixed mark was read, each geographical meridian must be regarded as corresponding only to the magnetic meridian which was observed nearly at the same time as itself. As it is often difficult to select a suitable fixed mark, we have, whether a fixed mark was observed or not, regarded each reading for the geographical meridian as corresponding to that for the magnetic meridian which was nearest to it in point of time.

In the case of the magnetic meridian (the reading for which is indicated by M.M.) we tabulate the following data

(1) The G.M.T. at which the observation was made reckoned from midnight.

(2) Half the difference between the readings when the magnet is erect and inverted is the angle between the magnetic and geometrical axes of the magnet If then from this quantity (ξ) we subtract its mean value as determined from all the observations made in the same year (ξ_0), we obtain a measure of the error due to inaccuracy of reading or adjustment, and to imperfect compensation of the diurnal variation or of any disturbance which may have been occurring at the time.

(3) ω is the error which would have been caused had the torsion of the thread not been allowed for.

(4) v and Δ are the deviations of the element from its mean value due to diurnal variation and disturbance respectively. The first, as has been already explained, is calculated for local time from the Kew curves of diurnal variation, the latter is the

disturbance registered by the Kew curves at the same G M T. as that at which the observations were taken

(5) δ (obs) is the value of the Declination obtained by subtracting the reading for the magnetic from that for the corresponding geographical meridian

(6) σ is the secular correction to Jan 1, 1886, obtained as is hereafter explained.

(7) δ (red) is the mean value of the Declination reduced to the epoch of the survey.

As an example we take Athlone, where observations were made on two different days On May 8, no back observations were taken, so that μ could not be calculated.

DECLINATION. Athlone. Observer, T E T. Magnetometer 61.

Geographical Meridian

Date, 1887	Σ	Alt	μ	G M
	h m s	° ′	′	° ′
May 8	+1 40 59	48 42		338 38 1
„ 9	−3 43 58	33 51	− 0 2	210 31 2

Magnetic Meridian

Date	G M T	$\xi - \xi_0$	ω	ι	Δ	M M
	h m	′	′	′	′	° ′
May 8	12 39	+0 2	−0 1	+6 3	+1 5	316 21 6
„ 9	9 9	−0 4	0 0	−3 9	0 0	188 14 4

Declination.

δ (obs)	Mean	σ	δ (red)
° ′	° ′	′	° ′
22 16 5 22 16 8	22 16 6	+10 1	22 26 7

In the case of the Horizontal Force observations the deflections were always taken when the distances between the centres of the magnets were 0 3 and 0 4 metre. If then l_1 and l_2 are the common logarithms of the values of m/H obtained at the two distances, the factor P used in the corrections may be calculated by the formula

$$P = 0\ 4737\ (l_1 - l_2) - 1\ 947\ (l_1 - l_2)^2$$

of which the second term is very small. ('Nature,' Aug 13, 1887, p 366; Sept. 29, p. 508, Dec 8, p 127, and Jan 19, 1888, p 272)

The following table gives the mean values of P obtained for each year during the survey.

	Magnetometer	
	60	61
1884	0 000817	
1885	0 000866	
1886	0 000828	0 000753
1887	0 000809	0 000692
1888	0 000800	0 000706

The ordinary method of determining the Inclination with two needles gives two independent values, the agreement of which furnishes a test of the accuracy of the observations It is, however, usual to give only one value of the Horizontal Force deduced from the means of the values of m/H and $m\mathrm{H}$ calculated from the deflection and vibration experiments The constancy or regular change of the value of m affords a means of determining whether the observations are sufficiently good, but it seems to us that nothing is for this purpose so satisfactory as the agreement or disagreement between two independent experiments The results of the ordinary deflection and vibration observations can readily be presented in such a form as to satisfy this condition with only an insignificant addition to the labour of reduction.

The vibration experiment furnishes twelve independent determinations of the time of 100 vibrations, six of which are taken when the scale of the vibrating magnet is apparently moving to the right, and six when it appears to be moving to the left. If we take the mean of the six observations furnished by the first and last three of each of these groups, we have two independent determinations of the vibration period, based in each case on six observations, of which three were taken when the movement appeared to be to the right, and three when it appeared to the left.

If we agree always to combine the first and second of these groups with the values of m/H obtained when the distance between the magnets is 0·3 and 0 4 metre respectively, we obtain from each experiment two values of m and H Except in so far as errors may arise in the determination of the temperature, or in the adjustment of the instrument (levelling, &c.), these are absolutely independent, and thus furnish a satisfactory check on the accuracy of the observations. It would, however, be hardly worth while for the sake of this advantage to go through the labour of repeating the reduction of the vibrations As the two vibration periods are very nearly the same, this is unnecessary. Let T be the time deduced from all the observations, and $\mathrm{T} \pm d\mathrm{T}$ the times given by the first and second groups of observations selected as above described Then if D be the value of m/H given by the deflection experiment,

$$H = \sqrt{\frac{\pi^2 K}{T^2} \div D},$$

$$\frac{dH}{H} = -\frac{dT}{T},$$

and in like manner

$$\frac{dm}{m} = -\frac{dT}{T}$$

Thus, if we use the mean of the times for deducing two values of H and m from the two deflection experiments (as is done in the ordinary method of reduction) we may obtain the independent values of H and m by altering the means in the ratio of the independent to the mean times

Thus, at Horsham the mean time of vibration was $4^s \cdot 1650$, and using this the deflections at the two distances gave

$$H = 1\cdot8365 \text{ and } 1\cdot8366$$

and

$$m = 0\cdot0091956 \text{ and } 0\cdot0091952$$

respectively.

The actual periods of vibration for 100 oscillations observed were—

Apparently—	
Moving to right	Moving to left
m s	m s
6 56·6	6 56·6
56·5	56·6
56·5	56·5
56·5	56·6
56·3	56·4
56·4	56·5

Taking the means of the observations recorded above and below the line we get for the time of one oscillation

$$4^s \cdot 1655 \qquad \text{and} \qquad 4^s \cdot 1645.$$

These independent times differ from the mean by 0·012 and − 0·012 per cent. respectively

Hence, altering the values of H and m in the same proportion, but in the opposite direction to that indicated by the signs of these quantities, we get

$$H = 1\cdot8363 \qquad \text{and} \qquad 1\cdot8368,$$

and

$$m = 0\cdot0091945 \qquad \text{and} \qquad 0\cdot0091963.$$

These quantities are quite independent, except as regards the temperature corrections and instrumental adjustments, and their difference is a measure of the errors of observation and imperfect compensation of the diurnal variation and disturbance In tabulating the Force observations we give the G.M T at which the deflection and vibration observations were made, and the extreme temperatures during the progress of the experiments, the observed deflections and the independent times calculated as above described

The agreement of the times of vibration serves to test the vibration experiments. As a measure of the accuracy of the deflections we take the ratio of the two values of m/H calculated by combining the mean time of vibration with each of the two deflections

The logarithms of these two quantities occur in the calculations, and all that is necessary is to subtract them and look out the corresponding natural number. As this quantity is very nearly unity we multiply by 100 and give the fractional part only. The symbol ϵ is thus the percentage difference between the results given by the two deflections when combined with the mean time of vibration.

The following form, which is that which has been used, explains itself.

The results of the deflections differed by 0 008 per cent., the times of vibration by $0^s{\cdot}001$ There was no disturbance, and the Horizontal Force, owing to diurnal variation, was 0·0007 and 0 0003 metric unit above the average, at the times which correspond to the middle of the deflection and vibration experiments respectively.

The corrected values are obtained from the mean values by applying the corrections for diurnal variation and disturbance.

HORIZONTAL FORCE. Horsham. Observer, A W. R. Magnetometer 60.

Deflection.

Date, 1888	G M T	t	Observed deflection	ϵ	v_1	Δ_1
	h. m.	°	° ′ ″			
April 21	14 44	11·8 to 10 7	21 51 55 9 0 8	+ 0 008	+ 0 0007	0

Vibration

Date	G M.T	t.	T	v_2	Δ_2.
	h. m	°	s		
April 21	13 34	16 5 to 17 4	4 1655 4 1645	+ 0 0003	0

Values of m

Observed	Mean	Corrected
0 0091945 0 0091963	0 0091954	0 0091965

Values of H

Observed	Mean	Corrected	σ	Reduced
1 8363 1 8368	1 8365	1 8360	− 0 0051	1 8309

At places where the deflection experiment was not made, the moment of the magnet was calculated from the other moments observed during the same tour It was assumed that the strength of the magnet diminished regularly, and a linear expression was determined by equations of condition which gave the rate of decrease A careful analysis of the Scotch observations, which were the first reduced, proved that this method was satisfactory.

The greatest difference between observation and calculation occurred at Kirkwall, and amounted to 0 234 per cent It is certain that in this case the difference is not due to errors of experiment, but to a real change in the moment, as the observations obtained at the next station (Lerwick) confirmed it We are also able to assign a probable cause for the alteration, as the "Coventina" was caught in a "roost" off Stromness, and experienced rather rough treatment, during which the magnet may have been jarred.

The mean difference (irrespective of sign) at all the stations was 0·057 per cent.; and if we put aside about one-fifth of the whole number of stations at which, owing to bad meteorological or magnetic weather, or some similar cause, the conditions under which the observations were taken were not very favourable, the mean difference at the remainder was 0 028 per cent.

It must, of course, be remembered that this quantity is not a measure of the error of experiment, but of the algebraical sum of that error, and of the deviation of the rate of decrease of the moment of the magnet from perfect uniformity. The conclusion we came to from the discussion of the Scotch observations was that if the observed moment differed from that calculated by the linear formula by 0·1 per cent, either a real change had taken place in the magnet or the observation had been affected by some disturbing cause, which it was in general easy to specify. In the later tours the moments of the magnets changed very slowly, and it was sufficient to take the mean

of the moments obtained at one or two stations immediately preceding and following that at which the deflection experiment had been omitted. The total number of stations at which vibrations only were observed was 42

In the case of the Dip observations the correction for diurnal variation was deduced from the Kew curves for the diurnal variation of the Horizontal and Vertical Forces, which were supplied to us by Mr WHIPPLE

The disturbance corrections were calculated as previously described.

The following form was used in working up the results —

INCLINATION Bunnahabhain Observers Needle 1, A. W R.; Needle 2, T. E T. Dip Circle, DOVER, 83

Date 1888	Needle	G M T.	Observed	v	Δ	Corrected
		h m	° ′	′	′	° ′
Aug 25	1	12 46	70 40 2	+ 0 1	+ 0 7	70 39 4
	2	14 38	70 39 4	− 0 4		70 39 8

Mean	σ	Reduced
70° 39′ 6	+ 3′ 4	70° 43′ 0

All the observations were tabulated in these forms, and we propose to deposit copies of them with the Royal Society. In this paper we give fewer data It will, however, be more convenient to describe these under the heading "Results of the Observations," p 93, and we now proceed to use the tabulated results in a discussion of the accuracy of our work.

Errors of Experiment

The completion of so large a number of observations as those involved in our survey, carried out for the most part by two observers with different sets of instruments, affords a good opportunity of testing the accuracy of experiment in the field, especially as we have followed a regular practice of determining both the geographical and magnetic meridians by means of independent experiments made at an interval of several hours. Again, our method of tabulating the results of the Force observations furnishes a good test of the various parts of the experiments, and, as will be seen hereafter, we are able, in the case of the Dip observations, to arrive at conclusions as to the small errors due to the imperfections even of the excellent needles supplied by Mr. DOVER.

In the first place then, we take as the error of a Declination, the difference from the mean of the values obtained at the same station at an interval so small that the secular correction is not involved (*i e*, a few hours or days), when both the geographical and magnetic meridians are determined independently

The error of a Force observation is taken as half the difference between the two *independent values*, calculated as above described

Lastly, the error of a Dip observation is half the difference between the results given with needles 1 and 2 In the case of the last two, it is evident that both quantities may be affected with small errors which are not thus detected, such as that due to uncertainty as to temperature, in the case of the Forces, and errors of setting in the magnetic meridian, in that of the Dips

Taking, however, these quantities and treating them as though all the results in each group were measures of the same quantity, we get the following values of the probable error —

	Number of observations	Probable error
Declination	97 5	0′ 699
Horizontal Force	196	0 00028 (M U)
Dip	190	0′ 15

In this Table, we count each double observation as one. The fraction, in the case of the Declination, is due to the fact that, at some stations, an odd number of observations was made.

It is useful to analyse the observations still further In so doing, we have generally treated Scotland as a whole During our earlier tours we observed together, and sometimes one observer would take the geographical and the other the magnetic meridian, one would observe the deflection and the other the time of vibration It is, therefore, difficult to separate the results In England and Ireland we always observed apart, and, with the exception of a short tour in 1886, when Dr. THORPE used Dip Circle No 74, we always used different instruments. The mean values of the quantities are taken *irrespective* of *sign*, so that they indicate the average value whether positive or negative.

The definitions of the quantities used as tests, are given on p 70 The symbol dS indicates the mean error of the element

DECLINATION Observations

	Observer	Instrument	Mean values of			
			μ	$\xi - \xi_0$	ω	dS
Scotland	R and T	60 and 61	±0′ 31	±0′ 39	±0′ 75	±0′ 77
England	T	61	±0 39	±0 32	±0 28	±0 71
	R.	60	±0 45	±0 35	±0 39	±0 85
Ireland .	T	61	±0 48	±0 24	±0 37	±0 51
	R	60	±0 39	±0 32	±0 42	±0 63

Roughly then, the average errors due to imperfect adjustment of the mirror, to inaccuracy in determining the magnetic axis of the magnet, and to torsion, ought not, in each case, to exceed about 0 4 of a minute of arc, assuming that no correction was attempted, as would be the case, for instance, with regard to μ, if no back observations of the sun were taken. As in the great majority of cases, all three were corrected by front and back observations on the sun, by observing the magnet erect and inverted, and by determining the torsion, we may be sure that the actual errors were much below these amounts. It seems, therefore, probable that the observations are affected with small errors which are not so capable of correction, for the mean and probable values of dS appear to prove that it is an even chance that each of two independent declinations differ from their mean by more than 0′·7

In the following Table we give a similar analysis of the Force observations The quantity $\epsilon/2$ is the percentage difference of either deflection experiment from their mean

dT is the average difference in seconds between each "independent" determination of the time of a single oscillation (as defined on p. 73) and their mean.

dH is the average difference between each independent determination of the Horizontal Forces and the mean of the two involved in each complete experiment. In all cases the averages are, of course, taken without reference to sign

HORIZONTAL Force.

	Observer	Instrument	Mean values of		
			$\epsilon/2$	dT	dH.
			Per cent.	s	M U
Scotland	R and T.	60 and 61	± 0 041	± 0 00043	± 0 00033
England	T.	61	0·031	0 00040	0 00030
	R	60	0·023	0 00035	0 00025
Ireland	T	61	0 033	0·00032	0 00028
	R	60	0·017	0 00040	0 00022

We have analysed the dips in a somewhat different way.

If one needle tends to give a slightly higher or lower result than the other, the mean value of the difference of the results, always taken in the same order—viz, dip given by Needle 1, minus dip given by Needle 2—ought to give the bias or measure of the difference due to imperfections in their construction. If this quantity be subtracted from the difference of the results in each observation, we get the error of experiment, which is the difference due to imperfection in observing and correcting. The mean of the last quantities, taken without reference to sign, is the mean error of experiment

Thus, if we take the difference as positive when Needle 1 is the higher, out of 23 stations in Ireland at which observations were made with the same instrument (74), it was positive at 15 and negative at 8 The mean of all the differences showed that, on the average, Needle 1 gave results 0′ 31 higher, and Needle 2 results 0′ 31 lower, than the mean of the two needles; so that there is evidence of a *bias* which would make Needle 1 give results 0′ 6 higher than Needle 2 That this number is worthy of credit is shown by the close agreement of the values obtained with the same instrument in England, and by the practical identity of all the numbers obtained for Dip Circle 83 in England, Scotland, and Ireland.

The axles of the needles of Dip Circle No 74 appear to have undergone some slight alteration between the end of 1885 and the beginning of 1886, but there seems no doubt that since that date Needles No 1 in the two instruments have read about 0′ 3 and 0′·2 respectively above the mean of the two needles.

The following Table gives a summary of the results —

INSTRUMENT 74

	Observers.	Number of stations, with difference		Bias Upper sign, Needle 1 Lower sign, Needle 2.
		Positive.	Negative	
Scotland	T, R, and L	18	31	∓ 0′ 25
England	T	8	2	± 0 25
	R	24	12	± 0 33
Ireland	R	15	8	± 0 31

INSTRUMENT 83.

	Observers	Number of stations, with difference		Bias Upper sign, Needle 1 Lower sign, Needle 2
		Positive	Negative	
Scotland	T and R	6	3	± 0′19
England	T	25	16	± 0 20
Ireland	T	11	7	± 0 20

An analysis of the errors and corrections such as we have carried out appears to constitute a very delicate test of the observer and instrument combined. Thus it is curious to note that (although the differences are small) one of us has on the whole been more successful with the Declinations and the other with the Force observations. As has already been remarked, the observations in Scotland were often divided between us, so that we do not attempt to analyse them, but both in England and Ireland Dr. THORPE's Declinations agree the better by about 0′·26, and Professor RUCKER's Forces by about 0·00011 metric unit. The facts that in Ireland Professor RUCKER's mirror appears to have been generally in the better adjustment, and that in England Dr. THORPE's observations of the period of oscillation of the needle were the better, indicate that the causes of the differences in the results were probably a more perfect management of the torsion of the thread in the Declination observations, and a slight superiority in the deflection observations in the Force measurements.

It will, we think, be granted that in all cases the observations are as good as the occasion requires, and we discuss these minute differences not because we think they produce any appreciable effect on the results of our survey, but as an illustration of the fact that our method of tabulating the observations, conveys information as to the personal equations of the observers which is not afforded by the results as ordinarily presented.

We think it possible that if we had been able to carry out the work in a more leisurely manner, to observe azimuths only when the Sun was low in the heavens, and to wait for fine weather, the results might have been in closer accord, but on the other hand we do not think that any improvement so obtained would have been of practical importance The uncertainty of the value of the secular correction at any given station, and the changes produced in disturbed districts by a slight alteration in the position of the instruments, are far more important than any residual errors by which our observations may possibly be affected.

Secular Corrections

The secular correction is often appreciably different at neighbouring stations Thus the annual change in the Dip in the years 1883–4 and 1884–5 was 1′ 8 and 1′ 7 at Greenwich, and 1′ 4 and 1′ 2 at Kew

The differences in the Declination change are still more marked, as the following Table shows

Year	Declination		Annual variation	
	Greenwich	Kew	Greenwich	Kew
1880	18° 32′ 6	18° 59′ 0	5′ 5	8′ 5
1881	27 1	50 5	4 8	5 7
1882	22 3	44 8	7 3	4 8
1883	15 0	40 0	7 4	8 0
1884	7 6	32 0	5 9	7 3
1885	1 7	24 7	-	

It will be noted that the annual variations differed in 1880 by 3′ 0, and that the average difference irrespective of sign is 1′ 4.

It does not therefore appear advisable to reduce the observations to one epoch by means of the annual variations as determined at a single observatory

On the whole a better result will probably be attained if we collect all the evidence at our disposal and deduce from it an average secular change

The comparison of our own observations with those of Mr WELSH affords the means of determining this quantity for Scotland, and proves that the rate of decrease of the Dip is less in the northern than in the southern stations, a result which is in accord with previous observations

The mean decrease in Scotland between 1837 and 1857 was 1′ 94 ('Brit Assoc. Rep,' 1859, p 169) In the nearly corresponding interval between 1837 and 1860 it was 2′ 05 on the northern border of England and 2′ 68 on the south coast ('Brit. Assoc Rep,' 1861, p 260)

Unfortunately very few Dip observations were made in England and none were made in Ireland in the 1857 survey, but there are a number of stations common to the surveys of 1837 and 1886 We can thus determine the secular decrease during the last 50 years In the following Table the stations are grouped according to their geographical distribution, the districts being North and South Scotland, North, Central, and South England and Wales, and North and South Ireland.

The data for the 1837 survey are taken from the paper by the late Sir EDWARD SABINE, already cited ('Phil Trans,' 1870, Vol. 160, p 271)

TABLE II.—Mean Annual Decrease of Inclination between 1837 and 1886.

Station	Mean Annual Decrease	Station	Mean Annual Decrease.
Lerwick	1′·22	Malvern	2′·33
Aberdeen	1·58	Brecon	2·24
Kirkwall	1·41	Aberystwith	2·27
Wick	1·47	Harwich	2·02
Golspie	1·48	Clifton	2·25
Inverness	1·57	Swansea	2·22
Fort Augustus	1·44	Kew	2·04
Edinburgh	1·43	Ilfracombe	2·04
Glasgow	1·53	Salisbury	2·28
Helensburgh	1·38	Dover	2·10
Campbelton	1·70	Exeter	2·06
Cumbrae	1·23	Ryde	2·26
Berwick	1·77	Weymouth	2·32
Alnwick	1·67	Plymouth	2·26
Newcastle	1·75	Falmouth	2·46
Carlisle	1·96	Londonderry	2·04
Whitehaven	1·77	Strabane	1·87
Thirsk	1·88	Bangor	1·98
Scarborough	1·80	Armagh	2·09
Manchester	2·10	Enniskillen	2·04
Birkenhead	1·98	Ballina	2·06
Holyhead	2·02	Westport	2·12
Cromer	1·81	Clifden	2·13
King's Lynn	1·96	Galway	1·99
Nottingham	2·06	Dublin	2·06
Pwllheli	2·04	Limerick	2·21
Shrewsbury	1·70	Waterford	2·31
Birmingham	2·10	Killarney	2·45
Cambridge	2·02	Cork	2·32
Lowestoft	1·84	Valentia	2·50

There is a steady increase as we pass from north to south, and from east to west. This is brought out in the next table, in which the mean decrease is given for each district, and also for the stations in the easterly and westerly parts of it.

District	Number of Stations	Mean annual decrease of inclination between 1837–1886 for		
		Whole district	Western half	Eastern half
North of Scotland	7	1′·45	1′·49	1′·42
South of Scotland	6	1·51		
North of England and North Wales	9	1·88	1·97	1·77
Midlands and Wales	14	2·05	2·10	2·00
South of England	9	2·20	2·24	2·17
North of Ireland	8	2·04	2·07	1·98
South of Ireland	7	2·26	2·37	2·15

In the case of twelve Scotch and six English stations we are able to compare the rate of change in the two periods 1837–1857 and 1857–1886 The data are taken from the sources above described

Station	Mean annual decrease of inclination.		Ratio
	1837–1857	1857–1888	
Lerwick	1′ 65	0′ 96	1 72
Aberdeen	2 02	1 33	1 52
Kirkwall	1 97	1 07	1 84
Wick	2 02	1 11	1 82
Golspie	1 68	1 38	1 22
Inverness	1 92	1 37	1 40
Fort Augustus	1 80	1 21	1 49
Edinburgh	1 82	1 14	1 60
Glasgow	1 80	1 42	1 27
Helensburgh	1 48	1 32	1 12
Campbelton	1 99	1 39	1 43
Cumbrae	1 53	0 94	1 63
Berwick	2 48	1 33	1 87
Scarborough	2 02	1 63	1 24
Stonyhurst	..	1 75	.
Cromer	2 22	1 44	1 54
Cambridge	2 50	1 58	1 58
Lowestoft	2 17	1 54	1 41
Kew	2 65	1 62	1 63
St Leonards		1 82	.
Plymouth	2 73	1 89	1 44

The mean ratio for North Scotland is 1·57, for South Scotland 1 49, and for England 1 47.

These figures prove that the ratio of the mean annual decrease, in the intervals 1837–57 and 1857–87 has been nearly constant all over Great Britain, but that the rate of change has been rather more rapid in the north.

If we assume that the ratio has been 1 5 for England and Ireland, we may deduce x, the mean annual decrease between 1857–87, from the corresponding quantity given in Table II , p 82, for the interval 1837–87, by the formula

$$1{\cdot}5 \times 20x + 30x = 50d,$$
$$x = 5d/6.$$

The same quantity is determined for stations in Scotland by direct experiment by the comparison of our own observations with those of Mr Welsh The results are given in the following Table

MEAN Annual Decrease of Inclination in Scotland between 1857 and 1884–8.

Station	Mean annual decrease	Station	Mean annual decrease
Lerwick	0′·96	Fort Augustus	1′·21
Kirkwall	1·07	Aberdeen	1·33
Stromness	1·27	Pitlochrie	1·40
Thurso	1·15	Oban	1·27
Wick	1·11	Port Askaig	1·38
Loch Inver	1·32	Lochgoilhead	1·12
Stornoway	0·90	Edinburgh	1·14
Callernish	0·99	Glasgow	1·42
Golspie	1·38	Berwick	1·33
Banff	1·33	Cumbrae	0·94
Elgin	1·30	Campbelton	1·39
Inverness	1·37	Ayr	1·56
Kyle Akin	1·12	Dumfries	1·45
Dalwhinnie	1·41	Stranraer	1·54

Using these figures and the numbers calculated as above described from Table II., p. 82, we get the mean annual decrease between 1857 and 1886 in different parts of the United Kingdom, as follows:—

District	Mean annual decrease between 1857–8 and 1884–7.
N. of Scotland	1′·20
S. of Scotland	1·36
N. of England and N. Wales	1·57
Midlands and Wales	1·71
S. of England	1·84
N. of Ireland	1·68
S. of Ireland	1·90

Although we give these numbers as showing the mean results for districts of considerable size, the figures obtained at the individual stations appear worthy of careful study.

The observed values of the annual decrease for 1857–87 for Scotland, and those calculated for the same period for England and Ireland from the 1837–87 surveys are inserted in the accompanying map (fig. 1).

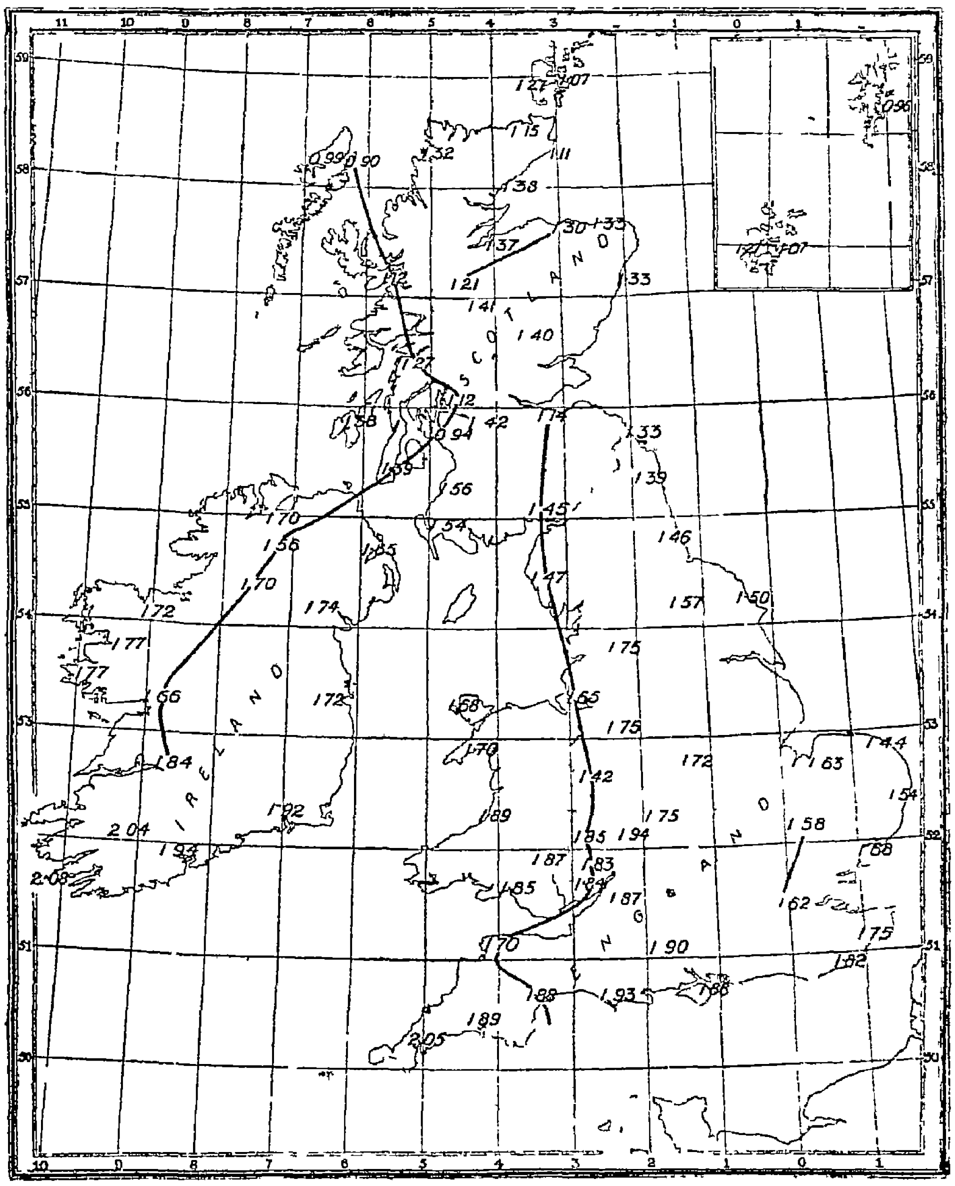
The general increase from east to west is evident, but in Great Britain there are some stations the annual decrease at which is less than at those which are nearest to them on the eastern side.

To test whether the differences were accidental a number of supplementary stations were taken on the Wye.

At Malvern and Brecon the calculated secular rates between 1857 and 1886 are $-1'{\cdot}94$ and $-1'{\cdot}87$, and at Clifton and Cardiff $-1'{\cdot}87$ and $-1'{\cdot}85$. In each case

the smaller value occurs at the western station. Between them lie Hereford, Ross, and Chepstow, and the rates obtained from observations at these in April, 1889, are $-1' 85$, $-1' 83$, and $-1' 84$, which are in all cases less than those at Malvern and Clifton. These observations then support the view that there is a real maximum followed by a minimum rate of change of Dip in this neighbourhood.

Fig 1



Secular change of Dip, 1858–86

This induces us to show in fig. 1 that if all stations at which there is a maximum rate of change are joined, we get two long lines running from Edinburgh to Exeter, and from Stornoway to Limerick The stations are not sufficiently numerous to enable us to draw any certain conclusions about them, but the point ought not to be lost sight of when the survey is repeated.

For the present we note only that of the three pairs of neighbouring abnormal stations at which observations were made in all three surveys, all three are abnormal (in the sense that the rate of change is greater at the easterly station) whether we take the interval 1837–87 or 1857–87. In both cases the rate at Inverness is greater than at Fort Augustus, at Glasgow than at Cumbrae, and at Berwick than at Edinburgh.

The distribution of these stations affords, at all events, *prima facie* evidence that the differences between them are not accidental That they cannot be due to errors of measurement is evident when we remember that an error of 0′ 1 in the annual rate of change determined from experiments separated by an interval of fifty years would require an error of 5′ in one or other of the observations, or of 2′ 5 in both We therefore decided to regard the figures in fig. 1 as giving the true rate of secular change of Dip between 1857 and 1887 at the stations indicated

The determination of the mean rate of decrease of Dip between the years 1857 and 1887 does not, however, give the value of that quantity during the period of our survey As the rate is diminishing it would be less in recent than in earlier years

The evidence we are able to collect on this head is confined to data from Greenwich, Kew, and Stonyhurst, and to a few stations at which we observed both near the beginning and near the end of our survey In the following Table we have entered such facts as were known to us at the time when the progress of the reductions made it necessary to come to a decision on this point

Our own observations and those taken at Stonyhurst are corrected for diurnal variation and disturbance. The Table illustrates the importance of these corrections when conclusions are to be drawn from observations between which the interval of time is small.

TABLE III.

Station	Date	Dip	Date	Dip	Interval in years	Secular rate of change	Assumed rate
Loch Aylort	Sept, 1884	71° 24′ 0	Aug, 1888	71° 22′ 3	4	− 0′ 4	− 1′ 0
Kerrera	Aug, 1884	70 51 1	Aug, 1885	70 48 7	1	− 2·4	− 1·2
Stranraer	Sept, 1884	70 14 1	Aug, 1888	70 11 1	4	− 0 8	− 1 4
Stonyhurst	1884	69 16 2	1887	69 9 0	3	− 2 4	− 1 6
Kew	1884	67 39 5	1888	67 36 4	4	− 0 8	− 1 5
Greenwich	1884	67 30 2	1886	67 27 0	2	− 1 6	− 1 5
Leeds .	Sept, 1886	69 9 7	Dec, 1888	69 6 9	2 2	− 1 3	− 1 5
					Mean .	− 1 4	− 1 4

NOTE.—Stornoway is omitted because the observations were made in 1884 in the Castle grounds and in 1888 on Ard Point

These figures illustrate the fact that there may be a considerable discrepancy at any particular station between the actual secular change at a given time and that deduced from the mean taken over a long series of years. On the whole, we think it probable that the present rate of change of Dip is somewhat less than its mean value between 1857–87, and we have, therefore, reduced the numbers for that period by one tenth, neglecting fractions of a tenth of a minute. This *assumed rate* gives a correct mean rate for the stations referred to above in Table III, though it is considerably too high for some and too low for others. It is probable that in the case of Kerrera the observed rate of change is not quite correct The site is subject to local

attraction, and a single pair of observations made at so short an interval as a year is hardly sufficient to rely on The results from Stonyhurst, however, prove that the assumed and actual rates of change may differ by 0′ 8, and that, thus, Dips reduced to January 1, 1886, from the end of 1888, may be about 2′ 5 wrong To prevent, as far as possible, any effect on the general run of the isoclinal lines from this uncertainty, we put down on a map all the assumed rates at places for which they could be determined, and took the rate at any other place to be that given by the nearest of these stations

The rates of decrease thus obtained for the south of England are considerably less than those assumed by M. MOUREAUX for the north of France, which do not differ much from — 2′ 7. These values are obtained for the interval 1858–1885 by comparing M MOUREAUX's observations with LAMONT's map We think, however, that the observations of the Rev S J PERRY, F R S reduced to the epoch, January 1, 1869, prove that in France as in England the rate of change of Dip has been diminishing

Twelve stations were common to Father PERRY and M MOUREAUX, at one of these, Marseilles, the station of the earlier observer was no doubt seriously affected by local disturbance The rate of change for the epoch 1858–69 is much larger than for any other station, and for 1869–85 it is much too small Rejecting this observation we think the following table shows that M MOUREAUX's numbers are too large

SECULAR Change of Dip in France

Station	Jan 1, 1869	Jan 1, 1885	Secular variation	
			1869–85	1859–85
	° ′	° ′	′	′
Amiens	66 40 3	66 8 0	— 2 0	— 2 7
Bordeaux	63 23 0	62 41 8	— 2 6	— 3 1
Clermont	63 36 4	62 52 1	— 2 8	— 2 8
Dijon	64 24 5	63 53 4	— 1 9	— 2 7
Grenoble	62 54 2	62 6 9	— 2 9	— 2 6
Monaco	61 22 1	60 41 2	— 2 5	— 2 8
Moulins	64 4 9	63 30 1	— 2 2	— 2 7
Paris	65 52 5	65 17 3	— 2 2	— 2 8
Périgueux	63 23 9	62 44 9	— 2 4	— 3 1
Poitiers	64 28 1	63 55 6	— 2 0	— 2 8
Toulouse	62 1 1	61 23 9	— 2 3	— 3 2
Mean			— 2 35	— 2 85

We turn next to the secular change of the Declination

A number of stations are common to our own survey and those of 1837 and 1857, and we have also observed at several places where the Declination was determined in 1872. The mean annual decrease calculated for the observations is exhibited in the following table —

TABLE IV.—Table of Mean Annual Decrease of Declination.

Station	1837 to 1886	1857 to 1886	1872 to 1886
Lerwick	8·4	10·5	8·2
Kirkwall	7·9	10·5	
Thurso		10·6	9·9
Loch Eriboll		10·1	9·4
Wick	8·1	10·5	9·3
Stornoway		8·2	
Callernish		9·8	
Loch Inver		11·8	
Golspie	8·1	10·4	
Elgin		8·8	
Banff		8·9	
Loch Maddy		9·8	
Kyle Akin		8·8	8·9
Inverness	7·9	9·0	
Aberdeen	7·7	8·9	8·9
Loch Boisdale			12·0
Fort Augustus		9·1	
Dalwhinnie		8·8	
Pitlochrie		8·9	
Oban		8·1	12·9
Lochgoilhead			
Edinburgh		9·1	8·4
Glasgow		9·0	
Cumbrae			
Ayr		8·7	
Campbelton		9·0	
Dumfries		8·4	
Stranraer		8·5	
Liverpool		8·2	
Cromer	7·3		7·0
King's Lynn			6·6
Lowestoft	7·1		
Harwich	7·3		7·3
Kew		7·8	
Greenwich	7·3	7·6	7·3
Milford			7·7
Plymouth		8·3	
Falmouth		8·4	
Jersey (St. Helier's)		7·6	
Londonderry	7·7		
Westport	7·6		
Dublin	7·5		
Limerick	6·9		
Wexford			9·1
Waterford	6·7		
Killarney	6·6		
Valentia	6·9		8·4
Cork	6·2		

These results indicate—(1) that the secular change is greater in the northern than in the southern parts of the United Kingdom, (2) that it was much larger between the years 1857–86 than during the interval 1837–57, and, lastly, (3) that, in the north at all events, the rate of annual decrease is again diminishing

The first of these conclusions differs from that arrived at by Sir E SABINE when collating the 1837 and 1857 surveys He assumed ('Phil Trans,' vol 160, 1870, p 268) "an annual decrease of West Declination of approximately 5′ 6 in Scotland and the north of England, increasing to 6′·2 in the middle and southern parts of England and to 6′ in Ireland."

Our conclusion that, at the present time, the rate is greater in the north is borne out by the values deduced by M MOUREAUX from his own observations and the maps of LAMONT (*loc cit*, p 162) The values he gives for stations on the Mediterranean and on the English Channel are about −6′ 5 and −7′ 7 respectively, which indicates an increase of about 0′ 2 on the rate for each degree of latitude as we go north

The truth of the second conclusion is rendered more apparent by calculating the annual rate for the epoch 1837–57 instead of 1857–86, as shown in the following Table for the Scotch stations, for which the requisite data are available —

MEAN Annual Decrease of Declination

Station	1837 to 1857	1857 to 1886	Station	1837 to 1857	1857 to 1886
Lerwick	− 5′ 5	− 10′ 5	Inverness	− 6′ 3	− 9′ 0
Kirkwall	− 4 5	− 10 5	Aberdeen	− 6 8	− 8 9
Wick	− 4 8	− 10 5	Loch Inver	− 2 9 (?)	− 11 8
Golspie	− 5 0	− 10 4			

The third conclusion is supported by a comparison of M MOUREAUX's rates for the epoch 1859–1885 with those obtained by collating as before his results and those of the Rev S J. PERRY (*loc cit*)

Station	Declination		Secular rate of change	
	Jan. 1, 1869.	Jan. 1, 1885.	1869–85.	1859–85.
Amiens	18° 19′·0	16° 34′·7	−6′·5	−7′·4
Bordeaux	18 12·5	16 45·7	−5·4	−7·0
Clermont	16 27·6	15 25·0	−3·9	−6·9
Dijon	16 36·7	14 45·2	−7·0	−7·2
Grenoble	15 49·3	14 11·0	−6·1	−6·8
Marseilles	15 41·5	14 0·0	−6·3	−6·7
Monaco	14 31·4	13 10·5	−5·0	−6·5
Moulins	16 29·2	15 25·6	−4·0	−6·9
Paris	17 50·5	16 10·2	−6·3	−7·4
Périgueux	17 40·9	16 8·6	−5·8	−7·0
Poitiers	18 18·4	16 40·8	−6·1	−7·1
Toulouse	17 7·3	15 41·4	−5·4	−6·8
Mean		..	−5·6	−7·0

This result—that the rate of decrease is diminishing—is in accord with the conclusions drawn by M. MOUREAUX himself from a comparison of his own observations with those on which a magnetic map of France for the epoch 1875 was based by M. MARIÉ-DAVY (*loc. cit.*, p. 166). He did not, however, make use of this fact, as the number of observations employed by MARIÉ-DAVY was small, and they were taken at stations irregularly distributed over France. He does not cite the observations of Father PERRY, which, as the above Table shows, strongly support the same view.

In the next Table we give such observations as were made at English observatories or by ourselves during the progress of the survey which bear on the question under discussion. No very definite conclusions can be drawn from them.

Two single observations at the interval of a year, as at Kerrera and Glasgow, are hardly sufficient for the purpose of deducing a secular rate. Loch Aylort and Stornoway are both disturbed stations and the results are thus less trustworthy. The general evidence that the secular change is greater in higher latitudes is opposed by the fact that Stonyhurst gives a less value than Kew and Greenwich. If the increase with latitude which obtains in France were maintained in England, the value at Paris being taken at −6′·3, we could deduce −6′·8 for London; −7′·7 for Edinburgh, and −8′·2 for Wick, of which the two latter are less than the values for the epoch 1872–86 given in Table IV., p. 88.

Station	Date	Declination	Date	Declination	Interval in years	Secular rate of change
		° ′		° ′		′
Stornoway (Aird Point)	Sept, 1884	24 20 6	Aug, 1885	24 12 4	0 96	− 8 5
			Aug, 1888	23 49 9	4	− 7 7
Loch Ayloit	Sept, 1884	23 40 2	Aug, 1888	22 41 8	4	− 14 6
Kerrera	Aug, 1884	22 25 4	Aug, 1885	22 15 2	1	− 10 2
Glasgow	Aug, 1884	21 21 3	Aug, 1885	21 18 2	1	− 3 1
Stranraer	Sept, 1884	21 46 2	Aug, 1888	21 13 1	4	− 8 2
Stonyhurst	1884	19 52 8	1887	19 35 2	3	− 5 9
Reading	April 30, 1886	18 13 1	May 30, 1888	17 53 9	2	− 9 6
Kew	1884	18 33 9	1888	18 9 3	4	− 6 1
Greenwich	1884	18 7 6	1888	17 40 0* (approx)	4	− 6 9

From Table IV we see that the rates at Greenwich for the three intervals, 1837–86, 1857–86, and 1872–86 are nearly the same, the differences between them and the mean rate during the period of our survey (6′·9) being 0′ 4, 0′ 7, and 0′ 4 respectively Assuming that if we subtract these numbers from the rates calculated for the same epochs for the other English stations in Table IV we obtain the present rates at the stations, we find as a mean rate 7′ 0, which agrees closely with that of Greenwich

On the whole we decided to take a secular rate of — 7′·0 in England south of a line joining Redcar and Barrow, and also in Wales, and in Ireland south of a line joining Dublin and Donegal

For the remainder of England and Ireland we have assumed the rate to be — 8′ 0 per annum. At the time when most of the Scotch observations were reduced we had not M. Moureaux's results before us, and we hardly felt justified in departing from the rates given by the comparison of our observations with Mr Welsh's, on the evidence of the small number of stations at which the Declination was determined in 1872 We, therefore, employed rates varying from — 10′ 4 in the north of Scotland to — 8′·8 in the south, the change on the east coast being somewhat greater than on the west These may be a little too large but the error certainly does not exceed the variation which occurs between neighbouring stations. For the later Scotch observations we used a rate of — 9′ 0.

The data for the determination of the secular change of the Horizontal Force are more scanty than in the case of the other elements.

At the time of the 1837 survey the measures were comparative only, and though Sir E Sabine afterwards reduced them to absolute values by means of the known absolute value and secular rate at Kew, we do not think any useful end would be attained by discussing them

M Moureaux found that for the interval 1848–85, the secular variation in France attains its maximum + 0 0027 (M.U.) at Bordeaux and decreases slowly towards the

* From information kindly supplied by the Astronomer-Royal.

east, being 0025 at Paris and + 0 0023 at Nice and Mezières (*loc cit*, p. 167) The mean value obtained by the Rev. S J PERRY in 1869 by comparison with LAMONT was 0 0023 ('Phil Trans,' Vol 160, 1870, p 48), which was increasing by about 0 000008 per annum and would correspond to 0 0025 at the present time

By comparison of our own observations with those of WELSH we obtain 0 0018 as a mean for Scotland between 1857 and 1886

The following Tables give these values and those which were obtained at the English observatories or by ourselves during our survey.

Secular Change in Horizontal Force

Station	WELSH (Uncorrected for v and Δ)	R and T (Uncorrected for v and Δ)	Difference	Interval	Secular rate of change
				Years	
Dumfries	1·6006	1 6522	0516	27	·0019
Stranraer	1 5853	1 6433	0580	27	0021
Ayr	1 5756	1 6317	0561	27¼	0020
Oban	1 5451	1 6083	0632	31	0020
Fort Augustus	1 5134	1 5643	0509	28	0018
Inverness	1 5093	1·5653	0560	28	·0020
Aberdeen	1 5166	1 5735	0569	27½	·0021
Pitlochrie	1 5341	1 5926	·0585	27¼	0021
Dalwhinnie	1 5377	1 5926	·0549	27¼	·0020
Edinburgh	1 5665	1 6162	·0497	27½	0018
	1 5697		·0465	26¼	·0017
Kyle Akin	1 5009	1 5407	0398	..	0015
Stornoway	1 4773, 1 4783, 1 4794 } 1 4783	1 5122	0339	$26\frac{1}{12}$	·0013
Callernish	1 4731	1 5236	0505	27	0019
Loch Inver	1 4499	1 4956	0457	$26\frac{1}{12}$	0018
Thurso	1 4763	1 5209	0446	26¾	0017
Lerwick	1 4313	1 4718	0405	27	·0015
Kirkwall	1 4669, 1 4725, 1 4753 } 1 4716	1·5111	·0395	27	0015
Wick	1 4706	1·5117	0411	26¾	·0015
Golspie	1 4893	1·5390	0497	26¾	0019
Mean	.	..	.. ..	..	0018

Station.	Date.	H.	Date.	H.	Interval in years.	Secular rate of change
Stornoway	Aug. 19, 1885	1·5200	Aug 14, 1888	1 5220	3	+ ·0007
Loch Aylort	Sept 12, 1884	1·5577	Aug. 2, 1888	1·5774	4	+ 0049
Stranraer	Sept. 18, 1884	1·6423	Aug. 28, 1888	1 6470	4	+ ·0012
Stonyhurst	1884	1 6954	1887	1·7024	3	+ 0023
Kew	1884	1 8056	1887	1·8091	3	+ ·0012
Greenwich	1883	1·8023	1887	1 8091	4	+ ·0017
	1883	1 8100	1885	1 8156	2	+ ·0028
	1883	1 8100	1886	1·8157	3	+ ·0019

The results at Kew and Greenwich show that the secular variation varies so much from year to year that it is practically impossible to draw any conclusions from so short a period as 2 or 3 years, and our own observations on Scotland show that if to this difficulty, that due to a slight variation in the position in a disturbed district is added, the results are still less trustworthy At the time when the reductions were made the Greenwich results for 1886 and the Stonyhurst results for 1887 had not been published We therefore took the mean of the values for Stonyhurst 1883–1886, Kew 1883–87, and Greenwich 1883–1885, or + 0 0022 the annual increase for England This value is in fair accord with M MOUREAUX's results In Scotland we took the number given by the comparison of our own and WELSH's observations, viz, + 0 0018, and for Ireland + 0 0020

Results of the Observations

Having thus described the observations and the methods of correction and reduction to epoch, we now proceed to give a more detailed account of the results at each station In doing this we have attempted to distinguish between facts which it is necessary to give for the information of most of those who may read our paper, and details which ought to be preserved, but which will nevertheless only be of interest to observers who may for any cause wish to undertake a detailed examination of our results We have therefore decided to publish in this paper only a description of each station, the hours at which the observations were taken, the results corrected for diurnal variation and disturbance, and the mean reduced to epoch. As Magnetic Observatories and provincial Colleges are now rapidly multiplying, it is not too much to hope that special studies of the districts in their immediate neighbourhood may from time to time be made by those connected with such institutions It is, therefore, we think, important that the descriptions of the stations should be readily accessible. In adding to these only the times at which the observations were taken, and the actual and reduced results, we are publishing far less than has been usual Thus, in the case of the Rev S J PERRY's survey of the east of France, he gave for the Force observations the date, hour, and temperature for both vibrations and deflections, the time of one vibration, log mX, the distances of the magnets in the deflection experiment, the observed deflections, and log m/X. M MOUREAUX has not given quite so many details of the Force observations, but in the case of the Declination he gives the individual readings for the determination of the geographical meridian, &c, so that the description of his work at each station occupies about two quarto pages. For our own part we have no fault to find with the publication of these details; on the contrary, we have found them to be useful, but the large number of stations included in our survey would, we fear, make this paper inordinately long if we adopted a similar plan We therefore purpose to place in the hands of the Royal Society bound copies of the details of the observations and calculations, and also of

the forms described on pp 71–74, in which the results are analysed and the corrections for diurnal and secular variation and disturbance are applied It will thus be possible for those who may desire to do so to inspect these, and the data used in the preparation of the Tables on pp. 77–80 will be on record, while this paper will, we hope, contain sufficient to enable future observers who are not specially interested in the details of the calculations and reductions to find the positions where we observed, to know when we observed there, and to judge from the final results of the general accuracy of the observations We have followed the plan adopted by M MOUREAUX of giving all the facts with respect to each station together, which we think the most convenient. Tables are also given on pp 251–258 in which the final results are entered in tabular form for comparison with the values obtained by calculation from formulæ to be hereafter discussed.

The stations are arranged in the following order —Three groups are formed, comprising Scotland, England and Wales, and Ireland, this being the chronological order of the bulk of the observations in each of these countries In each group the stations are arranged in alphabetical order, and they are numbered continuously throughout. These numbers are affixed to the positions of the stations as given on Plate I., which serves as an Index map

The Scotch stations, from Aberdeen to Wick, are numbered from 1 to 54, and it should be mentioned that the name of a Loch is regarded as determining the initial letter. Thus East Loch Tarbert is found under T

The English and Welsh stations, including the Isle of Man and the Channel Isles, from Aberystwith to Worthing, are numbered from 55 to 156

The Irish stations, from Armagh to Wicklow, are numbered from 157 to 200

Thus anyone desirous of looking up the observations at a particular place, can easily do so from a knowledge of its name, while the stations in any particular district can be found in Plate I., and then referred to by means of the corresponding numbers.

The data given in each case, are as follows ·—

(1) The number and name of the station

(2.) Date of the observations.

(3) Initials of the observer and numbers of the instruments.

(4.) Latitude and longitude of the station

(5) Verbal description of the station.

For the Declination we give —

(1.) The time from the southing of the sun (Σ), at which the geographical meridian was determined by sun observations, a positive sign indicating the afternoon

(2.) The G M T. of the determination of the magnetic meridian.

(3) The observed Declinations with all corrections applied (δ).

(4.) The mean observed Declination reduced to the epoch, January 1, 1886 (δ_0)

For the Inclination we give —

(1) The number of the needle
(2) The G M T at which the observation was made
(3) The observed Dip, with all corrections applied (θ)
(4) The mean observed Dip reduced to epoch (θ_0)

For the Horizontal Force we give —

(1) The G M.T at which the deflection (D) and vibration (V) were observed
(2) The corrected independent forces found as described on p 73 (H), but corrected for diurnal variation and disturbance
(3) The mean observed Horizontal Force reduced to epoch

Longitudes are to be taken as west of Greenwich, unless the contrary is expressly stated

Descriptions of Scotch Stations

1. Aberdeen April 6 and 7, 1885, A W R and T E T (60, 74) Lat 57° 9′ 50″; Long 2° 6′ 5″ In a field behind Professor Milligan's house, immediately opposite to King's College Tower of King's College E S E, hermitage in Miss Leslie's park W by S, and the centre of the gate of Miss Leslie's lodge S.E by S

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
April 6		14 49	20 24 3	20 16 3
„ 7	−2 42	9 53	20 21 6	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
April 6	1	14 46	71 12 4	71 12 3
	2	15 28	71 14 1	

Horizontal Force

Date	G M T		H	H_0
		h m		
April 6	D	15 56	1 5724	1 5734
	V	14 10	1 5719	

2. Arinagower (Coll). August 11, 1885; A. W. R. (60). Lat. 56° 37′ 5″, Long. 6° 31′ 12″. Near the landing-place on the west side of the bay This observation was taken during an unexpected detention of the steamer The declination was the only element determined.

Declination.

Σ	G M T	δ	δ_0
h m. −0 52	h m 12 1	° ′ 23 44 2	° ′ 23 40 4

3. Loch Aylort (Gobbar Island). September 12, 1884, A W R and T. E T (60, 74). August 2, 1888, T. E T. (61, 83) Lat. 56° 51′ 5″, Long. 5° 47′ 0″. On the east side of the island.

Declination.

Date	Σ	G M T.	δ	δ_0
	h m	h m	° ′	° ′
Sept 12, 1884	+ 2 48	15 42	23 40 2	23 27 2
Aug 2, 1888	+ 1 15	{ 12 16 13 24	22 40 5 22 43 1	23 5 9

Inclination

Date	Needle	G M T	θ.	θ_0
		h m	° ′	° ′
Sept 12, 1884	1 2	14 44 15 46	71 24 8 71 23 2	71 22 6
Aug 2, 1888	1 2	12 38 13 27	71 22 0 71 22 7	71 25·4

Horizontal Force

Date	G M T		H	H_0.
		h m		
Sept 12, 1884	D	16 39 17 14	1 5571 1 5583	1 5600
	V	15 9	1 5576 1 5578	1 5600
Aug 2, 1888	D V	14 28 13 0	1 5772 1 5776	1 5727

4 AYR May 25, 1885; A W. R (60, 74). Lat 55° 27′ 30″, Long 4° 37′ 10″ On the low green about 150 yards from the Castle. Identical with Mr. WELSH's station.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−3 55 −1 29	8 55	21 23 0 21 23 4	21 17 9

Inclination

Needle	G M T	θ	θ_0
	h m.	° ′	° ′
1 2	12 32 13 15	70 21 6 70 23 1	70 21 4

Horizontal Force

G M T		H	H_0
	h m		
D	11 21	1 6337	1 6345
V	9 25	1 6336	
D	..	1 6331	
V	10 28	1 6334	

5. BALLATER April 7 and 8, 1885, A W. R. and T E T. (60, 74) Lat 57° 2′ 53″, Long. 3° 2′ 6″. Fifty yards from the Invercauld Arms Hotel; near the river

Declination.

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
April 7 ,, 8	+ 3 30 − 2 51	16 26 .	20 36 9 20 36 1	20 29 5

Inclination

Date	Needle	G M T	θ	θ_0
April 7	1 2	h m 15 39 16 22	° ′ 71 14 9 71 17 8	° ′ 71 15 4

Horizontal Force.

Date	G M T		H	H_0
April 7	D V	h m 15 11 17 2	1 5703 1 5699	1 5714

6. BANAVIE. August 4, 1885 A. W R and A P L (60, 74) Lat. 56° 51′ 0″, Long 5° 5′ 40″. Field close to and on the North side of the Hotel.

Declination.

Σ	G.M T	δ	δ_0
h m + 4 16	h. m 17 39	° ′ 22 10 6	° ′ 22 6 7

Inclination

Needle	G M T	θ	θ_0
1 2	h m 17 11 18 5	° ′ 71 11 7 71 12 1	° ′ 71 11 4

Horizontal Force.

G M T		H	H_0
D V	h m 18 23 17 10	1 5928 1 5938	1 5940

7 BANFF July 9, 1885; A W R (60, 74) Lat 57° 39′ 57″; Long 2° 31′ 17″. In the grounds of the old Castle; on the lawn in front of the house formerly occupied by Dr BREMNER, now by Sheriff SCOTT-MONCRIEFF Same station as that at which Mr WELSH observed

Declination

Σ	G M T	δ	δ_0
h m −2 40	h m 10 28	° ′ 21 8 8	° ′ 21 4 5

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 12 58 13 39	° ′ 71 19 7 71 19 6	° ′ 71 19 0

Horizontal Force.

G M T		H	H_0
D V	h m 11 48 11 7	1 5669 1 5681	1 5684

8 BERWICK. April 2, 1885; A. W R. and T. E T. (60, 74). Lat. 55° 46′ 4″, Long 1° 59′ 52″. On a bastion Powder-magazine distant about 150 yards N. Church, N W, and works on sandspit at mouth of river S.

Declination.

Σ	G M T	δ	δ_0
h m −3 5	h m 9 50	° ′ 19 42 9	° ′ 19 36 4

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	10 57	70 15 8	70 15 9
2	11 34	70 17 8	

Horizontal Force.

G M T		H.	H_0.
	h m		
D	11 35	1 6474	1 6483
V	10 28	1 6467	

9 BOAT OF GARTEN July 31, 1885; A W. R and A. P. L. (60, 74) Lat. 57° 15′ 0″; Long. 3° 45′ 13″. On the Green to the North of the Station, about 200 yards from the Hotel.

Declination.

Σ	G M T	δ.	δ_0
h m	h m	° ′	° ′
+ 3 37	15 31	22 11 7	22 7 7

Inclination.

Needle	G M T	θ	θ_0.
	h. m.	° ′	° ′
1	15 51	71 15 0	71 16 3
2	18 0	71 19 3	
1	18 28	71 16 0	

Horizontal Force.

G M.T.		H.	H_0.
	h m		
D	16 43	1·5776	1 5786
V	15 6	1 5781	

10 LOCH BOISDALE (S Uist) August 31, 1884, A W R and T E T. (60, 74) Lat 57° 8′ 55″; Long 7° 18′ 0″ In the Bay on the Island of Kisgay.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 51	11 19	23 7 1	22 53 3
+4 56	16 45	23 7 1	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	11 29	71 41 0	71 39 3
2	12 30	71 39 9	

Horizontal Force

G M T		H.	H_0
	h m		
D	12 49	1 5284	1 5310
V	12 2	1 5288	

11. BUNNAHABHAIN (Islay) August 25, 1888, A. W. R. and T. E. T. (60, 61, 83). Lat 55° 53′ 0″; Long 6° 8′ 0″. At the bottom of the road, 150 yards N. of the Distillery and about 10 yards from the beach.

Declination.

Σ.	G M T.	δ	δ_0
h m	h m	° ′	° ′
−2 32	10 55	22 44 0	
−1 12	11 36	22 46 0	
−2 29	10 29	22 46 0	23 10 3
	12 25	22 47 3	
−1 16	14 30	22 47 9	
	15 9	22 47·3	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 46	70 39 4	70 43 0
2	14 38	70 39 8	

Horizontal Force.

G M T		H	H_0
	h m		
D	15 39	1 6292	1 6243
V	14 55	1 6291	

12. Callernish (Lewis) August 20, 1885, T. E. T. (60, 74) Lat. 58° 11′ 0″; Long. 6° 42′ 0″. Fifty yards S.S.W. (magnetic) from front of Garynahine inn.

Declination.

Σ	G M T.	δ	δ_0
h m.	h m	° ′	° ′
+ 5 2	18 3	23 44 1	23 40 6

Inclination.

Needle	G M T.	θ	θ_0
	h m	° ′	° ′
1	7 26	72 7 5	72 7 1
2	8 1	72 7 4	

Horizontal Force.

G M.T.		H	H_0
	h m		
V	18 21	1 5224	1 5231

13. CAMPBELTON. August 22, 1884, A W. R and T E T (60, 74) Lat 55° 25′ 30″, Long 5° 36′ 5″ 200 yards N E from Lime Crags, 400 yards E S.E. from old Parish Church. Same station as that at which Mr WELSH observed.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 20		22 21 1	
−1 13	14 36	22 21 0	22 8 1
+1 43		22 19 8	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 56	70 35 2	70 34 2
2	13 27	70 36 9	

Horizontal Force.

G M T		H	H_0
	h m		
D	12 58	1 6236	1 6244
V	11 49	1 6204	

14 CANNA August 30, 1884, A W R and T E. T (60, 74). Lat 57° 3′ 30″ Long 6° 29′ 20″ On the high land N E of the Harbour, and S W of Compass Hill

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−4 35	8 12	21 21 8	21 8 4

Inclination

Needle	G M T	θ	θ_0
2	h m 8 38	° ′ 72 46 3	° ′ 72 45 0

Horizontal Force

G M T		H	H_0
D V	h m 9 10 8 30	1 5065 1 5072	1 5092

15. CARSTAIRS May 25 and 26, 1885, A. W. R. (60, 74). Lat 55° 41′ 10″; Long 3° 40′ 11″. In Mr. MONTEITH's grounds, about 400 yards from the station, and about 50 yards from the road leading from the station to the house. Nearly the same position as that at which Mr. WELSH observed.

Declination.

Date	Σ	G M T	δ	δ_0
May 25	h m + 5 55	h m 18 35	° ′ 20 57 6	° ′ 20 52 2

Inclination.

Date	Needle	G M T	θ	θ_0
May 26	1 2	h m 9 49 10 23	° ′ 70 16 4 70 16 4	° ′ 70 15 7

Horizontal Force

Date	G M T		H	H_0
May 25	D V	h m 19 42 19 5	1 6434 1 6440	1 6448

16. CRIANLARICH September 17, 1884, A W R (60, 74). Lat. 56° 23′ 25″, Long 4° 37′ 6″. Near the inn, about 80 yards from the roads to Dalmally and Loch Lomond

Declination.

Σ	G M T	δ	δ_0
h m − 1 47 + 2 49	h m 11 10	° ′ 22 2 2 22 3 8	° ′ 21 50 6

Inclination

Needle	G M T	θ	θ_0
1 2	h m 8 31 9 8	° ′ 70 54 5 70 53 1	° ′ 70 52 5

Horizontal Force

G M T		H	H_0
D V	h m 12 52 11 32	1 5943 1 5971	1 5980

17. CRIEFF. July 28, 1885; A. W. R. and A P. L (60, 74). Lat. 56° 22′ 27″; Long 3° 50′ 22″ In the grounds of MORRISON's Academy, about half-way between the school and the gate.

Declination

Σ	G M T	δ	δ_0
h m + 1 46	h m 14 48	° ′ 21 37 6	° 21 33 6

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 36	70 53 4	70 53 6
2	14 49	70 54 8	

Horizontal Force

G M T		H	H_0
	h m		
D	15 55	1 6071	1 6079
V	15 19	1 6072	

18. CUMBRAE July 24, 1888; T. E. T. (61, 83). Lat 55° 47′ 45″, Long. 4° 53′ 40″. Eight yards to the N. of the Monument to the "Shearwater's" midshipmen, at the N.E. end of the Island. Mr WELSH's Station

Declination.

Σ	G M T.	δ	δ_0.
h m	h m	° ′	° ′
+ 4 19	17 0	21 13 2	21 37 2

Inclination.

Needle	G M T	θ	θ_0.
	h m	° ′	° ′
1	17 41	71 1 0	71 2 3
2	18 0	70 59 7	

Horizontal Force.

G M T.		H	H_0.
	h. m		
V	17 16	1 5957	1 5911

19. DALWHINNIE July 30, 1885. A W R and A. P L (60, 74). Lat. 56° 55′ 52″, Long 4° 14′ 12″. About 150 yards from the Hotel, on the opposite side of the river

Declination.

Σ	G M T	δ	δ_0
h m + 2 54	h m 16 10	° ′ 21 49 2	° ′ 21 45 5

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 9	70 59 7	71 0 1
2	17 35	71 1 6	

Horizontal Force

G M T		H	H_0
	h m		
D	14 42	1 5901	1 5909
V	15 49	1 5902	

20. DUMFRIES September 20, 1884 and July 11, 1885 A W R (60, 74) Lat 55° 2′ 10″, Long 3° 35′ 30″ On Mr. STOTT's farm at Lower Netherwood In a field about 100 yards W of the farm-house, 39 and 50 paces from the N and E walls of the field respectively Nearly the same station as that of Mr. WELSH.

Declination.

Date	Σ	G M T	δ	δ_0.
July 11	h m + 4 51	h m 16 33	° ′ 20 51 5	° ′ 20 47 4

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Sept 20	1 2	12 6 12 46	70 4 2 70 4 5	70 2 6

Horizontal Force.

Date	G M T		H.	H_0
		h m		
Sept 20	D V	13 38 14 19	1 6524 1 6514	1·6542

21. DUNDEE. April 9, 1885, A. W. R. and T E T. (60, 74). Lat. 56° 28′ 17″; Long 2° 56′ 58″ In the Baxter Park. The tower of Morgan's Hospital N by W., tower of lodge S.W. (100 yards); Park Pavilion N.N.E. (100 yards), Ogilvie Church N.W.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 33	12 45	20 51 2	20 44 5

Inclination

Needle	G M T	θ	θ_0.
	h m	° ′	° ′
1 2	11 56 12 38	70 51 7 70 54 6	70 52 2

Horizontal Force

G.M T		H	H_0
	h. m		
D V	11 31 12 16	1 5990 1 5989	1 6002

22 EDINBURGH April 3, 1885, A W. R. and T E. T. (60, 74). Lat. 55° 57′ 52″ Long. 3° 12′ 28″. In the Arboretum of the Botanic Gardens. Inverleith House N N.W. 100 yards. Cathedral S W. by S. Melville College S.S E Donaldson's Hospital W S.W. The small magnetic house in the gardens in which Mr WELSH probably observed had been removed

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+2 14	13 56	20 53 8	20 47 2

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	11 57	70 37 2	
2	12 36	70 39 5	70 38 5
2	14 21	70 40 8	

Horizontal Force

G M T		H	H_0
	h m		
D	12 22	1 6175	1 6183
V	12 51	1 6165	

23 ELGIN. July 8, 1885, A W. R. (60, 74). Lat. 57° 38′ 40″, Long 3° 19′ 0″ In the grounds of North College, the residence of G. SMITH, Esq., 100 yards N.N.E. of the Cathedral Tower and 80 yards E.S E. of the house.

Declination.

Σ	G M T	δ	δ_0
h m	h m.	° ′	° ′
−1 33	12 29	21 1 8	20 57 5

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 26	71 32 5	71 32 0
2	15 2	71 32 7	

Horizontal Force.

G M T		H	H_0
	h m		
D	13 8	1 5566	1 5577
V	12 11	1 5570	

24. Loch Eriboll (Camas Bay) August 23, 1885; T E. T (60, 74). Lat 58° 29′ 15″, Long. 4° 39′ 20″ Near the stream running into Camas Bay, ¼ mile N.E (magnetic) from the ruin in the bight

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 3 31	14 52	22 21 7	22 18 1
+ 4 34			

Inclination

Needle	G M T	θ	θ_0
	h m.	° ′	° ′
1	11 40	72 9 4	72 9 4
2	12 13	72 10 3	

Horizontal Force

G.M T		H	H_0
	h m		
D	13 7	1 5184	1 5198
V	14 29	1·5201	

25. FAIRLIE. August 14, 1884 A W. R and T E. T. (60, 74) Lat 55° 45′ 30″, Long 4° 51′ 5″. In a field on the high ground to the rear of the village

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 4	70 43 7	70 42 8
2	16 8	70 44 2	

Horizontal Force

G M T		H	H_0
	h m		
D	12 50	1 6150	1 6172
V	13 56	1 6143	

26 FORT AUGUSTUS. August 3, 1885 A. W R. and A P L (60, 74) Lat 57° 8′ 30″, Long 4° 40′ 32″. In a field on the south side of the Abbey, and about 150 yards from the building

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 1 37	14 38	21 49 4	21 45 6

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 47	71 29 4	71 27 7
2	15 51	71 27 1	

Horizontal Force

G M T		H	H_0
	h m		
V	15 47	1 5634	1 5641

27. GAIRLOCH. September 9, 1884; A. W R. and T. E. T (60, 74). Lat. 57° 42′ 40″; Long. 5° 40′ 55.″ On the rising ground behind the pier in Flowerdale Bay.

Declination.

Σ	G M T	δ	
h m + 3 23	h m 16 33	° ′ 22 28 0	22 14 4

Inclination

Needle	G M T	θ	$θ_0$
1 2	h m 16 39 17 36	° ′ 71 45 9 71 45 3	° ′ 71 44·3

Horizontal Force.

G M.T		H	H_0
D V	h m 17 37 17 2	1 5323 1 5336	1 5353

28. GLASGOW. August 13, 1884, and July 27, 1885; A. W. R. (60, 74) Lat. 55° 52′ 43″, Long. 4° 17′ 39″. In a field to the West of the Observatory; 48 paces from the building.

Declination.

Date		G M.T	δ.	$δ_0$.
Aug. 13, 1884 July 27, 1885	h m + 3 47 + 2 43	h m. 16 53 15 54	° ′ 21 21 3 21 18 2	° ′ 21 11·5

Inclination

Date	Needle	G M T	θ	θ_0
Aug 13, 1884	1 2	h m 12 12 12 18	° ′ 70 44 6 70 48 4	° ′ 70 44 7

Horizontal Force.

Date	G M T		H	H_0
Aug 13, 1884	D V	h m 14 17 15 5	1 6038 1 6038	1 6064

29 GOLSPIE. July 4, 1885, A. W. R (60, 74) Lat. 57° 58′ 20″, Long 3° 58′ 15″ The Dips were taken in a field by the road in front of the Sutherland Arms Hotel and about 50 yards to the West of it. The other observations in a field behind the Hotel about 80 yards S W of some cottages and 50 yards N.N.W of the Bank.

Declination

Σ	G M T	δ	δ_0
h m + 2 9 + 2 39	h m 15 33	° ′ 21 34 9 21 35 6	° ′ 21 30 2

Inclination

Needle	G M T	θ	θ_0
1 2	h m 12 20 12 55	° ′ 71 46 8 71 47 8	° ′ 71 46 7

Horizontal Force

G M T		H	H_0
D V	h m 16 37 16 1	1 5372 1 5374	1 5382

30. HAWICK March 31 and April 1, 1885, A. W. R and T E T (60, 74) Lat 55° 25′ 58″, Long 2° 47′ 58″ In the Park of Sillerbit Hall (T LAIDLAW, Esq) to the N of the town The Hall bears 400 yards E N E and the Lodge 120 yards S S.E

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
March 31	.	12 43	20 21 5	20 16 0
April 1	+ 1 50	13 34	20 23 5	

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
March 31	1	11 15	70 7 5	70 7 3
	2	11 54	70 9 0	

Horizontal Force.

Date	G M.T		H	H_0
		h m		
March 31	D	11 14	1 6473	1 6487
	V	12 1	1 6476	

31. LOCH INVER September 6, 1884, A. W R. and T E. T (60, 74) Lat 58° 9′ 30″, Long. 5° 14′ 40″. Near the Coast-guard Station, on the bank of the River Inver

Declination.

Σ	G M T.	δ.	δ_0
h m	h m.	° ′	° ′
+ 2 53	14 2	22 21 8	22 7 4

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 48	72 2 6	72 0 2
2	13 46	72 1 0	

Horizontal Force

G M T		H	H_0
	h m		
D	13 14	1 4968	1 4990
V	13 25	1 4965	

32. INVERNESS August 1, 1885, A W. R. and A P L (60, 74) Lat 57° 28′ 30″, Long 4° 13′ 20″. In the garden of Mr MITCHELL's house, neai the Castle Mr WELSH's station.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+2 32	16 18	21 47 2	21 43 3

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 44	71 30 0	71 31 1
2	16 45	71 33 2	

Horizontal Force

G M T		H.	H_0
	h m		
D	16 56	1 5638	1 5642
V	15 50	1 5631	

33. IONA September 15 and 16, 1884, A W R (60, 74). Lat 56° 20′ 0″; Long. 6° 23′ 40″. In a field behind the Inn, and about 50 yards from it.

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Sept 15	+ 4 6	17 4	23 41 0	23 28 6

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Sept 16	1	9 35	70 57 1	70 55 8
	2	10 10	70 57·5	

Horizontal Force

Date	G.M T		H	H_0
		h m		
Sept 15	V	17 47	1 6162	1 6185

34. KIRKWALL. August 27, 1885; T E T. (60, 74). Lat. 58° 59′ 12″, Long. 2° 57′ 15″. At Battery Point and close to the road Cathedral tower bearing S W. by W Cairn on Wideford Hill, bearing W N.W.

Declination

Σ.	G.M T	δ.	δ_0.
h m.	h. m	° ′	° ′
+ 1 46	12 19	21 32 9	21 29 3
+ 4 31	..	21 32 7	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 31	72 12 4	72 12 8
2	14 59	72 13 8	

Horizontal Force

G M T		H	H_0
	h m		
D	15 55	1 5104	1 5108
V	11 51	1 5100	

35. KYLE AKIN September 11, 1884, A.W.R and T.E T (60, 74). Lat. 57° 16′ 35″, Long. 5° 44′ 0″ Near the shore; between the Inn and Kyle House

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
— 2 49	10 29	23 23 4	23 10 4

Inclination.

Needle	G M T	θ	θ_0
	h. m	° ′	° ′
1	10 15	71 40 9	71 38 5
2	11 1	71 38 8	

Horizontal Force

G M T		H	H_0
	h m		
D	8 56	1 5432	1 5465
V	11 21	1 5452	

36 LAIRG. July 6, 1885, A. W R. (60, 74) Lat 58° 1′ 30″, Long 4° 24′ 0″ 40 yards S W. of Church, and about 200 yards N.E of the Hotel.

Declination.

Σ	G M T	δ	δ_0
h m +1 44	h m 15 56	° ′ 21 55 2	° ′ 21 50 3

Inclination.

Needle	G M T	θ	θ_0
1 2	h, m 15 13 18 5	° ′ 71 51 0 71 50 8	° ′ 71 50 3

Horizontal Force.

G M T		H	H_0
D V	h m 17 7 16 23	1 5347	1 5356

37. LERWICK. August 30, 1885, T E. T. (60, 74) Lat. 60° 8′ 53″; Long 1° 7′ 47″. Towards the South Ness, ½ mile due S. (mag) from Fort Charlotte. Mr WELSH's station.

Declination.

Σ.	G.M.T	δ	δ_0
h. m. +1 53	h m. 12 38	° ′ 20 33 1	° ′ 20 29 7

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 36	72 46 7	72 47 1
2	16 3	72 48 1	

Horizontal Force.

G M T		H	H_0
	h m		
D	14 39	1 4712	1 4710
V	13 34	1 4696	

38. LOCHGOILHEAD July 21, 1888, T E T (61, 83) Lat 56° 10′ 20″, Long. 4° 54′ 0″ In a field about 80 yards N.E. of the Church Mr WELSH's station.

Declination

Σ	G M T	δ	δ_0
h m	h m.	° ′	° ′
+ 4 40	15 48	21 29 2	21 54 2
	18 6	21 30 8	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 32	70 42 4	70 46 1
2	17 48	70 43 6	

Horizontal Force

G M T		H	H_0.
	h m		
D	16 29	1 6069	1·6021
V	16 4	1 6065	

39 Loch Maddy (N Uist), September 1 and 2, 1884, A W R and T E T. (60, 74). Lat. 57° 35′ 50″, Long 7° 9′ 0″ On the most westerly of the Reelee Islands

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Sept 1		18 34	23 31 3	23 18 0
„ 2	−3 14	10 29	23 32 3	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Sept 1	2	18 55	71 53 3	71 52 1

Horizontal Force.

Date	G M T		H	H_0.
		h m		
Sept 2	D	12 26	1 5346	1 5365
„ 1	V	19 13	1 5336	

40a. Oban. August 13, 1885, and July 30, 1888; T E. T (60, 61, 83). Lat. 56° 25′ 9″; Long 5° 28′ 30″ Same station as that of Mr. Welsh.

Declination.

Date.	Σ.	G.M.T	δ	δ_0
	h m	h. m	° ′	° ′
Aug 13, 1885	+3 38	16 37	22 12 0	22 8 7
July 30, 1888	−1 41	11 20	21 47 3	22 10 2
		12 59	21 45 2	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
July 30	1 2	12 43 13 7	70 50 7 70 49 5	70 53 2

Horizontal Force

Date	G M T		H	H_0
		h m		
July 30	D V	12 24 11 38	1 6090 1 6091	1 6044

40B —OBAN (Kerrera) August 26, 1884, and August 6, 1885. A. W. R. and T E T (60, 74) Lat. 56° 25′ 20″, Long. 5° 30′ 0″. Near the shore of Ardentraive Bay.

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Aug 26, 1884	+ 2 5 + 3 55	15 32 .	22 25 4 22 25 5	22 12 4
,, 6, 1885	− 1 25	12 32	22 15 2	22 11 4

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Aug 26, 1884	1 2	15 18 15 47	70 50 5 70 51 8	70 49 5
,, 6, 1885	1 2	13 14 14 11	70 49 1 70 48 3	70 48 2

Horizontal Force.

Date	G M T.		H.	H_0
		h m		
Aug 26, 1884	D V	17 30 16 44	1 6066 1 6071	1 6092
,, 6, 1885	D V	13 30 12 13	1 6110 1 6105	1 6114

41 PITLOCHRIE July 29, 1885, A W. R and A. P. L (60, 74) Lat 56° 42′ 7″, Long 3° 43′ 26″ In the grounds of the Hydropathic Establishment, about 100 yards S S W. of the building, near the lower road through the grounds.

Declination.

Σ	G M T	δ	δ_0
h m +3 9	h m 16 35	° ′ 21 12 2	° ′ 21 8 3

Inclination

Needle	G M T	θ	θ_0
1 2	h m 17 55 16 58	° ′ 70 58 7 70 57 1	° ′ 70 57 4

Horizontal Force.

G M.T		H	H_0
D V	h m 17 10 16 11	1 5891 1 5891	1 5899

42. PORT ASKAIG (Islay) August 25, 1884; A W. R and T. E T. (60, 74). Lat. 55° 50′ 40″; Long. 6° 6′ 55″. Between the Inn and the Quay.

Declination

Σ.	G M T.	δ	δ_0
h m. −2 44	h m 10 18	° ′ 23 13 7	° ′ 23 0 7

Inclination.

Needle.	G M.T.	θ	θ_0.
1 2	h m 12 23 12 22	° ′ 70 37 9 70 38 1	° ′ 70 36 2

Horizontal Force.

G M T		H	H_0
	h m		
D	12 2	1 6313	1 6340
V	10 38	1 6319	

43 PORTREE. T E T (61, 83)

(*a*) August 9, 1888, Lat. 57° 24′ 35″, Long 6° 11′ 40″. On the N shore of the bay. Boat-house 20° W.

(*b*) August 9 and 10, 1888 Lat 57° 24′ 10″, Long 6° 11′ 5″ On the S. side of the bay, on a rock close to the shore. Portree Landing Stage 12° W of N (mag). School-house 35° W of N Station (*a*) 22° E of N

(*c*) August 10, 1888. Lat 57° 24′ 15″, Long. 6° 11′ 50″ On the edge of the bay, S of the town, and within a dozen yards of the shore Station (*b*) bearing 100° E of N., School-house 2° E of N, Station (*a*) bearing 57° E of N. (approx), St Columba's Church steeple 24° E of N.

Declination

Date	Σ	G M T	δ.	δ_0
	h m	h m	° ′	° ′
Aug 9 (*a*)	−2 30	10 26	24 37 6	22 42 3
„ 9 (*b*)	+4 24	15 50	22 21 6	
„ 10 (*b*)	−2 42	11 25	22 22 7	
„ 10 (*c*)	+1 12	13 24	19 50 8	

Inclination

Date	Needle	G M T	θ.	θ_0
		h m	° ′	° ′
Aug 9 (*a*)	1	13 2	72 16 3	72 1 2
	2	13 21	72 14 7	
„ 10 (*b*)	1	10 34	71 5 9	
	2	10 51	71 5 5	
„ 10 (*c*)	1	14 31	72 33 9	
	2	14 56	72 33 6	

Horizontal Force.

Date	G M T		H	H_0
		h m		
Aug 9	D	(a) 12 9	1 5223	1 5177
	V	(a) 11 10	1 5226	
,, 10	D	(c) 12 38	1 5265	1 5211
	V	(c) 13 10	1 5252	
,, 9	V	(b) 15 36	1 5906	1 5859
,, 10	V	(b) 11 16	1 5941	1 5894

NOTE —The ground at Portree was known to be extremely bad as there is much basaltic rock in the neighbourhood The observations were made with a view of gaining information as to the magnitude of the disturbing forces.

44 Row (Gairloch). July 23, 1888, T E T. (61, 83) Lat. 56° 1′ 0″; Long 4° 46′ 50″ Near the shore of the loch, Pier end bears due W. (mag), Roseneath House 37° W. of S., and Roseneath Point 17° W. of S. (mag.).

Declination.

Σ	G M T.	δ	δ_0
h m.	h m	° ′	° ′
− 2 23	10 24	21 24 4	
	12 0	21 22 6	21 47 7
+ 1 16	13 13	21 24 0	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 28	70 48 4	70 51 0
2	12 52	70 47 5	

Horizontal Force.

G M.T		H	H_0.
	h m.		
D	11 31	1 6025	1 5978
V	10 53	1 6023	

45. SCARNISH (Tiree) August 8 and 10, 1885, A W R and A. P L. (60, 74). Lat. 56° 30′ 12″; Long 6° 47′ 20″. Dips about 15 yards in front of Hotel Declinations. Station I. 30 yards W S W of Inn. Station II Inn bears S.W. by S. (200 yards), Harbour Mouth bears S Station III. Inn bears S.W by S (250 yards) On returning to Oban, when a comparison could be made with Greenwich time, the rate of the chronometer was found to have altered suddenly during the visit to Tiree The difference between the values of the declinations obtained at Stations I and II on the 8th and 10th is probably due to this fact.

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Aug 8 (1)	− 1 43	11 36	24 49 8	24 46 9
,, 10 (1)	− 2 49	9 24	24 51 8	
,, 10 (3)	+ 5 57	17 0	24 52 4	24 47 3
		18 51	24 50 0	
,, 8 (2)	+ 3 27	16 34	23 52 1	23 49 6
,, 10 (2)	+ 4 13	16 30	23 54 9	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Aug 8	1	11 43	71 20 0	71 19 4
	2	12 50	71 19 9	

Horizontal Force.

Date	G M T		H	H_0
		h m		
Aug 8	D	13 6	1 5898	1 5909
	V	12 9	1 5907	

46. SOA (Skye) August 29, 1884, A W R and T E T (60, 74) Lat. 57° 9′ 45″, Long 6° 10′ 12″ On the N E point, about 300 yards E of the anchorage, and near the Ru Mhoil Dearg

Declination

Σ	G M T.	δ	δ_0
h m	h m	° ′	° ′
− 4 41	8 38	23 28 3	23 14 9

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	8 48	72 0 4	71 59 6
2	9 31	72 1 5	

Horizontal Force.

G M T		H	H_0
	h m		
D	10 16	1 5051	1 5072
V	9 21	1 5045	

47. STIRLING. July 10, 1885, A W R (60, 74) Lat 56° 7′ 2″, Long 3° 56′ 55″. In the King's Park, in the centre of the plain to the S. of the Castle.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 1 42	14 53	21 33 0	21 28 6

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 54	70 53 7	70 53 3
2	16 26	70 54 1	

Horizontal Force

G.M T		H.	H_0
	h. m		
D	17 11	1 5938	1 5945
V	14 32	1 5935	

48A. STORNOWAY (Ard Point) September 4 and 5, 1884, and August 19, 1885, A. W. R. and T. E. T. (60, 74); and August 14, 1888, T. E T (61, 83), A W. R. (60, 74) Lat 58° 12′ 10″, Long. 6° 23′ 40″ On the top of the hillock on the Point The Declinations taken on this spot agreed well with that obtained in the Castle Grounds, and were therefore regarded as normal. In 1888, simultaneous observations were made by both of us, which agree among themselves but differ considerably from each other Dr. THORPE occupied the old station near the top of the hillock, Professor RUCKER was about 50 yards distant on lower ground and nearer the mainland There can be little doubt that the station is disturbed

Declination.

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Sept 4, 1884	− 2 22	10 54	24 21·0	24 9·8
„ 5, „	− 1 42	11 3	24 20·2	
Aug 19 1885	+ 3 11	14 7	24 12·4	24 9·5
„ 14, 1888	− 1 52	11 2	T { 23 50·7	
		12 28	T { 23 48·4	24 13·9
	+ 3 24	15 15	T { 23 50·7	
	− 1 45	11 8	R { 24 8·7	
		12 37	R { 24 7·9	24 32·3
	+ 3 23	15 15	R { 24 8·3	

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Aug 14, 1888	1	13 3	72 8·4	72 10·5
	2	13 31	72 8·5	

Horizontal Force.

Date	G M.T		H	H_0
		h m		
Aug 19, 1885	D	16 44	1·5197	1·5210
	V	15 13	1·5210	
	V	17 42	1·5191	1·5205
Aug. 14, 1888	D	12 1	1·5205	
	V	11 25	1·5242 } R	1·5195
	D	12 0	1·5243 } R	
	V	11 20	1·5222 } T 1·5218 } T	1·5173

48B. STORNOWAY September 4, 1884, A. W. R and T E. T. (60, 74) Lat 58° 12′ 40″; Long. 6° 23′ 35″. In the Castle Grounds.

Declination

Σ	G M T	δ	δ_0
h m + 2 32	h m 14 20	° ′ 24 18 4	° ′ 24 7 6

Inclination

Needle	G M T	θ	θ_0.
1 2	h m 13 21 12 54	° ′ 72 9 4 72 11 0	° ′ 72 9 1

Horizontal Force.

G M.T		H	H_0
D	h m 13 5	1 5119 1 5124	1 5145
V	13 44 13 59	1 5122 1 5125	1 5147

49. STRACHUR. August 16 and 17, 1884; A. W R and T. E. T. (60, 74). Lat. 56° 10′ 20″; Long. 5° 4′ 40″ On the lawn in front of Strachur House.

Declination.

Date.	Σ.	G.M.T.	δ	δ_0
Aug 17	h m − 4 4	h m 8 59	° ′ 22 1 5	° ′ 21 48 9

Inclination.

Date	Needle	G.M.T.	θ.	θ_0.
Aug 16	1 2	h. m 12 17 12 19	° ′ 70 44 4 70 44 9	° ′ 70 42 9

Horizontal Force.

Date	G M T		H	H_0
		h m		
Aug 16	D	15 38	1 6068	1 6095
	V	16 19	1 6073	

50 STRANRAER September 18, 1884, A. W R (60, 74), and August 28, 1888; A. W R. (60, 83), and T E T (61, 83). Lat 54° 54′ 25″, Long 5° 2′ 10″. In a field 300 yards N W by W of Schuchan Church. The same position was occupied on both occasions Near Mr. WELSH's station.

Declination.

Date	Σ	G M T.	δ	δ_0
	h m	h m	° ′	° ′
Sept 18, 1884	+ 2 7	16 14	21 46 2	21 35 0
Aug 28, 1888	− 1 30	11 35	21 13 7	21 37 6 T
	− 2 10	10 54	21 12 5	21 36 6 R
		13 6	21 13 0	

Inclination.

Date	Needle.	G M T.	θ	θ_0
		h m	° ′	° ′
Sept 18, 1884	1	15 1	70 14 6	70 12 3
	2	15 37	70 13 7	
Aug 28, 1888	1	11 16	70 11 7	70 14 8
	2	12 43	70 10 5	

Horizontal Force

Date	G M T		H	H_0
		h m		
Sept 18, 1884	D	17 25	1 6421	1 6446
	V	16 39	1 6426	
Aug. 28, 1888	D	11 50	1 6466 } R	1 6420
	V	11 21	1 6471 } R	
	D	12 37	1 6473 } T	1 6425
	V	12 3	1 6473 } T	

51. STROMNESS (Orkneys) August 25, 1885; T. E. T (60, 74). Lat 58° 57′ 30″, Long. 3° 17′ 12″. 60 yards S E. from the door of the house on Holm of Stromness.

Declination

Σ	G M T	δ	δ_0
h m +1 9	h m 12 29	° ′ 21 31 5	° ′ 21 27 9

Inclination.

Needle	G M T	θ	θ_0
1 2	h m. 15 1 14 28	° ′ 72 12 3 72 11 9	° ′ 72 11 7

Horizontal Force

G M T		H	H_0
V	h. m 12 8	1 5143	1 5149

52. E. LOCH TARBERT (Loch Fyne) August 19 and 20, 1884; A. W. R. and T. E. T. (60, 74). Lat. 55° 51′ 56″; Long. 5° 24′ 25″. At the back of the town, near the Castle.

Declination.

Date	Σ.	G M.T.	δ.	δ_0.
Aug. 19 „ 20	h m +5 43 −3 20	h m. 18 50 9 46	° ′ 22 14 4 22 19 3	° ′ 22 4·3

Inclination.

Date	Needle	G M T	θ	θ_0
Aug 20	1 2	h m 10 2 10 26	° ′ 70 47 8 70 49 5	° ′ 70 46 8

Horizontal Force.

Date	G M T		H	H_0
Aug 20	D V	h m 11 33 10 22	1 6027 1 6031	1 6053

53. THURSO. July 3, 1885, A. W R. (60, 74). Lat 58° 35′ 30″, Long 3° 31′ 15″, On the green by the right bank of the river, about 30 yards above the bridge. The station occupied by Mr. WELSH had been built over.

Declination.

Σ	G M T	δ	δ
h m − 1 44	h m 11 11	° ′ 21 43 5	° ′ 21 38 4

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 13 7 13 43	° ′ 72 2 9 72 0 6	° ′ 72 1 1

Horizontal Force.

G M T		H.	H_0
D V	h m 14 38 11 54	1 5208	1 5217

54. WICK. July 2, 1885, A. W R (60, 74) Lat. 58° 26′ 20″, Long 3° 5′ 45″
On the lawn in front of Rosebank. Same station as that of Mr. WELSH

Declination

Σ	G M T	δ	δ_0
h m − 2 52	h m 11 6	° ′ 21 20·4	° ′ 21 15·3

Inclination.

Needle	G M.T	θ	θ_0
 1 2	h m. 13 25 14 0	° ′ 72 10·1 72 10·7	° ′ 72 9·8

Horizontal Force.

G M T.		H	H_0.
 D V	h m 11 56 10 44	 1·5139 1·5131	1·5144

Descriptions of English Stations.

55 Aberystwith. May 18, 1886, A W. R. (60, 74). Lat 52° 23′ 51″, Long 4° 5′ 13″. About a mile to the South of the town, between the River Ystwith and the sea, near the point marked Pen-y-ro on the Ordnance map.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 19	11 19 13 42	19 54 6 19 51 7	19 56 5

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1 2	13 2 13 4	68 34 4 68 33 9	68 34 7

Horizontal Force

G M T		H	H_0
	h m		
V V	12 16 12 34	1 7508 1 7485	1 7500 1 7477

56. Alderney. April 3 and 4, 1888; T. E. T. (61, 83) Lat. 49° 43′ 10″, Long. 2° 11′ 0″. In the N.E. corner of the garden of Scott's Hotel, about 70 yards from the house

Declination

Date	Σ.	G M T	δ	δ_0
	h m.	h m	° ′	° ′
April 3	+ 1 54	12 56	17 46 1	18 28
		15 9	17 46 5	
	+ 4 27	16 48	17 45 1	
,, 4	− 4 2	7 54	17 49 3	

Inclination.

Date	Needle	G M T	θ.	θ_0
		h m	° ′	° ′
April 3	1 2	16 1 16 23	66 35 2 66 33·7	66 38 9

Horizontal Force.

Date.	G M T.		H	H_0
		h. m		
April 3	D V V	14 45 13 41 15 21	1 8755 1 8756 1 8719 1 8735	1 8705 1 8677

57. Alnwick. September 16, 1886. A. W. R Lat. 55° 25′ 19″; Long 1° 43′ 53″. In the Deer Park, on the N. side of the drive, and about one-third of a mile from the reservoir. Alnwick Church S.E by E. Castle E. by S.

Declination

Σ	G M T	δ	δ_0
h m − 1 7	h m 11 26	19° 39′ 4	19° 45′ 0

Inclination.

Needle	G M.T.	θ	θ_0.
	h m	° ′	° ′
1 2	13 1 13 32	70 2 1 70 3·3	70 3 6

Horizontal Force

G M T		H.	H_0.
	h m		
P V	14 20 11 50	1 6529 1 6526	1·6511

58. ALRESFORD. April 28, 1888, A. W. R. (60, 74). Lat. 51° 4′ 46″, Long. 1° 9′ 56″. On Tichborne Down.

Declination.

Σ	G M T	δ	δ_0
h m + 2 39 3 45	h m 13 16 16 19 5	° ′ 17 52 8 17 53 2	° ′ 18 9 7

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 15 26 16 2	° ′ 67 17 8 67 19 0	° ′ 67 22 3

Horizontal Force.

G M T		H	H_0
D V	h m 14 42 14 2	1 8292 1 8293	1 8241

59 APPLEBY September 15, 1886, A. W. R (60, 74). Lat. 54° 34′ 10″, Long 2° 29′ 3″. On the Holme, a common close to the Eden, to the S. of the town.

Declination

Σ	G M T	δ	δ_0
h m −1 1 +2 7	h m 11 47 13 51	° ′ 20 1 5 19 59 0	° ′ 20 5 8

Inclination

Needle.	G M T	θ	θ_0
1 2	h m 12 46 13 15	° ′ 69 43 7 69 44 1	° ′ 69 44 9

Horizontal Force

G M T.		H.	H_0
	h m.		
V	12 2	1 6706	1 6690

60. BARROW. August 25, 1886, A. W R. (60, 74). Lat 54° 7′ 24″; Long 3° 13′ 0″. In a field about 200 yards W. from the road which leads to Barrow from Furness Abbey. About 1½ miles from the Abbey on the brow of the hill above Barrow.

Declination.

Σ	G M T	δ	δ_0
h m	h m.	° ′	° ′
+2 15	14 52	20 4 6	20 9 3
+5 18	17 5	20 2 9	

Inclination.

Needle	G M.T	θ	θ_0.
	h m	° ′	° ′
1	16 3	69 29·5	69 30 6
2	16 32	69 29 7	

Horizontal Force.

G M T		H	H_0.
	h m.		
D	17 59	1 6886	1·6875
V	15 13	1 6892	

61. BEDFORD. April 21, 1888 ; T. E. T. (61, 83). Lat. 52° 8′ 3″; Long. 0° 26′ 51″. On a road on the Bower estate, half-a-mile to the E. of the town-bridge, and about 80 yards from the river.

Declination.

Σ.	G M.T	δ.	δ_0.
h m	h. m		
− 1 24	9 51	18° 10′ 9	18° 27′ 4

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	11 13	68 4 4	68 7 3
2	11 30	68 3 3	

Horizontal Force.

G M T		H	H_0
	h m		
V	10 8	1 7756	1 7705

62 BIRKENHEAD. August 23, 1886; A. W R (60, 74) Lat. 53° 24′ 4″, Long 3° 4′ 18″. 1st station, 130 yards S of Observatory, 2nd station, 100 yards S. of Observatory

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 1	(1) 12 23	19 54 8	19 58 3
+ 5 2	(2) 17 32	19 52 1	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 37	69 4 0	69 4 3
2	16 4	69 2 6	

Horizontal Force.

G M T		H	H_0
	h m		
D	13 29	1 7188	1 7176
V	12 50	1 7192	

63 BIRMINGHAM May 7, 1886; A. W R (60, 74) Lat 52° 27′ 37″; Long 1° 53′ 40″ In Calthorpe Park, at the S end About 50 yards from the East, and 200 yards from the South railings

Declination.

Σ	G M T	δ	
h m	h m	° ′	
− 1 14	11 15	18 42 1	18 44 0
+ 2 44	14 29	18 39 7	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 24	68 21 3	68 21 3
2	13 55	68 20 1	

Horizontal Force.

G M T		H	H_0
	h m		
D	12 22	1 7672	1 7669
V	11 37	1 7683	

64 BRAINTREE September 22, 1888; T. E T (61, 83) Lat 51° 52′ 41″, Long. 0° 32′ 40″ E. In a field to the W. of the Church, distant 400 yards, 400 yards N. of the Railway to Dunmow

Declination.

Σ	G M T	δ.	δ_0.
h m	h m	° ′	° ′
+ 3 34	15 51	17 36 4	17 55·4

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 16	67 41 8	67 45 4
2	17 32	67 40 8	

Horizontal Force.

G M T		H	H_0
	h m		
D	16 40	1 8005	1 7942
V	16 15	1 7999	

65. BRECON. May 26, 1886, A. W R (60, 74) Lat 51° 56′ 56″, Long 3° 24′ 42″. In the fields (Newton Port) to the W of the town, and close to the river Church in centre of town S.E. by E. About $\frac{3}{4}$ of a mile E. by N of Castle Hotel

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 4 42	17 50	19 35 1	19 38 6

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
2	19 12	68 15 1	68 15 8

Horizontal Force

G M T		H	H_0
	h m		
D	18 28	1 7710	1 7701
V	17 51	1 7710	

66 BUDE HAVEN. T. E. T. (61, 83)

(*a*) April 11 and 15, 1887, Lat 50° 49′ 34″, Long 4° 32′ 37″. On the upper walk of the kitchen garden of the Falcon Hotel, and on the W. front of the house

(*b*) April 12, 1887, Lat. 50° 49′ 42″; Long 4° 32′ 54″. On Efford Down near the Compass Tower, which bore 75° W. of N. (mag) Belfry of Efford House 10° E. of S (mag). Entrance lock to Bude Canal 68° E of N. (mag).

Declination.

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
April 11	− 1 46	11 2	19 50 1	20 0 1
	+ 3 7	15 2	19 52 2	
" 22	− 1 32	11 13	19 43 9	
	+ 2 44	14 41	19 43 5	19 52 9
" 15	− 0 20	12 26	19 44 2	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
April 11	1	15 58	67 44 8	
	2	14 31	67 40 2	67 44 2
" 12	1	13 46	67 41 8	
	2	14 16	67 41 1	

Horizontal Force.

Date	G M T		H	H_0
		h m.		
April 11	D	12 14	1·8126	1 8085
	V	11 47	1 8100	
12	D	12 2	1 8103	1 8075
	V	11 36	1·8104	

67. CAMBRIDGE T E T (61, 83)

May 18, 1886; Lat 52° 11′ 40″, Long 0° 7′ 17″ E. In the lane off the Trumpington Road, past the Botanic Gardens, and on the right hand side

May 19 and 20, 1886, Lat. 52° 11′ 23″, Long 0° 7′ 16″ E In the Leys School Grounds

Declination.

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
May 18	+2 28			
	+2 37			
„ 19		11 47	18 2 1	
		15 35	18 0 5	18 5 0
„ 20	−1 51	10 34	18 2 1	

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 18	1	16 39	68 1 8	68 2 4
	2	17 11	68 1 9	

Horizontal Force

Date	G M T		H	H_0
		h m		
May 19	D	12 37	1 7799	1 7784
			1 7786	
	V	11 11	1 7797	
„ 20	V	10 58	1 7790	1 7785

68. CARDIFF May 25, 1886, A. W R (60, 74). Lat 51° 29′ 36″, Long 3° 10′ 33″. In a field W of the town, on Crwys Farm House E N E, about 80 yards distant; gate of barracks, 200 yards N. by W.

Declination

Σ	G M T.	δ	δ_0
h. m	h m	° ′	° ′
+3 0	16 1	19 16 6	19 19 7
+4 33	17 5	19 15 9	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 38	67 52 8	67 52 3
2	18 3	67 50 5	

Horizontal Force

G M T.		H	H_0
	h m		
V	16 22	1·7953	1 7944

69 CARDIGAN. May 29, 1886; A W R. (60, 74). Lat 52° 5′ 20″, Long. 4° 40′ 9″. In a field about half a mile to the west of the town.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 1 12	12 15	20 22 2	20 25 8

Inclination.

Needle	G M T.	θ	θ_0
	h m	° ′	° ′
1	13 6	68 30 3	68 31 3
2	11 43	68 31·0	

Horizontal Force

G M T		H	H_0.
	h. m.		
V	13 47	1 7544	1 7535

70. CARLISLE August 28, 1886, A W R. (60, 74). Lat 54° 53′ 55″, Long 2° 55′ 40″ On the Swifts, about 100 yards from the footpath on the S side, and 250 yards E. by S. of the Grand Stand Cathedral bore W.S.W. ½ S

Declination.

Σ	G M T	δ	δ_0
h m – 0 41 + 1 30	h m 12 1 14 41	° ′ 20 20 5 20 19 8	° ′ 20 25 8

Inclination

Needle	G M T	θ	θ_0
1 2	h m 13 17 14 14	° ′ 69 52 1 69 54 0	° ′ 69 54 0

Horizontal Force.

G M T		H	H_0
V	h m 12 25	1 6640	1 6625

71 CHESTERFIELD September 14 and 15, 1887, A W R (60, 74) Lat. 53° 14′ 3″, Long 1° 24′ 37″ In a field on Mr. PENISTONE'S farm to the N of the main road. Church bearing W.N W about a mile distant Cemetery S.W by W. about half a mile

Declination

Date	Σ	G M T	δ	δ_0
Sept 15	h m – 2 36 – 2 10	h. m 10 19 ..	° ′ 19 0 3 18 59 8	° ′ 19 11 9

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Sept 14	1 2	11 40 12 10	68 47 1 68 44 5	68 48 5

Horizontal Force

Date	G M T		H	H_0
		h m		
Sept 14	D V	14 16 13 26	1 7383 1 7393	1 7351

72. CHICHESTER. September 18, 1888, T E. T (61, 83) Lat. 50° 50′ 0″, Long 0° 47′ 2″. In the meadows between the Cathedral and the Railway Station, close to the ditch. Railway Station bearing S E. Cathedral tower N.E., distant a quarter of a mile.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 52	11 32	17 46 6	18 5 5

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1 2	13 27 13 43	67 7 6 67 7 0	67 11 6

Horizontal Force.

G M T.		H	H_0
	h m		
D V	12 36 11 54 .	1 8456 1 8455	1 8395

73 CLENCHWARTON August 2, 1888, A W. R (60, 74) Lat 52° 45′ 20″, Long 0° 21′ 20″ E In the grounds of a house on the road from King's Lynn to Sutton Bridge Not quite 3 miles from King's Lynn

Declination

Σ	G M T.	δ	δ_0
h m	h m	° ′	° ′
−0 29	11 58	17 51 4	18 10 3

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 45	68 14 3	68 17 9

Horizontal Force

G M T		H	H_0
	h m		
D	12 49	1 7722	1 7662
V	12 16	1 7717	

74. CLIFTON April 22, 1886, T E T (61, 74) Lat 51° 27′ 15″, Long 2° 37′ 4″. On the Down, on the N. side of the Stoke road, about 60 yards from it, and about ¼ mile from the Clifton Cricket Club Pavilion Steeple of Christ Church Congregational Chapel bearing E Pumping station S E. about 150 yards.

Declination

Σ	G M T	δ	δ_0
h. m	h m	° ′	° ′
+0 15	11 56	19 8 6	19 10 7

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 12	67 48 9	67 48 7
2	13 47	67 47 6	

Horizontal Force

G M T		H	H_0
	h m		
D	14 49	1 8001	1·7996
V	11 30	1 8006	

75. Clovelly April 16, 1887, T E. T. (61, 83) Lat 50° 59′ 48″, Long 4° 23′ 50″. To the W of the village, between it and Clovelly Court.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 33	12 9	19 44 2	19 53 8
+ 3 4	15 45	19 45 5	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 23	67 47 8	67 49 9
2	14 57	67 47 6	

Horizontal Force

G.M T.		H	H_0
	h m		
D	11 27	1 8027	1 8000
V	12 44	1 8030	

76. COALVILLE April 30, 1888, T E T (61, 83) Lat. 52° 43′ 41″, Long 1° 21′ 18″ In a field about 100 yards E of Coalville Station (L and N W Railway), and E of town.

Declination

Σ	G M T	δ	δ_0
h m + 1 26	h m 14 28	° ′ 18 24 6	° ′ 18 41 4

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 11 58 12 17	′ 68 20 3 68 20 6	° ′ 68 24 1

Horizontal Force

G M T		H	H_0
D V	h m 13 1 14 15	1 7583 1 7587	1 7534

77. COLCHESTER. September 24, 1888, T E T. (61, 83) Lat 51° 53′ 30″, Long. 0° 54′ 0″ E. In the Castle grounds, to the S. of the main tower, distant 20 yards

Declination

Σ	G M T	δ	δ_0
h m − 1 13 + 1 18	h m 10 52 12 52	° ′ 17 36 2 17 36 2	° ′ 17 55 2

Inclination.

Needle	G M T	θ	θ_0
1 2	h m. 11 21 11 37	° ′ 67 31 3 67 31 1	° ′ 67 35 3

Horizontal Force

G M T		H	H_0
	h m		
D	12 28	1 8078	1 8012
V	12 2	1 8066	

78. CROMER May 10, 1886, T E T (61, 83) Lat 52° 55′ 20″, Long 1° 18′ 24″ E In a field S S W of the Lighthouse, Railway Station about three-quarters of a mile away bearing W, town bearing N.W. by N

Declination.

Σ	G M T	δ	δ_0
h m	h m		
+ 1 0	13 21	17° 32′ 7	17° 35′ 8

Inclination.

Needle	G M.T	θ	θ_0
	h. m		
1	14 26	68° 19′ 6	68° 20′ 0
2	14 53	68 19 2	

Horizontal Force.

G M T		H	H_0
	h m		
V	13 40	1 7611	1 7603

79 DOVER. September 28, 1887, T. E. T. (61, 83). Lat. 51° 6′ 53″, Long 1° 17′ 52″ E In the Public Recreation Ground, near Archcliff Fort; between the town and Shakespeare Cliff. End of Dover Breakwater 63° E. of S. (mag); Flagstaff on Shakespeare Cliff 65° W. of S. (mag.).

Declination.

Σ	G M T.	δ	δ_0.
h m	h m		
− 1 58	10 20	16° 44′ 1	16° 57′ 2
+ 0 56	12 26	16 46 2	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	11 15	67 5 8	67 8 0
2	11 31	67 4 7	

Horizontal Force.

G M T		H	H_0
	h m		
D	11 59	1 8377	1 8336
V	10 34	1 8372	

80 Falmouth April 8, 1887, T E T. (61, 83). Lat 50° 8′ 47″; Long 5° 4′ 21″
At the Observatory

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 51	11 10	19 43 3	19 53 4
+2 39	15 42	19 45 8	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 31	67 12 3	67 15 0
2	13 45	67 13 1	

Horizontal Force.

G M.T		H	H_0
	h m		
D	11 59	1 8354	1 8323
V	11 35	1 8348	

81. GAINSBOROUGH. September 19, 1887, A W R (60, 74) Lat 53° 23′ 23″; Long 0° 44′ 51″ In a field to the W. of the Upton Road, and about 1 mile S (mag) of the town.

Declination.

Σ	G M T	δ	δ_0
h m +3 56	h m 16 16	° ′ 18 24 5	° ′ 18 36 4

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 53	68 46 4	68 49 3
2	18 12	68 46 7	

Horizontal Force.

G M T		H	H_0
	h m		
D	17 14	1 7362	1 7321
V	16 31	1 7356	

82. GIGGLESWICK. September 14, 1886, A. W. R. (60, 74). Lat. 54° 4′ 18″, Long. 2° 17′ 48″ In the cricket field of Giggleswick School, 150 yards N W. of the Pavilion, and near the W. end of the field.

Declination.

Σ	G M.T	δ	δ_0
h m	h m	° ′	° ′
+1 33	12 2	19 31 2	19 35 3
+3 16	15 8	19 29·8	

Inclination

Needle	G M T	θ	θ_0
	h. m	° ′	° ′
1	10 59	69 21 4	69 22 3
2	11 19	69 21 0	

Horizontal Force

G M T		H	H_0
	h m		
D	13 16	1 6976	1 6962
V	12 37	1 6979	

83 GLOUCESTER May 24, 1886, A W R (60, 74) Lat 51° 52′ 12″, Long 2° 14′ 50″ In the fields to the E of the town, about 300 yards E by N of Alexandra Terrace, Cathedral bearing W.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 12	.	19 9 4	19 12 9
+3 26	16 4	19 9 7	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 46	68 4 5	68 4 3
2	14 19	68 2 8	

Horizontal Force

G M T		H	H_0
	h m		
D	15 1	1 7823	1 7811
V	12 46	1 7817	

84. Grantham May 22, 1886, T E T (61, 83) Lat. 52° 54′ 54″, Long 0° 37′ 46″ W. In a field to the E of the town, and across the river. The Church bore about S.W., distant 400 yards

Declination

Σ	G M T	δ	δ_0
h m +1 36	h m 13 13	° ′ 18 25 6	° ′ 18 29 0

Inclination.

Needle	G.M T	θ	θ_0
1 2	h m 14 9 14 30	° ′ 68 26 6 68 28·3	° ′ 68 28·0

Horizontal Force.

G M T.		H	H_0
V	h m 12 56	1 7536	1 7527

85 and 86. Guernsey. T. E T. (61, 83).

85. L'Erée. Lat 49° 27′ 50″, Long. 2° 35′ 40″ Fifty yards due S. of L'Erée Hotel, and about 15 yards from the road to Roquaine Castle.

86 Peter Port Lat 49° 27′ 45″; Long 2° 31′ 45″ In the Gardens behind the Royal Hotel, and about 60 yards from the house.

Declination

Date	Σ	G M T	δ	δ_0.
April 2 " 4	h m + 2 32 + 2 43	h. m 14 22 14 19	° ′ 18 16 6 18 2 3	° ′ 18 32 7 18 18 4

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
April 2	1	13 2	66 28.9	66 34.1
	2	13 38	66 30.3	
,, 4	1	16 2	66 29.0	66 32.4
	2	16 17	66 26.8	

Horizontal Force

Date	G M T		H	H_0
		h m		
April 2	D	15 20	1.8798	1.8746
	V	14 10	1.8795	
,, 4	D	15 28	1.8843	1.8796
	V	14 37	1.8849	

87 HARWICH May 8, 1886, T E T (61, 83) Lat 51° 56′ 48″; Long. 1° 17′ 5″ E. In a field to the W of the town, Harwich Church bearing E. by N., Great Eastern Hotel, N E, and Railway Station, E S E

Declination

Σ.	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 10	11 38	17 15.5	17 18.8
	13 41	17 16.0	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 30	67 37.0	67 38.4
2	13 6	67 38.7	

Horizontal Force.

G M.T		H	H_0
	h m		
D	14 17	1.8040	1.8031
V	11 13	1.8039	

88 HARPENDEN October 5 and 6, 1887, and May 1, 1888; T. E T (61, 83). Lat. 51° 47′ 27″, Long 0° 21′ 15″ 1st Station. At the N end of the Common, 200 yards S of the Railway Hotel, and W of the Railway Station 2nd Station In Dr GILBERT'S garden. To the W. of the lower Common

Declination

Date	Σ	G M T	δ	δ_0
May 1, 1888	h m + 5 13	h m 14 33	° ′ 17 59 6	° ′ 18 16 5

Inclination

Date	Needle	G M T.	θ	θ_0
		h m.	° ′	° ′
Oct 6, 1887	1	16 13	67 53 4	67 53·5
	2	16 32	67 48 4	
May 1, 1888	1	18 7	67 48 2	67 51 4
	2	17 48	67 47 6	

Horizontal Force.

Date	G M T		H	H_0
		h m		
Oct 5, 1887	D	15 36	1 7961 }	1 7926
	V	15 13	1 7970 }	
May 1, 1888	V	14 50	1 7971	1 7920
	V	15 6	1 7960	1 7909

89 HASLEMERE. September 30, 1887, T E T (61, 83) Lat 51° 6′ 4″, Long. 0° 44′ 40″ On the tennis-lawn of Professor WILLIAMSON'S house, High Pitfold, and about 50 yards S. of it About 600 feet above sea level

Declination.

Σ	G M.T	δ	δ_0
h. m. −1 6	h m. 11 12	° ′ 17 55·6	° ′ 18 7 7

Inclination

Needle	G M T	θ	θ_0
1	h m 12 57	° ′ 67 17 8	° ′ 67 20 6

Horizontal Force

G M T		H	H_0
D V	h m 12 17 11 46	1 8321 1 8319	1 8282

90 HOLYHEAD May 4, 1887, T E T (61, 83) Lat 53° 17′ 53″; Long 4° 38′ 22″. On the road to Porth Dafarch, 20 yards from the roadside, and about a mile from the Railway Station which bore 50° E of N, Monument on Black Bridge, 55° E of N., Spire of New Church, 30° E of N., Windmill, 80° E. of N., Summit of Holyhead Mountain, 30° W. of N.

Declination

Σ	G M T	δ	δ_0
h m − 1 51 + 1 42 + 2 0	h m 10 50 13 42	° ′ 20 40 3 20 41 8	° ′ 20 51 1

Inclination

Needle	G M T.	θ	θ_0
1 2	h m 12 46 13 10	° ′ 69 22 2 69 20 1	° ′ 69 23 1

Horizontal Force

G M T		H	H_0
D V	h m 11 50 11 11	1 6988 1 6986	1 6958

91. HORSHAM. April 21, 1888, A. W. R. (60, 74). Lat. 51° 4′ 16″, Long. 0° 21′ 54″. In a field on the N.W. side of Barber's Green, 1½ miles W. of Horsham.

Declination.

Σ	G. M. T.	δ	δ_0
h m	h m	° ′	° ′
+ 0 44	13 14	17 47·0	18 3·3
+ 3 16	15 37	17 47·3	
+ 5 4	17 26	17 46·1	

Inclination.

Needle	G. M. T.	θ	θ_0
	h m	° ′	° ′
1	16 17	67 11·6	67 15·2
2	16 39	67 11·4	

Horizontal Force.

G. M. T.		H	H_0.
	h m		
D	14 44	1·8358	1·8309
V	13 34	1·8363	

92. HULL. Sept. 16 and 17, 1887, A. W. R. (60, 74). Lat. 53° 44′ 40″, Long. 0° 22′ 5″. In the Botanic Gardens.

Declination.

Date.	Σ	G. M. T.	δ.	δ_0
	h m	h m	° ′	° ′
Sept. 16	+ 5 38	18 1	18 45·7	18 57·8
„ 17	− 3 3	9 20	18 46·2	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Sept 16	1	16 1	69 1 8	69 3 9
	2	16 44	69 1 3	

Horizontal Force

Date	G M T		H	H_0
		h m		
Sept 16	V	17 18	1 7163	1 7125

93. ILFRACOMBE April 25, 1886, T E T (61, 83) Lat 51° 12′ 38″, Long 4° 7′ 36″. In a field near the Tori's Walk, between the town and the sea Parish Church, bearing S by E (½ mile) Ilfracombe Hotel, E by N

Declination

Σ	G M T	δ	δ_0
h m	h m	h m	° ′
− 1 32	11 12	19 45 5	19 46 5
+ 1 51	13 48	19 43 3	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 15	67 53 3	67 53 8

Horizontal Force

G M T		H	H_0
	h m		
D	14 48	1 7962	1 7952
V	11 33	1 7957	

94, 95, 96. JERSEY. T. E. T. (61, 83).

94. Grouville Lat. 49° 12′ 5″, Long 2° 1′ 20″. In the eastern ditch of the Fort. Railway Station bore about W

95 S Louis Lat 49° 12′ 0″; Long 2° 5′ 40″.

96 S Owen. Lat 49° 13′ 30″, Long 2° 12′ 5″. Within 300 yards of the shore, on the flat ground near the Marée Corbière Lighthouse, 40° W. of S. (mag)

Declination

Date	Σ	G M T	δ.	δ_0
	h m	h m	° ′	° ′
Grouville March 30	+1 19	13 16	18 19 9	18 36 0
	+2 20	14 24	18 19 9	
S Louis „ 31	+1 20	12 12	17 34 4	17 49 7
		13 56	17 32 9	
S Owen April 1	+2 21	12 37	18 7 6	18 21 7
		15 1	18 9 7	

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Grouville March 31	1	9 49	66 3 4	66 8 7
	2	9 9	66 5 1	
S Louis „ 31	1	14 32	66 7 8	66 13·0
	2	15 0	66 9 2	
S Owen April 1	1	14 4	66 9 8	66 15 5
	2	14 20	66 12·2	

Horizontal Force.

Date	G M T		H.	H_0
		h m.		
Grouville March 30	D	15 13	1 9099	1 9053
	V	14 7	1 9106	
S Louis „ 31	D	13 6	1 8938	1 8887
	V	12 28	1 8934	
S Owen April 1	D	13 28	1 8952	1 8904
	V	12 51	1 8955	

97. KENILWORTH April 16, 1888, T. E T. (61, 83). Lat 52° 20′ 51″, Long 1° 34′ 43″ In a field near the Abbey Hotel, 60 yards from the road, Castle bearing 65° W. of N., and S Nicholas' Church steeple 24° W of N. (all magnetic)

Declination

Σ	G M T	δ	δ_0
h m −1 23 +1 16	h m 11 9 13 42	° ′ 18 45 2 18 44 8	° ′ 19 1 4

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 12 47 13 5	° ′ 68 24 1 68 26 2	° ′ 68 28 8

Horizontal Force

G M T		H	H_0
D V	h m 12 4 11 28	1 7628 1 7624	1 7576

98 KETTERING May 1, 1888, T. E T (61, 83). Lat. 52° 23′ 44″; Long. 0° 44′ 10″ In the second field past the Railway Arch, W. of the Church, and 70 yards from the Northampton Road.

Declination

Σ	G M T	δ	δ_0
h m − 2 37	h m 9 40	18° 19′1	18° 36′0

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	10 56	68 7 7	68 10 7
2	11 12	68 6 3	

Horizontal Force.

G M T		H	H_0
	h m		
V	10 0	1 7707	1 7656

99 Kew Lat. 51° 28′ 6″, Long. 0° 18′ 45″. In the Magnetic House at the Observatory

Declination.

Date	Observer	Instrument	G.M T.	δ	δ_0
			h. m.	° ′	° ′
July 17, 1884	T	60	16 9	18 23 7	18 15 8
„ 18, „	R	60	11 59	18 21 9	18 14 0
April 2, 1886	T	61	17 14	18 11·0	18 12 5
September 30, 1887	R	60	12 33	18 3 4	18 14 0
			15 38	18 5 1	18 15 7
October 11, 1887	T	61	12 11	18 3 9	18 13 8
			15 33	18 4 0	18 13 9
„ 12, „	R	60	12 14	18 4 8	18 14·7
			14 21	18 4 3	18 14 2
„ 13, „	T	61	10 33	18 5 3	18 15 2
			15 30	18 1 6	18 14 5
„ 18, „	T	61	10 47	18 4·1	18 14 0
			14 56	18 2 6	18 12 5
„ 19, „	R	60	11 9	18 4·8	18 14 7
			16 1	18 6 7	18 16 6
				Mean	18 14 4

It will be observed that the differences between these results are not exactly in accord with those which would be deduced from the table on p. 58.

This is due to the fact that in the above table the observations have been treated as though Kew were an ordinary station, whereas, on p. 58, the corrections have been applied in the most accurate way.

In the first place, no attempt is made on p. 58 to reduce the observations to epoch,

whereas in the above table they are reduced by the mean coefficient of secular change at Kew (6′ 1) during the period of the survey, together with a mean correction for monthly variation

In the next place in the reduction on p 58, no distinction was made between the diurnal variation and disturbance, the total divergence from the mean being read off directly The corrections applied in the above table were obtained by reading off the disturbances only and afterwards adding the value of the diurnal variation

The differences are a little larger than we should have expected, but this is probably due to the fact that, as the corrections were wanted quickly, the constants of the curves were only determined provisionally.

That the accuracy attained was practically sufficient is proved by a comparison of the numbers in the above table with those on p 58

The mean difference between our results and those given by curves treated as accurately as possible is $\pm 0' 81$, while the mean difference between the numbers given above and their mean is $\pm 0' 83$

The Kew value for January, 1886, is 18° 16′ 3, and ours, from the above table when reduced to the Kew instrument by adding the mean difference, is $18° 14' 4 + 2' 5 = 18° 16' 9$.

If from each of our uncorrected observations given on p 58, we subtract the difference between the corresponding Kew value and 18° 16′ 3, we reduce to January, 1886, by one operation, in which no distinction is made between disturbance and the diurnal, monthly, and secular variations The largest of the numbers so obtained is 18° 16′ 1, and the smallest 18° 11′ 9, while the largest and smallest in the above table are 18° 16′·6 and 18° 12′ 5 Thus, not only are the mean results obtained by the two methods in close accord, but the range of variation of the results is in each case practically identical; in other words, they agree to within the limits of the error of experiment.

Horizontal Force.

Date	Observer	Instrument	H	H_0
July 17, 1884	R	60	1 8075	1 8100
April 2, 1886	T	61	1 8082	1 8078
„ 2, „	R	60	1 8098	1 8094
„ 22, „	R	60	1 8101	1 8096
September 30, 1887	R	60	1 8112	1 8082
October 11, 1887	T	61	1 8114	1 8084
„ 12, „	R	60	1 8115	1 8085
„ 13, „	T.	61	1 8114	1 8084
„ 18, „	T	61	1 8123	1 8092
„ 18, „	T	61	1 8125	1 8094
„ 19, „	R	60	1 8125	1 8094
			Mean	1 8089

Inclination.

Date	Needle	Observer	Dip Circle	G.M.T.	θ	θ_0
				h m		
July 17, 1884	1	R	74	15 38	67° 36′·0	67° 34′·8
,, 18, ,,	1	T	74	12 9	67 35·7	67 34·5
	2	T	74	12 38	67 36·0	67 34·8
,, 19, ,,	1	R	74	12 12	67 36·7	67 35·5
	2	R	74	12 47	67 36·1	67 34·9
September 30, 1887	1	R	74	16 43	67 35·4	67 36·8
	2	R	74	17 10	67 34·2	67 35·6
October 11, 1887	1	T	83	14 33	67 34·2	67 35·6
	2	T	83	15 7	67 35·4	67 36·8
,, 13, ,,	1	T	83	14 27	67 35·0	67 36·4
	2	T	83	15 0	67 35·0	67 36·4
,, 18, ,,	1	T	83	13 52	67 34·2	67 35·6
	2	T	83	14 28	67 34·3	67 35·7
,, 19, ,,	1	R	74	13 38	67 35·3	67 36·7
	2	R	74	14 2	67 34·5	67 35·9
					Mean .	67 35·8

In the following Table the values for January 1, 1886, deduced above from the survey instruments, are given in Column I. In Column II. are the mean differences between the survey and Kew instruments (see p. 58), and hence in Column III. the Kew (calculated) values are deduced. In Column IV. are the published elements for that epoch ('Roy. Soc. Proc.,' vol. 41, 1887, p. 416), the Declination being corrected for diurnal variation.

I.	II.	III.	IV.
18° 14′·4	+ 2′·5	18° 16′·9	18° 16′·3
1·8089	− 0·0029	1·8060	1·8093
67° 35′·8	+ 2′·7	67° 38′·5	67° 37′·4

The agreement between Columns III. and IV. is satisfactory in the cases of the Declination and Dip. In that of the Horizontal Force the actual value in January, 1886, seems to have been rather high. As a good deal will hereafter be said about the relative values of the elements at Kew and Greenwich, and as we have not compared our instruments with those at Greenwich, it has been thought better to treat both these observatories in the same way, and therefore to use the numbers in Column IV. as the Kew values.

100 KING'S LYNN

(*a*) May 20, 1886, T E T (61, 83) Lat 52° 45′ 43″, Long 0° 24′ 30″ E. In a field to the E of the town, S Nicholas' Church half-a-mile away, bearing W S W, Waterworks, S W, quarter of a mile away

(*b*) August 2, 1888, A W R (60, 74) Lat 52° 45′ 20″, Long 0° 26′ 0″ E In a field on the S side of the road from Gaywood to Gayton Road Station About half a mile from the village of Gaywood

Declination

Date	Σ	G M T	δ	δ_0
	h m	h. m.	° ′	° ′
May 20, 1886	+5 12	16 14	17 54 5	17 57 9
(Gaywood) Aug 2, 1888	+3 26	15·47	17 42 8	18 1 7

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 20, 1886	2	17 15	68 17 3	68 17 8
Aug 2, 1888	1	16 39	68 13 2	68 16 8

Horizontal Force

Date	G M T		H	H_0
		h m		
May 20, 1886	V	15 51	1 7664	1 7656
Aug 2, 1888	V	16 5	1 7703	1 7646

101. KING'S SUTTON April 14, 1888; T E T. (61, 83) Lat 52° 1′ 9″, Long 1° 16′ 19″. In a field 25 yards E of the road to Aynho, and parallel to the Great Western line; King's Sutton Church bearing 35° W. of N., and distant 300 yards.

Declination.

Σ.	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 27	13 45	18 35·4	18 51 8

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 35	68 26	68 6 4
2	16 1	68 2 4	

Horizontal Force.

G M T		H	H_0
	h m.		
D	14 55	1 7825	1 7778
V	14 23	1 7832	

102. LAMPETER. May 31, 1886, A. W. R. (60, 74). Lat. 52° 6′ 38″, Long 4° 4′ 36″. In a field to the S. of the town Church bearing a little to the W. of N. Distance from the Church, about two-thirds of that between it and the river, 250 yards from the nearest point of a bend in the river.

Declination.

Σ	G M T	δ	
h m	h. m	° ′	
−1 35	11 2	19 51·7	19 55 3

Inclination.

Needle	G.M T	θ	θ_0.
	h m	° ′	° ′
1	12 34	68 24·6	68 25·5
2	12 59	68 25 1	

Horizontal Force.

G M.T		H	H_0.
	h. m		
D	13 38	1 7569	1·7559
V	11 26	1 7567	

103 LEEDS September 22 and 24, 1886, A W R (60, 74) Lat 53° 50′ 58″, Long 1° 35′ 3″. In the centre of a field behind St Helen's, the residence of O. Eddison, Esq, at Adel The Dip observations were repeated in the same position on December 31, 1888

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Sept 22, 1886	+ 4 2	16 38	19 3 9	19 8 9

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Sept 24, 1886	1	9 58	69 10 7	69 10 8
	2	10 21	69 8 8	
Dec 31, 1888	1	10 57	69 7 5	69 11 4
	2	11 30	69 6 3	

Horizontal Force.

Date	G M T		H	H_0
		h. m		
Sept 22, 1886	V	16 51	1 7098	1 7082

104. LEICESTER June 19, 1886; T E T (61, 83) Lat 52° 37′ 38″, Long 1° 6′ 41″. Twenty yards E of Evington Road, Evington, 1½ mile away, bearing S.E. Railway Station (Midland) ¾ of a mile away, bearing N.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 57	13 4	18 18 0	18 23 6
	16 42	18 21 0	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 48	68 23 0	68 24 0
2	16 16	68 23 5	

Horizontal Force

G M T		H	H_0.
	h m		
D	14 46	1 7548	1 7538
V	13 42		

105. LINCOLN. April 26, 1888; T. E. T (61, 83). Lat. 53° 12′ 27″; Long. 0° 31′ 14″. On the South Common, S of the Cathedral, and about 60 yards from the road.

Declination.

Σ	G M T	δ	δ_0.
h m	h m	° ′	° ′
+3 39	15 35	18 2 2	18 18 9

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 43	68 40 3	68 43
2	16 57	68 38 3	

Horizontal Force.

G.M T		H	H_0.
	h m.		
V	16 0	1 7446	1 7395

106. LLANDUDNO May 14, 1886, A. W R (60, 74) Lat 53° 19′ 5″, Long 3° 50′ 23″ Near the Beach to the W. of the Town, steeple bearing E by N In a little Bay about 300 yards S of the road round Great Orme's Head

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 2 45	9 1	20 49 5	20 51 5
+ 1 37	14 2	20 46 9	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 36	69 12 5	69 12 0
2	13 8	69 10 4	

Horizontal Force.

G M T		H	H_0
	h. m		
D	11 15	1 7087	1 7084
V	10 29	1 7098	

107 LLANGOLLEN. May 10, 1886; A W R (60, 74) Lat 52° 58′ 23″, Long. 3° 10′ 13″. On a hillock in a field on the N. of the town, near the road which runs parallel to the canal, and through which the footpath to Castle Dinas Bran passes Llangollen Church bearing S Bridge over Canal S by E (80 yards distant). Castle Dinas Bran N.E by N

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 30	11 15	20 5 3	20 8 4

Inclination

Needle	G M T	θ	θ_0
	h m		° ′
1	13 39	68 50·5	68 49·4
2	14 9	68 47·3	

Horizontal Force

G T M		H	H_0
	h m		
D	12 40	1·7338	1·7331
V	11 51	1·7340	

108 LLANIDLOES. May 19, 1886, A W R. (60, 74). Lat. 52° 56′ 57″; Long. 3° 32′ 20″. In the garden of the Inn.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 2 31	9 57	19 52·2	19 53·8
+ 2 21	14 53	19 48·7	

Inclination

Needle	G M T	θ	θ_0.
	h. m	° ′	° ′
1	7 3	68 33·2	68 33·8

Horizontal Force

G M T		H	H_0
	h m		
D	17 56	1·7504	1·7501
V	15 13	1·7515	

109 LOUGHBOROUGH April 30, 1888, T E T. (61, 83) Lat. 52° 46′ 37″, Long 1° 13′ 3″ In a field to the N. of the London and North-Western Railway Station, distant 100 yards.

Declination.

Σ	G M T	δ	δ_0
h m – 2 22	h m. 10 3	° ′ 18 2 4	° ′ 18 18 7

Inclination.

Needle	G M T	θ	θ_0
1	h m 10 47	° ′ 68 24 0	° ′ 68 27 7

Horizontal Force

G M T		H	H_0
V	h m 10 16	1 7582	1 7531

110. LOWESTOFT. May 9, 1886, T E T (61, 83) Lat 52° 27′ 54″; Long 1° 43′ 56″ E. In a field W.S.W. of the town 1¼ mile away S John's Church, S. Lowestoft bore E N.E. Kirkley Old Church, E by S.; Cemetery, half a mile away, S E by E

Declination

Σ	G M T	δ	δ_0
h m –1 13 +2 2	h m 11 3 13 34	° ′ 17 21 4 17 20 5	° ′ 17 24 0

Inclination

Needle	G M T	θ	θ_0
1 2	h m 12 21 12 53	° ′ 68 00 0 67 59 7	° ′ 68 00 4

Horizontal Force

G M T		H	H_0
	h m		
D	14 28	1 7804	1 7797
V	11 22	1 7806	

111. MABLETHORPE. May 23, 1886, T. E T. (61, 83). Lat 53° 20′ 20″; Long 0° 15′ 41″ E. In a field 60 yards from the Victoria Road, and 400 yards S of the "Book in Hand" Hotel Flagstaff of coastguard station bearing N E (mag). Convalescent Home bearing S E by S (mag) Railway Station bearing N.W (mag.)

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 20	11 0	18 13 5	18 16 5
+2 15	13 53	18 12 7	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 59	68 42 0	68 42 7
2	13 28	68 42 5	

Horizontal Force.

G M T		H.	H_0
	h m.		
D	12 5	1 7381	1 7370
V	11 20	1 7378	

112 Malvern. A W R (60, 74)

(*a*) Colwall Green, May 22, 1886; Lat 52° 4′ 0″, Long 2° 21′ 40″ About 250 yards S of the bridge over the railway, in the space between two roads which bifurcate about 100 yards away

(*b*) Great Malvern, May 21, 1886, Lat 52° 6′ 22″, Long 2° 18′ 10″ At the west end of Barnard's Green, about 20 yards S of the south side of a triangle formed by three roads, near a pond

(*c*) Malvern Wells, May 22, 1886, Lat 52° 4″ 49″; Long 2° 18′ 30″ In a field to the N of the Hanley Road from Malvern Wells; and N W of the cross-roads which are E of the Midland Station, about 10 yards from the road.

(*d*) Mathon, May 22, 1886; Lat. 52° 6′ 40″, Long 2° 22′ 6″ In a field to the S of the road to Mathon, a large pond on the other side of the road, 250 yards N.E. by N., small church at West Malvern nearly due E.

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
May 21 Great Malvern	+ 1 33	12 40	19 31 3	19 33 0
	+ 4 11	16 57	19 28 0	
,, 22 Mathon	+ 1 19	12 45	18 43 1	18 46 5
,, 22 Colwall Green	+ 2 9	14 37	19 0 3	19 3 6
	+ 3 5		19 0 1	
,, 22 Malvern Wells	+ 4 12	16 32	19 19 0	19 22 4

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 21 Great Malvern	1	12 57	68 14 8	68 14 2
	2	16 34	68 12 3	

Horizontal Force

Date	G M T		H	H_0
		h m		
May 21 Great Malvern	D	14 28	1 7691	1 7687
	V	13 12	1 7700	
, 22 Mathon	V	13 1	1 7664	1 7655
,, 22 Colwall Green	V	14 53	1 7636	1 7627
,, 22 Malvern Wells	V	16 47	1 7691	1 7682

113. MANCHESTER June 20 and 21, 1886; T. E. T (61, 83). Lat. 53° 27′ 40″; Long. 2° 16′ 54″. Old Trafford On the lawn of the Royal Botanic Gardens, near the Winter House Second series of Dip Observations, close to the lake, and about 100 yards S.E. of the first station.

Declination.

Date	Σ	G M T	δ	δ_0
June 20	h m + 2 25	h m 14 13	° ′ 19 12 5	° ′ 19 16 7

Inclination.

Date	Needle	G M T	θ	θ_0
June 20	1 2	h m 13 3 12 13	° ′ 69 3 2 69 3 0	° ′ 69 3 9

Horizontal Force

Date	G M T		H	H_0
June 21 ,, 20 ,, 21	D V V	h m 16 33 13 51 17 8	1 7138 1 7133	1 7128 1 7123

114. MANTON. April 21, 1888; T. E T. (61, 83). Lat. 52° 37′ 32″, Long 0° 41′ 51″. On the hill about 400 yards from the Railway Station.

Declination.

Σ	G M T	δ.	δ_0
h m + 3 16	h. m 15 13	° ′ 18 5·2	° ′ 18 21 7

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 1	68 13·8	68 17·1
2	16 46	68 13·0	

Horizontal Force.

G M T		H	H_0
	h. m		
D	16 15	1·7711	1·7661
V	15 34	1·7714	

115. MARCH. July 27, 1888, A. W. R. (60, 74) Lat 52° 32′ 0″; Long 0° 4′ 0″ E. About a mile from March Church, on the south side of the road which runs west towards Barrow Moor

Declination.

Σ	G M T	δ	δ_0
h m	h. m	° ′	° ′
+3 10	15 36	17 43·9	18 2·8

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 46	68 6·8	68 10·3
2	18 14	68 6·0	

Horizontal Force.

G M T.		H	H_0.
	h m		
D	16 36	1·7780	1·7719
V	15 53	1·7772	

116. MELTON MOWBRAY. T E T (61, 83).

(*a*) April 22, 1888 , Lat 52° 44′ 54″, Long 0° 52′ 46″. In a field about a mile S (mag) of the Midland Station 20 yards E of the Sandy Road.

(*b*) April 30, 1888 ; Lat 52° 45′ 47″ , Long 0° 53′ 20″ In a field by the river, 400 yards W of the Church Egerton Lodge bearing about N.

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
April 22	− 0 30	11 2	18 55 0	
	− 0 12	11 2	18 54 3	19 10 9
	− 0 5	13 8	18 53 7	
,, 30	+ 5 22	17 44	18 47 5	19 4 2

Inclination.

Date	Needle	G M.T	θ	θ_0
		h m	° ′	° ′
April 22	1	13 43	68 29 0	68 31 5
	2	14 0	68 26 7	
,, 30	1	18 20	68 36 1	68 39 7
	2	18 32	68 36 0	

Horizontal Force

Date	G M T		H	H_0
		h m.		
April 22	D	12 26	1 7673	1 7620
	V	11 17	1 7670	
	V	12 55	1·7669	1 7620
			1 7674	
,, 30	V	17 55	1 7466	1 7415

117. MILFORD HAVEN. July 29, 1887, A W. R. (60, 74) Lat 51° 42′ 24″; Long. 4° 56′ 47″. In a field belonging to Mr CARON 100 yards S W by W. of the main road which leads inland, and about 50 yards N W. of a short lane leading to Mr CARON's house. Barracks due S on the other side of the inlet; about $\frac{1}{3}$ of a mile N N W of the quay

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 13	10 29	19 56 5	20 8 8
+ 1 51	13 58	19 56 8	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	11 46	68 7 9	68 9 9
2	12 12	68 6 6	

Horizontal Force

G M T		H	H_0
	h m		
D	12 58	1 7844	1 7808
V	13 41	1 7843	

118. NEWARK April 27, 1888, T E T. (61, 83) Lat 53° 4′ 35″, Long. 0° 48′ 43′ In the middle of the Castle grounds

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 1 29	14 18	18 29 5	18 46 2

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 8	68 30 5	68 33 4
2	12 25	68 28 9	

Horizontal Force

G M T		H	H_0
	h m		
D	13 1	1 7505	1 7464
V	14 0	1 7525	

119 NEWCASTLE September 17, 1886, A W R (60, 74). Lat 54° 59′ 25″, Long 1° 39′ 44″. At the W end of the moor, about 200 yards from the road. First Station about 150 yards from a Colliery, second, about 200.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 59	11 37	19 25 6	19 30 3
+ 1 44	13 32	19 23 9	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 10	69 48 0	69 49 5
2	14 17	69 49 0	

Horizontal Force.

G M T		H.	H_0.
	h m.		
V	12 22	1 6681	1 6665

120 NORTHAMPTON. October 7, 1887, T E T (61, 83) Lat 52° 13′ 1′, Long 0° 53′ 52″ In a field on the W side of the High Road, and nearly opposite Queen Eleanor's Cross, distant about 30 yards

Declination

Σ	G M T	δ	δ_0
h m − 1 48 + 1 25	h m 10 27 13 37	° ′ 18 29 7 18 30 5	° ′ 18 41 7

Inclination

Needle	G M T	θ	θ_0
1 2	h m. 12 15 12 47	° ′ 68 6 3 68 7 2	° ′ 68 9 4

Horizontal Force

G M T		H	H_0
D V	h m 11 22 10 46	1 7708 1 7713	1 7671
V	13 51	1 7701 1 7705	1 7664

121. NOTTINGHAM April 28, 1888, T E T (61, 83) Lat 52° 57′ 0″, Long 1° 9′ 20′ To the W. of Tunnel Road, due W of the Castle, towards the Bowling Green

Declination.

Σ	G M T	δ	δ_0
h. m − 0 28 + 3 20	h. m 9 35 15 9	° ′ 18 27 1 18 29 4	° ′ 18 44 9

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 52	68 33 7	68 37 6
2	16 9	68 34 1	

Horizontal Force

G M T		H	H_0
	h m		
D	11 0	1 7520	1 7471
V	9 54	1 7524	
		1 7515	
V	10 24	1 7524	1 7468

122 OXFORD May 5 and 6, 1886, A W R (60, 74) Lat. 51° 45′ 34″; Long. 1° 15′ 6″ In the Parks, near the Physical Laboratory Ventilating flue in the middle of the Physical Lecture Room bears magnetic S., Cricket Pavilion E by N., 29 paces W from Plantation

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
May 5	+5 15	17 51	18 29 0	18 33 7
„ 6	−1 15	11 5	18 32 2	

Inclination.

Date	Needle	G M T		θ_0
		h m	° ′	° ′
May 6	1	14 24	67 57 1	67 57 5
	2	14 55	67 56 8	

Horizontal Force

Date	G M T.		H.	H_0
		h m		
May 6	D	12 18	1 7900	1 7890
	V	11 34	1 7894	

123 PETERBOROUGH May 21, 1886, T E T (61, 83) Lat 52° 34′ 16″, Long 0° 15′ 57″. In the meadows to the W of G N Railway Station, and about three quarters of a mile from it, 200 yards N of the River Nen, and 50 yards S of the Thorpe Road

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 23	12 30	18 18 6	18 21 9
+4 10	15 40	18 18 5	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 50	68 14 3	68 14 8
2	15 18	68 14 1	

Horizontal Force

G M T		H	H_0
	h m		
D	13 56	1 7692	1 7692
V	12 52	1 7708	

124 PLYMOUTH April 7, 1887, T E T (61, 83) Lat 50° 21′ 59″, Long 4° 8′ 35″. On the West Hoe, S of the Drake Monument W end of Breakwater 40° W. of S, E end of Breakwater, 10° W of S Old Eddystone Tower, 42° E of S, 70 yards away Grand Hotel, 55° W of N

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 33	11 13	19 22 6	19 31 6
	14 25	19 22 8	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 27	67 12 9	67 14 7
2	13 56	67 12 1	

Horizontal Force

G M T		H	H_0
	h m		
D	12 5	1 8343	1 8309
V	11 39	1 8331	

125 Port Erin (Isle of Man) August 8, 1887, T E T (61, 83) Lat 54° 5′ 4″, Long 4° 46′ 7″ On the S side of the Harbour near the Breakwater.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 5	11 46	20 40 3	20 55 4
+2 37	13 47	20 44 2	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 26	69 46 2	69 48 1
2	14 45	69 45 7	

Horizontal Force

G M T		H	H_0
	h m		
D	15 36	1 6714	1 6678
V	12 38	1 6712	

126 PRESTON. August 24, 1886, A W R (60, 74) Lat 53° 42′ 46″; Long 2° 43′ 18″ On Farrington Moor, about 2 miles S W of the town, and 200 yards from a railway crossing. Close to cross-roads running N. and S and E and W Spire of St James' Layland, S. by W, ½ W Farrington Factory, E S E

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 1 45	14 25	19 49 7	19 52 3
+ 4 48	17 24	19 45 2	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 29	69 13 7	69 14 7

Horizontal Force

G M T		H	H_0
	h m		
D	15 30	1 7070	1 7053
V	14 48	1 7064	

127 PURFLEET April 14, 1888; A W R (60, 74) Lat. 51° 29′ 7″, Long 0° 14′ 58″ E. Near a footpath which leads to the woods behind the village

Declination

Σ	G M T	δ	δ_0
h m	h. m	° ′	° ′
+ 1 32	15 30	17 38 9	17 54 5
+ 4 44	16 55	17 37 4	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 40	67 27 2	67 30 9
2	15 6	67 27 8	

Horizontal Force

G M T		H	H_0
	h m		
D	16 21	1 8178	1 8134
V	15 48	1 8190	

128 . PWLLHELI May 15, 1886, A W R (60, 74) Lat 52° 52′ 55″, Long 4° 24′ 35″ On the beach to the S of the town and harbour About 100 yards E of the road which leads to the beach. The first declination was taken on the shore of the harbour about 100 yards off

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 46	11 0	20 40 3	20 41 9
+ 2 9	14 6	20 36 9	

Inclination

Needle	G M T	θ	θ_0
	h. m	° ′	° ′
1	12 15	68 49 7	68 50 9
2	12 48	68 51 0	

Horizontal Force

G M. T		H	H_0
	h m		
V	13 54	1 7415	1 7407

129 RAMSEY (Isle of Man) August 3, 1887, T E T (61, 83) Lat. 54° 19′ 22″, Long 4° 22′ 48″ On the N shore of the harbour and about 60 yards W S W of the end of the pier Ramsey Church bearing due S, end of North pier bearing E S E.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 31	11 19	20 40 9	20 54 9
+ 1 52	13 54	20 42 8	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 22	69 54 0	69 55 0
2	12 44	69 51 6	

Horizontal Force

G M T		H	H_0
	h m		
D	14 47	1 6653	1 6617
V	11 40	1 6651	

130. RANMORE May 21, 1888, T E T. (61, 83). Lat 51° 14′ 38″, Long 0° 21′ 36″ On Ranmore Common, half-way between the Post Office and the Church, and 20 yards N. of the road.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 57	11 20	17 53 1	18 8 9
+ 1 18	13 6	17 49 9	

Inclination.

Needle	G M T	θ	θ_0
	h m.	° ′	° ′
1	9 10	67 16 7	67 20 0
2	10 27	67 16 1	

Horizontal Force

G M T		H	H_0
	h m		
D	12 11	1 8310	1 8261
V	11 36	1 8318	

131 READING. T E T (61, 74).

(*a*) April 20, 1886 Lat 51° 27′ 57″; Long 0° 58′ 46″. On the river-bank opposite Caversham Church.

(*b*) May 30, 1888 Lat 51° 27′ 56″, Long 0° 58′ 50″. Close to the former station

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
April 20, 1886	− 1 21	11 14	18 13 6	18 15 2
	+ 2 24	14 1	18 12 7	
	+ 4 12			
May 30, 1888	+ 1 4	12 1	17 53 9	18 11 6

Inclination

Date	Needle	G M T.	θ	θ_0
		h m	° ′	° ′
April 20	1	15 47	67 41 6	67 40 7
	2	16 45	67 38 9	

Horizontal Force.

Date	G M.T		H	H_0
		h m		
April 20	D	12 40	1 8112	1 8110
	V	13 27	1 8121	
May 30	V	12 15	1 8194	1 8141

132 REDCAR September 18, 1886, A W R (60, 74)

(a) Lat 54° 34′ 33″; Long 1° 0′ 22″ In a field on the spur of the hills due S of Marske Church, and about three-quarters of a mile from it, about 20 yards E of the road which runs inland from Marske

(b) Lat 54° 35′ 46″ Long 1° 1′ 15″ In a straight line between Marske Church and the Sea About 250 yards from the Church.

Declination

Σ	G M T	δ	δ_0
h m + 1 21 + 3 45	h m 11 22 15 11	° ′ 19 2 0 18 58 0	° ′ 19 5 6

Inclination.

Needle	G M T	θ	θ_0
1	h m 14 2	° ′ 69 30 5	° ′ 69 31 5

Horizontal Force

G M T		H	H_0
D V	h m 12 15 11 42	1 6865 1 6861	1·6847

133. RYDE April 28, 1886, T E T. (61, 74) Lat 50° 43′ 13″, Long 1° 10′ 44″. In a field W of Pellshurst, near Ryde Parish Church bore E by N, Mr. Hunter's house W.N.W About 40 yards N of the road

Declination

Σ	G M T	δ	δ_0
h m − 1 29 + 2 6	h. m 11 4 13 50	° ′ 18 1 1 17 57 8	° ′ 18 1 6

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 24	67 6 1	67 7 8
2	12 58	67 8 5	

Horizontal Force.

G M T		H	H_0
	h m		
D	14 39	1 8398	1 8391
V	11 22	1 8399	

134 St Cyres April 26, 1886, T. E T. (61, 74) Lat. 50° 46′ 30″, Long 3° 35′ 26″ In the middle of a field to the S. of the S W Railway, near the road, and N. of the river Avon.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 2 43	13 25	19 26 4	19 28 6

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 27	67 26 4	67 26 2
2	14 6	67 25 1	

Horizontal Force

G M T		H	H_0
	h. m.		
D	15 27	1 8269	1 8260
V	13 10	1 8266	

135 St. Leonards. August 9, 1886, A W R (60, 74) Lat 50° 50′ 56″, Long 0° 31′ 5″ E On the Common N of the road three-quarters of a mile W of Bo Peep Station About 100 yards N W of Coastguard Station.

Declination

Σ	G M T	ϵ	ϵ_0
	h. m	° ′	° ′
+ 2 41	15 13	17 21·4	17 24·8
+ 5 35		17 19·0	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 21	66 59·1	66 58·9
2	17 58	66 56·7	

Horizontal Force

G M T		H	H_0
	h m		
D	16 17	1·8452	1·8437
V	15 37	1·8449	

136 Salisbury April 27, 1886, T E T. (61, 74), Lat 51° 5′ 3″; Long 1° 48′ 6″ To the N of the town, and S of Old Sarum, in a field E of the Avon, 60 yards from the Old Castle or Stratford Road.

Declination

Σ	G M T	ϵ	ϵ_0
h m	h m	° ′	° ′
− 1 3	10 46	18 22·5	18 23·9
+ 1 59	13 46	18 21·0	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 38	67 25 9	67 25 6
2	13 15	67 24 4	

Horizontal Force

G M T		H	H_0
	h m		
D	14 35	1 8253	1 8242
V	11 34	1 8246	

137 SCARBOROUGH September 21, 1886, A W R (60, 74), Lat 54° 15′ 55″, Long 0° 23′ 24″ In a field near the edge of the cliff on the south side of the town About 200 yards from the residence of Alderson Smith, Esq

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 2 57	13 17	18 43 4	18 48 3

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 31	69 14 3	69 15 6
2	17 9	69 15 2	

Horizontal Force.

G M T.		H	H_0
	h m		
D	12 38	1 7032	1 7017
V	14 19	1 7035	

138 SHREWSBURY May 8, 1886, A W. R (60, 74) Lat 52° 42′ 11″, Long 2° 45′ 36″. In the School Grounds, about 100 yards S W. by W. of the new buildings. On grass W of road and E of the Cricket Ground.

Declination

Σ	G M T	δ	δ_0
h m – 1 26	h m 11 7	° ′ 19 38 1	° ′ 19 41 2

Inclination

Needle	G M T	θ	θ_0
1	h m 12 40	° ′ 68 35 9	° ′ 68 36 4

Horizontal Force

G M T		H	H_0
	h m		
V	11 27	1 7352	1 7344
V	11 53	1 7349	1 7341

139 SOUTHEND May 24, 1887, T. E T (61, 83) Lat 51° 32′ 49″, Long 0° 43′ 11″ E In a field, about 20 yards from the point where the Sutton Road bends at right angles towards Prittlewell, half a mile N of Southend Station, and about half a mile E of Prittlewell Church

Declination

Σ	G M T	δ	δ_0
h m +2 32	h m 14 3	° ′ 17 34 0	° 17 44 4

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 12	67 28 9	67 30 8
2	14 56	67 28 5	

Horizontal Force.

G M T		H	H_0
	h m		
D	13 12	1 8135	1 8112
V	13 46	1 8152	

140 SPALDING. April 26, 1888 ; T E. T (61, 83). Lat 52° 47′ 5″; Long 0° 8′ 48″. Seventy yards W. of the church.

Declination.

Σ	G M T	δ	δ_0
h. m	h m	° ′	° ′
−1 5	11 10 5	17 34 9	17 51 6

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 36	68 19 9	68 23 1
2	12 50	68 19 4	

Horizontal Force

G M T		H	H_0
	h. m		
D	12 0	1 7576 }	1 7512
V	11 25	1 7541 }	
V	11 25	1 7487	1 7436

141 STOKE-ON-TRENT September 13, 1887, A W R (60, 74) Lat 52° 57′ 47″, Long 2° 12′ 20″ In Trentham Park, about 350 yards S W of the house.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+2 29	11 22	19 11 8	19 22 7
	14 14	19 10 0	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 33	68 41 4	68 43 6
2	13 58	68 40 5	

Horizontal Force

G M T		H	H_0
	h m		
D	12 35	1 7443	1 7404
		1 7439	
V	11 37	1 7435	1 7398
V	11 55	1 7436	

142. SUTTON BRIDGE July 31, 1888, A W R (60, 74). Lat 52° 45′ 40″, Long 0° 11′ 50′ E In a field on the E. bank of the River, about a quarter of a mile from the Railway

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 39	14 8	17 36 2	17 54 1
+5 34	17 24	17 34 3	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 30	68 18 2	68 21 1
2	15 54	68 16 8	

Horizontal Force.

G M T		H	H_0
	h m		
V	14 29	1 7677	1 7620

143. SWANSEA. May 28, 1886; A W. R. (60, 74) Lat 51° 36′ 50″, Long 3° 58′ 57″. In Cwmdonkin Park, one and a half mile N W of the town, and 150 yards N W. of the Reservoir

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 4	11 39	19 43 2	19 45 6
+2 22	13 15	19 41 0	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 22	67 59 5	67 59 7
2	16 59	67 58 6	

Horizontal Force

G M.T		H	H_0
	h m		
D	14 0	1 7939 1 7943	1 7932
V V	12 26 13 0	1 7939 1 7941	1 7931

144 SWINDON April 21, 1886, T E T (61, 74) Lat 51° 34′ 12″, Long 1° 46′ 54″ In a field N N E (mag) of railway station, about a quarter of a mile from railway line, and near a farm

Inclination

Needle	G M T	θ	
	h m	° ′	° ′
	10 44	67 49 8	67 51 4
	11 18	67 52 1	

Horizontal Force

G M T		H	H_0
	h m		
D	12 5	1 7934	1 7930
V	9 48	1 7941	

145 TAUNTON April 23, 1886, T E T (61, 74) Lat 51° 0′ 52″, Long 3° 5′ 32″ In a field to the S E. of the town, King's College, about quarter of a mile away, bearing S W. by W. (mag) Trinity Church a quarter of a mile away, N N W (mag)

Declination

Σ	G M T	δ	δ_0
h m	h. m	° ′	
−2 35	12 32	19 8 8	19 10 7
+1 30		19 8 3	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	10 34	67 32 0	67 32 7
2	11 11	67 32 5	

Horizontal Force

G M T		H	H_0
	h m		
D	13 15	1 8151 1 8156	1 8146
V V	11 50 12 7	1 8169 1 8170	1 8162

146 THETFORD September 20, 1887, A W R (60, 74) Lat 52° 23′ 57″, Long 0° 43′ 12″ E In the Hundred-acre Field, on the estate of the Maharajah Duleep Singh

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 3 32 + 3 48	13 44	17 29 8 17 28 8	17 41 2

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1 2	16 5 16 31	67 58 9 67 58 7	68 1 4

Horizontal Force.

G M T		H.	H_0
	h m.		
D V	14 43 13 55	1 7831 1 7828	1 7791

147 THIRSK September 20, 1886; A W R (60, 74) Lat 54° 14′ 35″, Long 1° 20′ 38″ In the Kilvington fields, about half a mile N of the Church About 50 yards W S W of the high road to York, and between it and the stream

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 11	11 16	19 20 6	19 21 7
+ 2 21	15 14	19 13 1	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 51	69 27 6	69 28 3
2	14 44	69 26 8	

Horizontal Force

G M T		H	H_0
	h m		
D	12 42	1 6930	1 6912
		1 6926	
V	11 35	1 6926	
V	12 1	1 6932	

148 TILNEY August 1, 1888, A. W. R Lat 52° 42′ 10″, Long. 0° 17′ 11″ E In a field behind a farm house on the main road from Wisbech, and about half way between Tilney Buck and the point where the road crosses the Five-Mile Drain.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 1 27	12 10	17 39·2	17 58 1

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 57	68 17 1	68 20 7

Horizontal Force

G M T		H	H_0
	h m		
D	11 20	1 7707	1 7655
V	12 1	1 7717	

149. TUNBRIDGE WELLS September 27, 1887, T E T (61, 83) Lat 51° 7′ 36″, Long 0° 15′ 37″ E On the Common, about 200 yards from the London Road; Trinity Church bearing 45° E of N (mag) St Peter's Church bearing 95° E. of N. (mag). St Mark's Church bearing 35° W. of S. (mag)

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 49	10 31	17 29 1	17 41 3
+1 39	13 52	17 29 3	

Inclination.

Needle.	G M T	θ	θ_0
	h m	° ′	° ′
1	12 9	67 7 8	67 10 8
2	12 32	67 8 3	

Horizontal Force.

G M T.		H.	H_0
	h. m		
D	11 34	1 8340	1 8299
		1 8335	
V	10 54	1 8333	1 8296
V	14 7	1 8335	

150 WALLINGFORD May 29, 1888, T E T (61, 83) Lat 51° 36′ 3″, Long 1° 7′ 21″ Below Wallingford Bridge, 70 yards away, on the left bank, opposite the Church

Declination

Σ	G M T	δ	δ_0
h m + 0 49	h m 13 18	° ′ 18 3 9	° ′ 18 21 6

Inclination

Needle	G.M T	θ	θ_0
	h m	° ′	° ′
1 2	14 48 15 1	67 44 9 67 44 2	67 48 4

Horizontal Force

G M T		H	H_0
D V	h m 14 7 13 31	1 8040 1 8039	1 7986

151. WEYMOUTH April 6, 1887, T. E. T (61, 83) Lat 50° 36′ 16″, Long 2° 26′ 52″. On the S side of the Nothe, close to the shore. End of Portland Breakwater, 28° E of S (mag.). Bincleaves House, 60° W of S. (mag) Coastguard Station on Nothe, 55° W. of N (mag)

Declination

Σ	G M T	δ	δ_0
h. m − 1 26 + 2 4	h m 11 17 14 3	° ′ 18 37 3 18 38 6	° ′ 18 46 7

Inclination

Needle.	G M T	θ	θ_0
1 2	h m 13 25 13 46	° ′ 67 10 3 67 8 9	° ′ 67 11 7

Horizontal Force

G M T		H	H_0
	h m		
D	12 24	1 8357	1 8329
V	11 48	1 8358	

152 **Wheelock.** February 18, 1887, T E T. (83) Lat 53° 7′ 50″, Long 2° 22′ 30″ In the field behind Wettenhall Cottage

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 37	68 46 8	68 49 0
2	14 42	68 47 9	

153. **Whitehaven** August 27, 1886; A W R (60, 74) Lat 54° 32′ 37″, Long 3° 34′ 20″. In Midgey Mount, S E of the Castle, and in the grounds About 300 yards from Cukickle Gate

Declination.

Σ	G M T	δ	δ_0
h m	h. m	° ′	° ′
− 1 40	11 2	20 35 5	20 41 6
+ 2 22	14 21	20 36 6	

Inclination

Needle	G M T	θ	θ_0
	h. m	° ′	° ′
1	13 19	69 45 8	69 47 6
2	13 52	69 47 7	

Horizontal Force

G M T.		H	H_0
	h. m		
D	12 14	1 6740	1·6727
V	11 23	1 6743	

154 WINDSOR May 31, 1888, T E T (61, 83) Lat. 51° 29′ 22″, Long 0° 37′ 25″ To the W of the town, near the point where the river makes a bend towards the Railway Bridge

Declination

Σ	G M T	δ	δ_0
h m + 3 47	h m 16 10	° ′ 18 12·2	° ′ 18 29·9

Inclination

Needle	G M T	θ	θ_0
1 2	h m 17 20 17 30	° ′ 67 35·5 67 34·3	° ′ 67 38·8

Horizontal Force

G M T		H	H_0
D V	h m 16 48 16 22	1·8132 1·8142	1·8084

155 WISBECH July 31 1888, A W R (60, 74) Lat 52° 40′ 30″, Long. 0° 8′ 20″ E In a field belonging to Mr SHARP to the S of the Leverington Road About 50 yards E of an octagonal pigeon house

Declination

	G M T		
h m	h m 9 57	° ′ 17 46·7	° ′ 18 5·6

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 17 53 18 12	° ′ 68 17·2 68 13·7	° ′ 68 19·0

Horizontal Force

G M T		H	H_0
	h m		
D	16 47	1 7707	1 7653
V	16 1	1 7713	

156 WORTHING September 29, 1887, T. E T (61, 83) Lat 50° 48′ 35″, Long 0° 23′ 22″ In a field adjoining a new road W of West Worthing, about 300 yards from the beach and three-quarters of a mile from the Pier, about half-way between the railway line and the beach Tarring Church 10° E of N (mag) S Botolph's (W Worthing) 55° E of N (mag)

Declination

Σ	G M T	δ	δ_0
h. m	h m	° ′	° ′
−1 10	11 6	17 46 2	17 59 0
+2 56	14 12	17 47 6	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 46	67 4 3	67 6 4
2	13 12	67 2 9	

Horizontal Force

G M T		H	H_0
	h m		
D	11 59	1 8443	1 8403
		1 8439	
V	11 20	1 8445	1 8401
V	14 25	1 8433	

Descriptions of Irish Stations

157 Armagh Observatory August 15, 1887, T E T (61, 83) Lat. 54° 21′ 10″, Long 6° 38′ 53″ In a field belonging to the Observatory about 100 yards S of the house, and close to the position on which Lloyd and Ross had made observations.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 14	14 13	22 3 8	22 16 5
+3 15	16 1	22 2 5	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 9	69 55 3	69 57 6
2	15 29	69 55 2	

Horizontal Force

G M T		H	H_0
	h m		
D	16 40	1 6659	1 6625
V	14 26	1 6656	

158 Athlone T E T (61, 83)

(*a*) May 8, 1887 Lat 53° 26′ 8″, Long. 7° 57′ 22″ On the N. bank of the Shannon and about 50 yards from the river's edge About 1 mile from the town

(*b*) May 9, 1887. In the middle of the kitchen garden, 150 yards behind the Prince of Wales' Hotel, and near the Protestant Church

Declination

Date	Σ	G M T	δ	δ_0
	h. m	h m	° ′	° ′
May 8	+1 41	12 39	22 16 5	22 26 7
,, 9	−3 44	9 9	22 16 8	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 8	1 2	13 25 13 55	69 38 5 69 37 6	69 40 0

Horizontal Force

Date	G M T		H	H_0
		h m		
May 8	D V	15 18 12 14	1 6874 1 6885	1 6852

159 BAGNALSTOWN September 8, 1887; A. W. R (60, 74) Lat 52° 41′ 37″, Long. 6° 57′ 36″ In a field to the W of the Enniscorthy-road, about a quarter of a mile from the point where it crosses the railway

Declination

Σ.	G M T	δ	δ_0
h m	h m	° ′	° ′
− 3 27	9 9	21 43 3	21 55 0

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	9 47	69 2 2	69 5 1

Horizontal Force.

G.M T.		H.	H_0
	h m		
V	9 20	1 7242	1 7208

160 BALLINA September 2, 1887, A W. R (60, 74). Lat 54° 7′ 10″, Long 9° 9′ 0″ About 400 yards S of Belleck Castle. In a field on the west side of the road which runs north from the town, about a quarter of a mile from the river.

Declination.

Σ	G M T	δ	δ_0
h m −2 1	h m 11 6	° ′ 23 15 3	° ′ 23 26 9

Inclination

Needle.	G M T.	θ	θ_0
1 2	h. m 13 20 13 42	° ′ 70 22 8 70 23 8	° ′ 70 25 8

Horizontal Force.

G M T		H	H_0
D V	h m 12 3 11 23	1 6356 1 6357	1 6323

161 BALLYWILLIAM. September 8, 1887; A W R (60, 74) Lat 52° 26′ 37″; Long 6° 52′ 0″ In a field W of the road, nearly S. of the station, and about one-third of a mile from it

Declination

Σ	G.M T	δ	δ_0
h m. −0 49	h m 11 56	° ′ 21 25 6	° ′ 21 37 3

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 49	69 2 7	69 5 6

Horizontal Force

G M T		H	H_0
	h m		
D	13 7	1 7247 1 7268	1 7223
V V	12 13 12 29	1 7255 1 7258	1 7222

162 BANGOR August 18, 1887, T E T. (61, 83), Lat 54° 39′ 57″; Long. 5° 39′ 50″. Station 200 yards E (mag) from the Harbour.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 27 + 2 5	11 39 13 53	21 30 6 21 31 5	21 44 4

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1 2	12 48 13 15	69 59 7 69 58 1	70 1 3

Horizontal Force

G M T		H	H_0
	h m		
D	15 7	1 6636 1 6633	1 6601
V V	11 53 14 10	1 6624 1 6632	1 6595

163 BANTRY August 9, 1887, A. W R (60, 74), Lat. 51° 40′ 25″, Long 9° 28′ 53″ In a field opposite to Ivy Cottage, about a mile and a quarter W. of the town

Declination

Σ	G M T	δ	δ_0
h m	h. m	° ′	° ′
+ 2 35	15 53	22 29 7	22 40 3
+ 5 50	18 13	22 27 8	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	17 9	68 43 1	68 46 0
2	17 41	68 42 8	

Horizontal Force.

G M T		H	H_0
	h m		
D	18 59	1 7561	1 7529
V	16 10	1 7561	

164. CARRICK-ON-SHANNON May 12, 1887, T E T. (61, 83) Lat 53° 56′ 36″; Long 8° 5′ 46″ To the N of the Shannon Bridge, about 70 yards from the road, and close to the Quay-side, nearly opposite the door of the Royal Irish Constabulary Office

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−2 49	10 29	22 52 7	23 3 1
−1 24		22 53 1	

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 11 55 11 58	° ′ 69 51 7 69 50 9	° ′ 69 53 3

Horizontal Force.

G M T		H	H_0
V	h m 10 42	1 6727	1 6700

165. CASTLEREAGH May 9, 1887, T. E T (61, 83). Lat 53° 45′ 34″, Long 8° 29′ 7″ About 150 yards W of the Castlereagh Railway Station.

Declination

Σ	G M T	δ	δ_0
h m —0 32	h m 12 22	° ′ 23 1 0	° ′ 23 11 1

Inclination

Needle	G M T	θ	θ_0
1 2	h m 13 16 13 36	° ′ 69 53 9 69 54 7	° ′ 69 56 3

Horizontal Force.

G M T		H	H_0
V	h m 12 35	1 6743	1 6716

166 CAVAN. September 6, 1887, A W R (60, 74) Lat 53° 59′ 4″, Long 7° 21′ 43″. In a field about 100 yards N of the College.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 3 29	9 13	22 26 1	
+ 1 29	14 10	22 26 7	22 37 8
	15 45	22 27 4	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 18	69 54 6	69 57 3
2	12 52	69 55 1	

Horizontal Force

G M T		H	H_0
	h. m		
D	13 22	1 6660	1 6629
V	9 24	1 6666	

167. CHARLEVILLE August 6, 1887; A W. R (60, 74) Lat. 52° 20′ 54″, Long. 8° 40′ 22″. In a field about halfway between the Station and the Town.

Declination.

Σ	G M T	δ	δ_0.
h. m.	h. m	° ′	° ′
+ 3 56	15 6	22 19 4	22 30 8
+ 4 37	17 52	22 19 1	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 43	69 4 1	69 5 3
2	17 30	69 0 8	

Horizontal Force

G M T		H	H_0
	h m		
V	15 36	1 7261	1 7226
V	15 20	1 7255	

168 CLIFDEN. August 30, 1887, A. W R (60, 74) Lat 53° 29′ 35″, Long 10° 4′ 10″ In a field on the S side of the road which runs from Clifden to Ballymaconry Three-quarters of a mile from the upper gate to the grounds of Clifden Castle.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 35	11 30	24 8 2	24 20 7
+1 49	14 44	24 9 2	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 24	70 2 3	70 4 8
2	13 48	70 1 9	

Horizontal Force.

G M T.		H	H_0
	h m		
D	12 20	1 6660	1 6631
V	11 40	1 6668	

169. COLERAINE August 21, 1887, T E T. (61, 83) Lat 55° 7′ 31″, Long 6° 40′ 24″ On the West Side of the River and 200 yards S of the Bridge

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	
+ 0 39	13 32	22 24 3	22 36 9
	15 23	22 22 5	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 50	70 45 2	70 47 7

Horizontal Force

G M T		H	H_0
	h m		
D	14 35	1 6123	1 6091
		1 6125	
V	14 2	1 6111	1 6080
V	15 8	1 6115	

170 COOKSTOWN JUNCTION August 19, 1887, T E T. (61, 83). Lat 54° 44′ 56″; Long. 6° 16′ 1″. About 300 yards W of Cookstown Junction Station, on the road to Randalstown.

Declination

Σ.	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 0 28		21 19 6	21 32 8
+ 1 38	13 26	21 19 3	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 38	69 31 8	69 34 5
2	15 4	69 32 0	

Horizontal Force

G M T		H	H_0
	h m		
D	15 43	1 6862	1 6830
V	13 52	1 6865	

171 CORK August 8, 1887, A. W R (60, 74). Lat 51° 53′ 30″, Long 8° 29′ 30″. In the quarry behind Queen's College. The second Declination was taken in a field about 250 yards to the E of the first station. The difference between the two results is larger than was anticipated.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 3 44	10 40	22 3 8	22 14 3
	14 36	22 1 7	
+ 2 49	15 49	22 10 3	22 21 9

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 52	68 44 2	68 46 4
2	13 25	68 43 2	

Horizontal Force.

G M.T		H	H_0
	h m.		
D	12 6	1 7537	1 7506
V	11 26	1 7540	

172 DONEGAL August 23, 1887, T E T (61, 83) Lat 54° 39′ 5″, Long 8° 6′ 57″
On the N bank of the Port, nearly opposite the ruins of the Abbey, which bore S.S W Village bore due E.

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 0 11	11 42	23 6 9	23 20 1
+ 1 53	15 31	23 6 4	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 46	70 12 5	70 15 3
2	14 32	70 13 1	

Horizontal Force

G M T		H	H_0
	h m		
D	15 7	1 6482	1 6449
V	12 2	1 6483	

173. DROGHEDA. May 15, 1887, T E T (61, 83) Lat 53° 42′ 48″, Long 6° 22′ 4″.
In a field S S W of Rathmullan House, about 1 mile west of Drogheda Cathedral, and on the S side of the River Boyne

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 38	11 29	21 43 8	21 54 7
+ 1 0	13 4	21 42 3	

Inclination

Needle	G M T	θ	θ_0
	h. m	° ′	° ′
1	13 59	69 34·5	69 36·3
2	14 14	69 33·8	

Horizontal Force

G M T		H	H_0
	h m		
D	12 19	1·6921	1·6894
V	11 11	1·6921	
D		1·6924	1·6897
V	12 50	1·6924	

174. Dublin (Trinity College) May 5 and 6, 1887, T E T (61, 83) Lat. 53° 20′ 35″, Long 6° 15′ 24″. 1st Station On the path before the Magnetic House, between it and the University Buildings About 20 yards from the former. 2nd Station On the middle walk and almost due E of the Provost's House, to the N.E. of the entrance of the Magnetic Observatory.

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
May 5	+1 30	13 28	21 29·5	21 40·8
„ 6	−1 6	11 46	21 29·4	

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 5	1	15 30	69 13·9	69 15·7
	2	15 57	69 13·3	

Horizontal Force

Date	G M T		H	H_0
		h m.		
May 5	D	12 5	1·7108	1·7079
	V	13 11	1·7105	
,, 6	V	12 5	1·7125	1·7095
			1·7120	

175 ENNISKILLEN

May 14, 1887, T. E. T. (61, 83) Lat. 54° 21′ 18″, Long. 7° 39′ 50″ At Silverhill, at the S. end of lower Lough Erne Portora Old Castle, bearing due E. (mag.), Devenish Round Tower, 38° E. of N. (mag.), Royal Schools, Portora, 35° E. of S. (mag.)

September 3, 1887, A. W. R. (60, 74) Lat. 54° 21′ 7″, Long. 7° 39′ 8″ In a field close to the river, and about 150 yards east of the Royal Schools

Declination

Date	Σ	G M T	δ	δ_0
	h m.	h m	° ′	° ′
Sept 3 (R.)	+2 39	15 44	22 53·1	23 5·3
	+2 49			
	+5 44	18 41	22 51·0	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 14 (T.)	1	13 5	70 12·8	
	2	13 22	70 12·0	
Sept 3 (R.)	1	17 9	70 12·3	70 14·2
	2	17 29	70 11·5	
	2	18 24	70 10·8	

Horizontal Force

Date	G M T		H	H_0
		h m.		
May 14 (T.)	D	11 8	1·6526	1·6489
	V	11 42	1·6507	
Sept 3 (R.)	V	16 17	1·6496	1·6465
	V	15 58	1·6501	

176. GALWAY A W R (60, 74)

(a) August 25, 1887, Lat 53° 16′ 37″, Long 9° 3′ 42″ On the Lawn in front of Queen's College

(b) August 26, 1887, Lat 53° 17′ 30″, Long 9° 3′ 21″ At the lower end of Loch Cool, on the S side, near the entrance of the river The first station was chosen as being near to the College but as it is on the granite, and was found to be highly disturbed, the second position was selected on the limestone

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Aug 25	+ 2 15	15 18	23 55 3	24 6 3
	+ 4 28	17 49	23 53 6	
„ 26	+ 2 14	15 26	23 18 0	23 29 8
	+ 2 31		23 18 5	
	+ 5 14		23 17 3	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Aug 25	1	16 51	69 53 5	69 55 5
	2	17 23	69 52 6	
„ 26	1	17 1	69 40 2	69 44 1
	1	17 31	69 42 3	
	2	17 16	69 42 3	

Horizontal Force

Date	G M T		H	H_0
		h m		
Aug 25 (1)	D	16 12	1 6602	1 6568
	V	15 33	1 6600	
„ 26 (1)	V	12 55	1 6631	1 6598
			1 6631	
(2)	D	16 22	1 6898	1 6867
	V	16 46	1 6903	

177. GORT August 24 and 25, 1887, A W R (60, 74) Lat. 53° 4′ 20″, Long 8° 49″ 15″ In a field to the E of the road which runs N from the town, due N of the Church, and about 500 yards from it

Declination

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Aug 24	+ 4 28	15 41	22 38 2	22 50 5
„ 25	− 4 17	8 46	22 39 1	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Aug 24	1	16 47	69 29 5	69 31 7
	2	17 17	69 28 7	

Horizontal Force

Date		G M T	H	H_0
		h m		
Aug 24	V	16 6	1 7046	1 7013

178 GREENORE August 14, 1887, T E T (61, 83) Lat 54° 1′ 35″, Long 6° 7′ 47″. On the shore near the Coast Guard Station.

Declination.

Σ	G M T	δ	δ_0
h. m	h m	° ′	° ′
− 0 10	12 43	22 1 3	22 14 5
+ 3 41	15 17	22 1 0	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 42	69 41 7	69 42 2
2	15 1	69 37 5	

Horizontal Force.

G M T		H	H_0
	h m		
D	15 50	1 6868	1 6833
V	13 13	1 6863	

179 KELLS. May 16, 1887, T E T (61, 83). Lat 53° 43′ 8″, Long 6° 52′ 46″. In a field half a mile S of the Kells Railway Station and about 50 yards E. of the main road

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 14	12 30	21 56 7	22 7 0

Inclination

Needle.	G M T	θ	θ_0
	h m	° ′	° ′
1	13 51	69 36 9	69 38 7
2	14 6	69 36 2	

Horizontal Force.

G M.T		H	H_0
	h m.		
D	13 18	1 6949	1 6925
V	12 42	1 6955	

180 KILDARE May 6 and 7, 1887, T E T (61, 83) Lat 53° 9′ 25″, Long 6° 54′ 30″ In the centre of the enclosed ground attached to the ruins of the Castle—a plot of ground given by the Duke of LEINSTER to the town

Declination

Date	Σ.	G M T	δ	δ_0
	h m	h m	° ′	° ′
May 6	+6 7	18 56	21 50 2	22 0 4
„ 7	−2 38	11 7	21 50 4	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 7	1	12 8	69 15 0	69 17 2
	2	12 28	69 15 1	

Horizontal Force

Date	G M T		H	H_0
		h m		
May 7	D	10 39	1 7101	1 7071
„ 6	V	19 16	1 7096	
7	V	11 20	1 7105	1 7076
			1 7101	

181. KILKENNY September 7, 1887, A W. R (60, 74) Lat 52° 38′ 41″; Long. 7° 15′ 30″. In the centre of the grounds of the Catholic College.

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+1 50	14 42	21 49 0	21 58 7
+4 11	16 20	21 45 0	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 38	69 1 8	69 5 0
2	15 57	69 2 4	

Horizontal Force

G M T		H	H_0
	h m		
D	13 38	1 7290	1 7258
V	14 58	1 7295	

182. KILLARNEY August 11, 1887, A. W R (60, 74) Lat 52° 3′ 50″; Long 9° 32′ 20″ In the grounds of the Victoria Hotel To the E of the road leading from the Hotel to the Lake

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	
− 1 0	11 28	22 44 1	22 55 8
+ 3 25	15 23	22 44 2	

Inclination.

Needle.	G M T	θ	θ_0
	h m	° ′	° ′
1	13 56	68 54 2	68 56 5
2	15 49	68 53 1	

Horizontal Force.

G M T		H	H_0
	h m		
D	12 23	1 7435	1 7405
V	12 59	1 7436	

183 KILRUSH August 22, 1887, A W R (60, 74) Lat 52° 37′ 56″, Long 9° 29′ 40″. In a field to the W. of the road from the village to the quay, and about half-way between them

Declination

	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 4 24	16 32	23 0 1	
	17 43	22 59 1	23 11 4
+ 6 12	18 33	22 59 3	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	15 50	69 2[illegible] 5	69 23
2	16 10	69 20 3	

Horizontal Force

G M T		H	H_0
	h m		
D	18 5	1 7124	1 7090
V	17 31	1 7123	

184. LEENANE August 31. 1887, A W R. (60, 74) Lat 53° 35′ 43″, Long. 9° 42′ 28″ On the Green, immediately in front of the Hotel

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 1 34	14 31	23 21 0	23 36 2
+ 3 10	15 3	23 24 5	

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 2	70 5·3	70 8·0

Horizontal Force

G M T		H	H_0
	h m		
V	15 50	1·6545	1·6512

185. LIMERICK. August 20, 1887, A W R (60, 74). Lat. 52° 39′ 13″; Long 8° 38′ 46″ In a field at Summerville, the residence of J. BANNATYNE, Esq. The same position as in the survey of 1838

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
−1 27	11 50	22 24·2	
	12 42	22 24·8	22 36·6
+2 48	13 57	22 24·8	
	15 13	22 25·4	

Inclination.

Needle	G M T.	θ	θ_0
	h m	° ′	° ′
1	14 32	69 5·9	69 8·8
2	14 53	69 6·1	

Horizontal Force

G M T		H	H_0
	h. m.		
D	13 25	1·7266	1·7235
V	12 27	1·7271	

186 LISDOONVARNA. August 23, 1887, A. W R (60, 74) Lat 53° 1′ 47″, Long 9° 17′ 34″ In a field behind the Queen's Hotel

Declination

	G M T		
h m	h m	° ′	° ′
+ 4 17	16 17	22 48 1	22 58 5
+ 5 25	18 26	22 45 2	

Inclination.

Needle	G M T	θ	
	h m	° ′	° ′
1	17 25	69 30 3	69 31 8
2	17 46	69 28 4	

Horizontal Force

G M T		H	H_0
	h m		
D	18 51	1 7078	1 7042
V	16 35	1 7073	

187. LISMORE August 1, 1887, A W R (60, 74) Lat 52° 8′ 15″, Long 7° 55′ 12″ In a field, quarter of a mile E. by S. of the Cathedral

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 3 29	10 23	21 54 1	22 5 5
	13 24	21 54 0	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	11 52	68 46 2	68 48 6
2	12 16	68 45 6	

Horizontal Force.

G M T		H	H_0
	h m		
D	12 52	1 7463	1 7427
V	10 49	1 7455	

188 LONDONDERRY August 20, 1887, T E T (61, 83) Lat 55° 1′ 24″, Long 7° 18′ 6″ In a field adjoining the Moville Road, about 2 miles from Derry, and between the Road and the Lough. Boom Hall Lodge Gate 70° E. of N., distant 50 yards Boom Hall 35° W of S, distant 400 yards

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 25	12 52	22 35 5	22 50 5
+ 1 52	14 13	22 38 7	

Inclination

Needle	G M T	θ	θ_0
	h m.	° ′	° ′
1	15 28	70 24 6	70 26 9
2	15 56	70 24 2	

Horizontal Force

G M T		H	H_0
	h m		
D	16 21	1 6368	1 6333
		1 6365	
V	13 22	1 6376	1 6338
V	13 49	1 6367	

189. OUGHTERARD August 27, 1887, A W R (60, 74) Lat. 53° 26′ 10″; Long 9° 19′ 1″. On the shore of Lough Corrib, to the E of the Town, on limestone

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 3 11	17 0	23 28 0	23 40 6
+ 3 25		23 29 5	

Inclination

Needle	G M T	θ	θ_0
	h. m	° ′	° ′
1	18 1	69 54 1	69 56 7
2	18 28	69 54 2	

Horizontal Force

G M T		H	H_0
	h m		
V	17 15	1 6795	1 6762

190. PARSONSTOWN. August 4 and 5, 1887, A W R (60, 74). Lat. 53° 5′ 47″; Long. 7° 54′ 57″ In the grounds of Birr Castle, near the meridian mark used for the telescope of the Earl of ROSSE

Declination

Date	Σ	G M T	δ	δ_0
	h m	h. m	° ′	° ′
August 4	(18 20 G M T)	18 49	22 14 8	22 27 0
„ 5	(10 40 „)	11 7	22 16 3	

Inclination

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
August 5	1	13 8	69 27 8	69 30 3

Horizontal Force

Date	G M T		H	H_0
		h m		
August 5	D V	12 8 11 27	1 7022 1 7020	1 6989

191. SLIGO. May 13, 1887 ; T E T (61, 83). Lat 54° 16′ 34″, Long. 8° 28′ 36″ On a promontory in the harbour, to the W. of the town, and about a quarter of a mile distant from the bridge

Declination.

Σ	G M T	δ	δ_0
h. m	h m	° ′	° ′
−1 46 +1 44	12 24 13 53	22 55 0 22 53 9	23 4 6

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1 2	11 35 12 2	70 15 4 70 16 3	70 17 8

Horizontal Force.

G M.T.		H.	H_0
	h m		
D V	13 23 12 39	1 6453 1 6461	51 6430

192 STRABANE August 22, 1887, T. E T (61, 83) Lat 54° 49′ 48″, Long 7° 28′ 20″ In a field midway between the Railway Station and the Bridge at Lifford Railway Station bore 300 yards E.S E (mag), Bridge, W.N.W. (mag), and Church at Lifford, N.N W (mag)

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 0 17	12 33	22 34 3	22 46 9
+ 1 1	16 21	22 32 6	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	14 44	70 20 4	70 23 5
2	15 20	70 21 7	

Horizontal Force

G M T		H	H_0
	h m		
D	15 55	1 6391	1 6352
V	12 46	1 6380	

193 TIPPERARY August 3, 1887, A W R (60, 74) Lat. 52° 28′ 36″, Long. 8° 9′ 12″. 1st Station In a field on the W side of the road to Limerick Junction, about a third of a mile from the point at which it crosses the railway. 2nd Station In a field opposite the Hotel

Declination.

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 5 52	19 1	22 11 1	22 21 6

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	12 42	69 2 2	69 4 9
2	13 13	69 2 6	

Horizontal Force

G M T		H	H_0
	h m		
D	11 59	1 7317	1 7287
V	11 11	1 7321	
D	20 15	1 7292	1 7258
V	19 41	1 7289	

194 TRALEE August 19, 1887, A. W R. (60, 74). Lat 52° 16′ 15″, Long. 9° 44′ 30″. In a field about a mile to the W. of the town.

Inclination.

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 36	69 7 7	69 9 4
2	14 10	69 5 6	

195. VALENTIA. August 16 and 17, 1887; A W. R. (60, 74) Lat 51° 55′ 34″, Long 10° 17′ 40″ Lat. 51° 55′ 22″, Long 10° 17′ 25″ Station 1. On the cliffs to the N of the town. Station 2 In a field to the S of the town

Declination.

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Aug 16 (1)	+ 5 19	18 44	23 4 1	23 16 0
	+ 5 35		23 3 3	
„ 17 (2)	− 1 25	13 2	23 5 1	
	+ 1 54	13 57	23 4 2	

Inclination

Date	Needle	G M T	θ	θ_0
Aug 17 (2)	1 2	h m 12 8 15 11	° ′ 68 52 9 68 50 4	° ′ 68 54 7

Horizontal Force

Date	G M T		H	H_0
Aug 17 (2)	V V	h m 13 25 13 48	1 7485 1 7478	1 7452 1 7445

196. WATERFOOT. August 25, 1887, T. E T (61, 83) Lat 55° 3′ 11″, Long 6° 2′ 32″. The Declination was taken on the shore, and before the Chapel, the Forces in the road to the N of the Chapel

Declination.

Σ	G M T	δ	δ_0
h m + 2 11	h m 14 59	° ′ 22 1 5	° ′ 22 15 1

Inclination.

Needle	G M T	θ	θ_0
1 2	h m 16 38 16 55	° ′ 70 26 3 70 26 5	° ′ 70 29 6

Horizontal Force.

G.M T		H	H_0
V	h. m 16 3	1 6326	1 6293

197. Waterford August 2, 1887, A. W. R (60, 74). Lat 52° 16′ 54″, Long 7° 8′ 41″ In a field on the E. of the Kilmacow Road near the junction of the Lower Limestone and Shale About a third of a mile from the junction of the Limerick and Kilkenny Railways

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
− 1 44	12 7	21 15 1	21 27 9
+ 2 17	14 40	21 17 7	

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	11 44	68 49 6	68 53 7
2	14 12	68 52 5	

Horizontal Force.

G M T		H	H_0
	h m		
D	.	1 7362	1 7329
V	12 31	1 7361	

198. Westport. May 9 and 10, 1887; T.E.T (61, 83) Lat. 53° 48′ 8″; Long 9° 31′ 17″. In the demesne of the Marquis of Sligo, and about 100 yards from the main entrance from the town.

Declination.

Date	Σ.	G M T	δ	δ_0
	h m	h. m.	° ′	° ′
May 9	+ 4 48	17 54	23 5 5	
„ 10	− 1 55	11 10	23 4 2	23 15 1
	+ 2 4	14 13	23 5 4	

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
May 10	1	13 25	70 16 9	70 17 9
	2	13 42	70 15 5	

Horizontal Force.

Date	G M T		H	H_0
		h m		
May 10	D	12 21	1 6536	1 6505
			1 6529	
„ 9	V	18 10	1 6545	1 6514
„ 10	V	11 25	1 6537	

199. WEXFORD September 8, 1887, A. W. R. (60, 74) Lat. 52° 21′ 27″, Long 6° 27′ 18″ On the limestone on the opposite side of the river (Ardcavan) to the Town. In a field on the Western side of the main road, about a quarter of a mile N. of the point where the roads to the Wooden Bridge and Ferry Bank unite

Declination

Σ	G M T	δ	δ_0
h m	h m	° ′	° ′
+ 4 24	17 12	21 6 4	21 18 1

Inclination.

Needle	G M T.	θ.	θ_0
	h m	° ′	° ′
1	18 22	68 55 0	68 56·2
2	18 41	68 51 6	

Horizontal Force.

G M T		H	H_0
V	h m 17 49	1 7358	1 7324

200 WICKLOW September 9 and 10, 1887, A W R. (60, 74) Lat. 52° 58′ 53″; Long 6° 3′ 30″. Half a mile W of the town, in a field on the N side of the road through Ballynerrin, opposite to and a little beyond the old Roman Catholic Chapel.

Declination.

Date	Σ	G M T	δ	δ_0
	h m	h m	° ′	° ′
Sept 9	+5 18	18 5	21 5 0	21 21 4
„ 10	−1 55	10 55	21 14 4	

Inclination.

Date	Needle	G M T	θ	θ_0
		h m	° ′	° ′
Sept 10	1	11 44	69 7 6	69 9 9
	2	12 11	69 6 9	

Horizontal Force.

Date.	G M T.		H	H_0
		h m.		
Sept 9	D	18 49	1 7155	1·7126
	V	18 18	1 7165	

SUPPLEMENTARY STATIONS

(Dips only observed. T. E. T. (83))

201 CHEPSTOW April 22, 1889 In the courtyard of the Castle.

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	18 9	67 51 9	67 57 5
2	18 26	67 51 9	

202. GOODRICH CASTLE. April 19, 1889 On the Barbican To the East of the entrance to the Castle

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	16 47	68 3 3	68 9 0
2	16 52	68 3 5	

203. HEREFORD April 17, 1889 On the slope of Aylstone Hill overlooking Lugg Meadow About a mile from Hereford, and down a short lane which leads past Lugg Vale, the residence of Mr HAWKINS.

Inclination

Needle	G M T	θ	θ_0
	h. m	° ′	° ′
1	16 1	68 10 2	68 15 6
2	16 20	68 9 9	

204. Ross April 18, 1889 In the garden of the Royal Hotel. South of the house and between the large yew and the wall of the Prospect.

Inclination

Needle	G M T	θ	θ_0.
	h m.	° ′	° ′
1	12 29	68 2 2	68 6 9
2	13 5	68 0 5	

205. Tintern. April 22, 1889 On the Monmouth Road, above the Railway Station

Inclination

Needle	G M T	θ	θ_0
	h m	° ′	° ′
1	13 30	67 56 2	68 1 3
2	13 49	67 55 2	

Calculation of the Isomagnetic Lines

(1.) *The Isogonal Lines*

The isogonals, isoclinals, isodynamics and lines of equal Horizontal Force are drawn through points, at which the values of the magnetic elements to which they refer are equal. A general term which shall include them all would be useful, and we venture to suggest *isomagnetic* as the most obvious and convenient. In nothing do different magnetic surveys differ more widely than in the methods employed of drawing these lines. Some observers have calculated them by least squares; others give maps on which they are exhibited, but say nothing about the principles in accordance with which they have been drawn. But, inasmuch as the main object and result of a survey is the delineation of these isomagnetics, it seems to us that it is most important that they should be drawn with all the accuracy that the observations will allow. Before proceeding to discuss the method of doing this, it will be well to consider the exact meaning to be attached to the operation, and to attempt to give greater precision to the language in which the various curves, and the physical phenomena which they represent, are described.

If we suppose that a series of magnetic curves are drawn, in which all distortions due to local magnetism are neglected, except those which are on a scale comparable with the dimensions of the earth itself, they would be the *terrestrial isomagnetic lines*; on the other hand, lines which showed every disturbance, however large or small, would be the *true isomagnetic lines*.

The object of a survey is to determine as nearly as possible the forms of the true lines, and to deduce from them the directions of the terrestrial lines in the district under investigation. Between these two extremes various grades of accuracy of detail intervene, and the terrestrial lines may be regarded as affected with distortions of different orders due to disturbances of various magnitudes. It is of course impossible to frame definitions which shall accurately distinguish between them, but it is nevertheless convenient to recognise three classes, into which they may be divided with respect to any particular survey.

We may regard a disturbance as being of the third, second, or first class according as its range is less than the average distance between the stations, greater than this distance, but small compared with the dimensions of the entire area under survey, or such as to involve the whole or a considerable fraction of that area.

The term *local* may be reserved for disturbances of the third class, which affect only a single station or its immediate neighbourhood, and are represented by minor bends or small loops in the isomagnetic curves.

Those of the second class may be called *regional* disturbances. They are represented by considerable distortions of the curves, but do not seriously interfere with the determination of the general direction or average distance apart of the terrestrial

lines A disturbance of the first class is however an obstacle to any inference as to the relation between the true and the terrestrial curves It may distort them similarly, and so lead to false conclusions as to their undisturbed directions, or it may introduce such widespread and important irregularities that any conclusion deduced from the data afforded by the survey would be manifestly uncertain

It is to be noticed that the same cause may produce disturbances of different classes in the lines corresponding to different elements Thus a broad band of magnetic rock at right angles to the magnetic meridian, the extremities of which lay outside the district under investigation, would produce little effect on the Declination, but might affect the Dip to a much larger extent

If an equation can be found to a family of curves which represents, as closely as possible, the general direction of the true isomagnetics of a particular kind throughout the whole country, the curves are the nearest approach to the terrestrial lines which can be deduced from the observations The undulations of the district lines on each side of these are evidence of regional disturbances, while the still more sinuous lines obtained by joining the points calculated as corresponding to any particular value of the element from the values in two neighbouring stations are influenced by the local disturbances also

If the district under survey is very small, the assumption that the terrestrial lines are straight is very approximately true, but the nature of the curve indicated by a straight line on a map depends on the projection on which the map is drawn

The English observers have generally determined the position of the station by its distance north or south of a particular line of latitude, and east or west of a particular meridian, both distances being expressed in geographical miles That this system is open to objection is evident by considering its application to a simple ideal case in which the geographical meridians are themselves supposed to be the isogonal lines It would evidently be better under such conditions to take the latitude and longitude as coordinates, and inasmuch as in the neighbourhood of the United Kingdom there is an approximation to such an arrangement, it is probable that the latitude and longitude are at least as convenient as any other data for the determination of the position of the stations By this plan also, we are saved the trouble of converting the longitude east or west of Greenwich into miles east or west of the prime meridian We have therefore taken the latitude and longitude as our coordinates

In the next place it is, we think, important to find the general equations to the isomagnetic lines, but it would be difficult to determine their form with sufficient accuracy by a graphical process We therefore decided to trace them by a preliminary process of calculation which was carried out as follows.

The country was divided into nine overlapping districts bounded by lines of latitude and longitude. If the stations within any district were not uniformly distributed, they were weighted, so that the weighted number of stations per unit of area should be everywhere about the same. In speaking of the mean value of any quantity in a

district, it will be understood that the values appropriate to any station are throughout properly weighted The means of the latitudes, longitudes, and declinations determined the central station of the district and the declination at that station (δ_0') It was then assumed, if l' and λ' be the *district coordinates* of a station, *i e*, the differences between its latitude and longitude and those of the central station, that the declination is connected with these quantities by the linear equation—

$$\delta = \delta_0' + xl' + y\lambda'.$$

Two equations of condition were then formed by adding the equations thus obtained—(1), for all stations to the north of the central station, (2), for all stations to the east of it, and dividing by the number of stations employed, each multiplied by its proper weight.

By solving these for x and y, the rates of change of the Declination per degree of latitude and longitude respectively were obtained *

To test this method, it was applied in the case of the Dips to the whole of Scotland. This portion of Great Britain furnished a severe test, as from the irregularity of its form it is not particularly well adapted for the application of the method of equations of condition. The calculation was also made by the method of least squares. Both calculations were repeated twice, viz, with and without the inclusion of Soa and Canna, at which the local disturbance is very considerable The central station was not that given by the mean latitude and longitude, but that obtained from WELSH's Dip observations, viz, lat 56° 48′ N., long. 4° 19′ W In this particular again, the conditions of the selected example were unfavourable to the method of equations of condition

In the following Table, u is the angle made by the lines of equal Inclination with the geographical meridian, r is the change in Inclination (expressed in minutes) per geographical mile, measured at right angles to the isoclinals, and θ_0' is the Dip at the central station —

		θ_0		
Including Soa and Canna	Least squares	71 9 7	67 23	0 625
	Equations of condition	71 9 1	68 7	0 670
Excluding Soa and Canna	Least squares	71 8 0	72 49	0 595
	Equations of condition	71 7 8	72 42	0 609

* A plan very similar to that above described was employed by Dr. VAN RIJCKEVORSEL in working up the results of his survey of the Indian Archipelago ('Magnetische Opneming van den Indischen Archipel in de Jaren 1874–77, gedaan door Dr. VAN RIJCKEVORSEL' Amsterdam, J MULLER, 1879)

These results prove that in this case, at all events, the differences between the results of the two methods of calculation are not greater than those produced in the numbers given by the method of least squares according as stations affected with considerable disturbances (23′ and 76′ respectively), and amounting to 4 per cent only of the total number, are included or excluded

We do not think, therefore, that it is in general advisable to use so cumbrous a method as that of least squares, when the addition of a station or two may modify the results to an extent far exceeding the error with which numbers obtained by the equations of condition are likely to be affected If, however, the district under investigation is of such a shape that the effects of change of latitude and longitude respectively cannot be easily separated, it may be desirable either to modify the rule for obtaining the equations, or to employ least squares

In the case of a district so large as Scotland there is another objection to the use of least squares, viz, that the fundamental assumption on which that method is based is almost certainly not true when applied to it We shall show hereafter, as is indeed already known for other districts, that the errors are not irregularly distributed over the entire area, but that large fractions of the whole are affected with errors of a particular kind We cannot, therefore, regard the employment of least squares as theoretically better, while it is certainly practically more inconvenient than the method of equations of condition The following Table contains the boundaries of the nine districts, the latitudes and the longitudes of the central stations, the values of the Declination at the central stations (δ_0'), and of the change in Declination per degree of latitude and longitude ($d\delta'/dl$ and $d\delta'/d\lambda$) both expressed in minutes of arc

TABLE V

District	Boundaries		Central Station		δ_0'	$x = \frac{d\delta'}{dl}$	$y = \frac{d\delta'}{d\lambda}$
	Lat N	Long	Lat N	Long W			
	° ° ′	° °	° ′	° ′	° ′	′	′
I	All Scotland		56 48 0	4 19 0	21 38 8	14 5	40 1
II	54 to 57	0 to 6 W	55 27 3	3 41 6	20 55 6	16 7	36 4
III	52 55	0 5 W	53 26 7	2 26 0	19 39 0	15 5	33 6
IV	50 53	2 E 3 W	51 47 7	0 17 5	18 6 6	17 4	28 9
V	53 55 30	5 W. 10 W	54 2 9	7 36 5	22 41 3	17 2	32 5
VI	52 55	3 W 8 W	53 29 0	5 43 0	21 25 6	20 9	31 6
VII	49 52	1 W 6 W	50 47 0	3 1 1	19 6 2	17 8	28 9
VIII	51 54	5 W 11 W	52 57 1	8 13 1	22 35 0	27 3	30 1
IX	50 53	3 W 8 W	51 49 5	4 47 4	20 19 7	22 4	29 2

In District I, on account of its irregular form, the method of equations of condition is not very suitable, and the method of least squares has been used In order to compare this with the formula obtained by BALFOUR STEWART from WELSH's observation the

same coordinates have been employed as he used (the geographical-mile system), hence the values of $d\delta'/dl$ and $d\delta'/d\lambda$ can only be considered as applying to the central stations By means of the first the Declination at any point on the meridian through that station can be calculated, and for other points on the parallel of latitude passing through such a point, the formula $y = y_0 \cos l / \cos l_0$ must be employed where y_0 and l_0 refer to the central station In all the other districts the values of x and y are valid for the whole district

By means of these formulæ the Declination was calculated for all points within the United Kingdom defined by whole degrees of longitude and half degrees of latitude, *e g*, for lat 50° 30′, long 2° E, 1° E, 0, 1° W, and so on Where the districts overlapped the means of the numbers thus obtained were taken All these values are given in the following table The figures in brackets at the end of a row indicate the number of the district from which it was deduced Where two or more districts overlap, the individual declinations are given in italics and the mean in ordinary type

Throughout the central parts of the kingdom the agreement between the numbers given by the linear formulæ proper to different districts is sufficiently close to leave little doubt that the mean cannot be more than 1′ or 2′ wrong Where greater differences appear it is generally easy to account for them Thus, lat 53° 30′, long 10° W is in the highly disturbed region in the west of Galway Lat 52° 30′, long 0°, is close to a remarkable and hitherto unsuspected disturbance in the eastern counties, of which we shall have more to say hereafter The large differences on the border of Districts I and II (lat 56° 30′) are, perhaps, in part due to the irregularities in the shapes of these districts owing to which the formulæ are not obtained under favourable conditions. Both the other elements, however, agree with the Declination in indicating violent local disturbance in this region, and there can be no doubt that the discrepancies are due to a physical cause

TABLE VI.—Declinations Calculated

Lat.	Longitude 10° W	9°	8°	7°	6°	5°
° ′	° ′	° ′	° ′	° ′	° ′	° ′
60 30						
59 30						
58 30				(1) 23 46 2	23 7 9	22 29 6
57 30			(1) 24 13 7	23 34 4	22 55 1	22 15 8
56 30			(1) 24 2 8	23 22 4	*22 42 0* (2) *22 38 1* 22 40 0	*22 1 6* *22 1 7* 22 1 6
55 30			(5) 23 18 7	22 46 3	(2) *22 21 4* *22 13 8* (5) 22 17 6	21 45 0
54 30		(5) 23 34 0	23 1 5	*22 29 0* (6) *22 27 4* 22 28 2	(2) *22 4 7* *21 56 5* (5) *21 55 8* 21 59 0	*21 28 3* *21 24 2* 21 26 2
53 30	(5) *23 49 2* (8) *23 43 7* 23 46 4	*23 16 8* *23 13 6* 23 15 2	*22 43 3* *22 43 5* 22 43 4	*22 11 8* (6) *22 6 5* *22 13 4* 22 10 6	*21 39 3* (5) *21 34 9* *21 43 3* (8) 21 39 2	21 3 4
52 30	(8) 23 16·4	22 46 3	22 16 2	(6) *21 45 6* *21 46 1* (9) *21 39 4* 21 43 4	*21 14 0* *21 16 0* (8) *21 10 2* 21 13 4	*20 42 4* *20 41 0* 20 41 7
51 30	(8) 22 49 1	22 19 0	21 48 9	*21 18 8* (9) *21 17 0* 21 17 9	*20 48 7* (8) *20 47 8* 20 48 2	*20 18 6* (7) *20 16 3* 20 17 4
50 30				(9) 20 54 6	20 25 4	*19 56 2* (7) *19 58 5* 19 57 3
49 30						(7) 19 40 7

from the District Lines

Longitude							Lat
4°	3°	2°	1°	0°	1°E	2°	
° ′	(1) 21 46 6	21 10 0	20 34 0	19 58 0 (1)	° ′	° ′	60 30
(1) 22 6 2	21 29 2	20 52 0	20 14 9 (1)				59 30
21 51 4	21 13 2	20 34 9	19 56 7 (1)				58 30
21 36 5	20 57 2	20 17 9	19 38 6 (1)				57 30
21 21 2 *21 25 3* 21 23 2	*20 40 8* *20 48 9* 20 44 8	*20 0 4* *20 12 5* 20 6 4	*19 20 0* (1) *19 36 1* (2) 19 28 0				56 30
21 8 6	20 32 2	19 55 8	19 19 4 (2)				55 30
20 51 9 (3) *20 47 9* *20 52 7* (6) 20 50 8	*20 15 5* *20 14 3* 20 14 9	*19 39 1* *19 40 7* 19 39 9	*19 2 7* (2) *19 7 1* 19 4 9	18 33 5 (3)			54 30
(3) *20 32 4* *20 31 8* (6) 20 32 1	19 58 8	19 25 2	18 51 6	18 18 0 (3)			53 30
20 10 8 (6) (3) *20 16 9* *20 11 8* 20 13 2	*19 43 3* (4) *19 37 1* *19 42 6* (9) 19 41 0	*19 9 7* *19 8 2* 19 9 0	*18 36 1* *18 39 3* 18 37 7	*18 2 5* (3) *18 10 4* 18 6 4	17 41 5	17 12 6 (4)	52 30
19 49 4 *19 47 4* 19 48 4	*19 20 2* (9) (4) *19 19 7* *19 18 5* 19 19 5	*18 50 8* *18 49 6* 18 50 2	*18 21 9* *18 20 7* (7) 18 21 3	17 53 0	17 24 1	16 55 2 (4)	51 30
19 27 0 *19 29 6* 19 28 3	*18 57 8* (9) (4) *19 2 3* *19 0 7* 19 0 3	*18 33 4* *18 31 8* 18 32 6	*18 4 5* *18 2 9* (7) 18 3 7	17 35 6	17 6 7	16 37 8 (4)	50 30
19 11 8	18 42 9	18 14·0	17 45 1 (7)				49 30

Isogonal lines were next drawn by the aid of the mean values of the Declination given in this Table, the points at which they intersect any particular line of latitude or longitude being calculated on the assumption that the rate of change of the Declination with latitude or longitude may be regarded as constant over a single degree The curves thus obtained are shown in dotted lines in Plate II As they were deduced from the linear district formulæ they may be called the *district curves*

By drawing a smooth curve to coincide as nearly as possible with any one of the longer of these broken district curves, a close approximation to the corresponding isogonal could be obtained, but in order that the intervals between the curves might be properly spaced out, it was thought better to obtain a general formula by which they could be expressed

It appears from Table V, p 236, that y increases with the latitude, but is nearly independent of the longitude The values of x are more irregular After several trials it was found that the mean Declinations at the central stations of the various districts could be reproduced very accurately by means of the formula,

$$\delta = 19^\circ\ 11' + 19'\ 1\ (l - 49\ 5) - 3'\ 5 \cos \{45^\circ\ (l - 49\ 5)\}$$
$$+ \{26'\ 6 + 1'{\cdot}5\ (l - 49{\cdot}5)\}\ (\lambda - 4),$$

where l and λ are the numerical values of the latitude and longitude expressed in degrees and fractions of a degree

It must be distinctly understood that this formula has no theoretical value except in so far as it expresses satisfactorily the equation to smooth curves drawn according to a definite rule to represent the general form of the broken district curves

In applying it to the various stations the curves given by the periodic terms were drawn, and the values corresponding to the latitude and longitude of each read off It is possible that errors of $0'{\cdot}2$ or $0'\ 3$ may have occurred in this process The mean values of the Declinations at the central stations of each district are in the following Table compared with the results given by the formula

District	Declination at central station: Mean of values observed in district	Declination at central station: Calculated	Difference
	° ′	° ′	′
I	21 38 8	21 39 4	− 0 6
II	20 55 6	20 54 1	+ 1 5
III	19 39 0	19 38 9	+ 0 1
IV	18 6 6	18 4 4	+ 2 2
V.	22 41 3	22 41 5	− 0 2
VI.	21 25 6	21 26 5	− 0 9
VII.	19 6 2	19 5 6	+ 0 6
VIII	22 35 0	22 34 4	+ 0 6
IX	20 19 7	20 19 8	− 0 1

In District III. we included Stonyhurst; in District IV Greenwich, Kew, and Berck-sur-Mer, and in District VII, Cherbourg The data for the French stations were those given by M MOUREAUX, reduced, of course, to January 1, 1886

The values of the constants in the equation were not finally chosen until the Declination had been determined by means of a preliminary formula closely agreeing with that given above for all the points indicated in Table VI, pp 238–9, and the values given by it and by the district curves compared Over the greater part of the country the agreement is extremely close It is not necessary to reproduce the numbers here, as in Plate II. we have plotted down both the broken district curves, and also the smooth curves given by the equation The former are dotted, the latter are continuous lines The agreement is all that can be desired, except, perhaps, on the coast of Norfolk, and on the west and north coasts of Scotland

Taking, however, the English Channel first into consideration, we have calculated the points at which our 16° 53′, 17° 53′, etc., isogonals cut lat 50°. When secular change is thus allowed for, these correspond to the positions of the 17°, 18° isogonals on January 1, 1885, *i e*, with those of M. MOUREAUX. We have also measured the corresponding points from his map. The result is shown in the following Table —

Isogonal Jan 1, 1885	Longitude from Greenwich	
	R and T	MOUREAUX
°	° ′	° ′
17	1 17 E	1 9 E
18	0 55 W	0 57 W
19	3 7 W	2 55 W
20	5 19 W	4 58 W

The 20° isogonal does not pass through France, and, therefore, M. MOUREAUX's map is not of special authority on its direction, but there seems no doubt that while the 18° isogonals cut latitude 50° at the same point, there is a considerable divergence between the others

According to the general formula, 27′ 35 is the change of Declination per degree of longitude on lat 50° The two Channel Districts in Table V, p 236, viz, VII and IV agree in giving 28′ 9 for this number, and if we take this as correct and adhere to 55′ W. for the 18° isogonal, we obtain the following values :—

Isogonal	R and T	MOUREAUX
°	° ′	° ′
17	1 10 E	1 9 E
18	0 55 W	0 57 W.
19	3 0 W	2 55 W
20	5 5 W	4 58 W

These are in much better accord than those obtained from the general formula, and the conclusions which may be drawn from them are that in latitude 50°,

(1) M MOUREAUX's lines are certainly not too far to the west,

(2) And our 17° isogonal is not too far to the west

Both these conclusions are important M MOUREAUX believes that there is what we should call a considerable regional disturbance in Brittany The isogonals drawn by him sweep out to the west in the western part of France and do not resume their normal course until they have reached the English coast In the English Channel, therefore, they are deflected to the west, and as both our district lines, and to a much more marked extent our terrestrial lines are still further west, it is evident that M. MOUREAUX has not in any way exaggerated the westerly tendency of the lines in the western parts of the Channel, and that, therefore, our observations tend to confirm his view It is probable that our terrestrial curves are in this district a little too far to the west, but it must be remembered that lat 50° is almost outside the region of our survey, and that in the Channel Isles, which with Cherbourg, are our only stations to the south of it, there are considerable disturbances

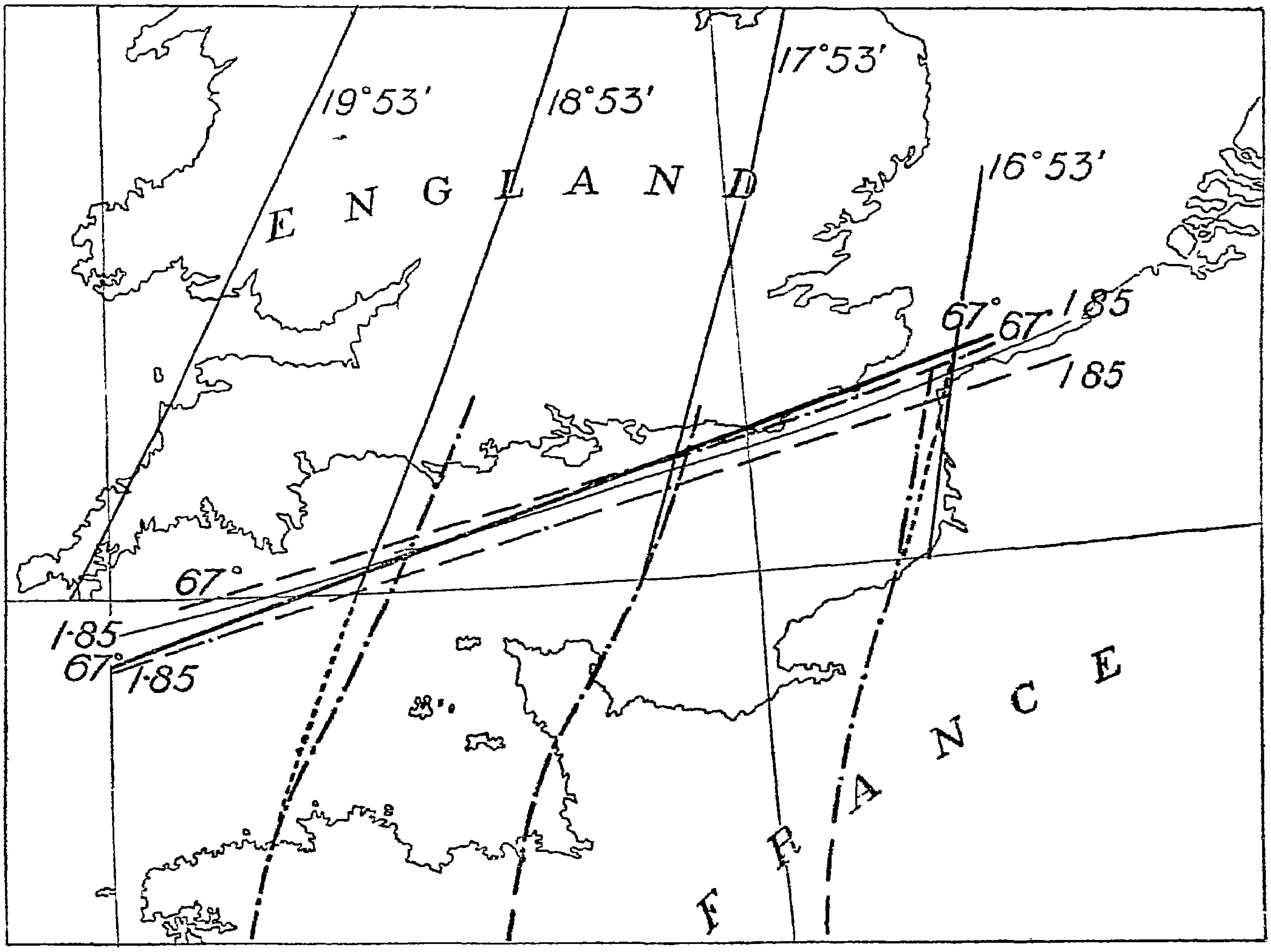
In fig 2 the continuous lines are our terrestrial isogonals, which on January 1, 1886, corresponded to M. MOUREAUX's 17°, 18°, etc, isogonals on January 1, 1885, the secular change being about 7′. M MOUREAUX's curves are shown with dashes and dots, and the hypothetical connections in dots A satisfactory comparison on the borders of the areas of two surveys cannot, however, be made unless there is a closer agreement in the methods of working up the results of the observations than there is between our own and M MOUREAUX's.

Coming next to the 17° isogonal, we find that on the coast of Norfolk the calculated Declinations are generally much lower than the observed values This would be remedied by moving the 17° isogonal further to the east, but the fact that it is already to the east of M. MOUREAUX's line is a strong argument against such a course

It is noticeable that Sir FRED EVANS appears to have found some difficulty in this part of the country ('Phil Trans.,' 1872, vol. 162, p 330) His curves are shown in Plate II. The points at which the isogonals cut lat 52° are given below, and it will be seen that the distance between them is a maximum in the centre of England.

Isogonal (1872).	Long	Difference of Long
°	°	°
25	9 54 W	
24	7 86	1 68
23	6 16	1 70
22	4 40	1·76
21	2 40	2 00
20	0 40	2 00
19	1 47 E	1·87

Fig 2



Isomagnetics in the border district of the English and French Surveys,
—— R and T, — —·— — MOUREAUX, hypothetical connections

We think the agreement between our calculated and observed curves is too close to leave any doubt that in this and neighbouring latitudes the law that the distance between the points of intersection with a line of latitude is constant is very approximately true At all events, there is no trace of a change in this distance amounting to a quarter of a degree as is shown by the 1872 lines. As the observations on which Sir F. EVANS's map were based were comparatively few in number and were made exclusively at coast stations the accurate delineation of the isogonals was not easy He makes the observed Declinations too small in the eastern counties. If the distances between the isogonals had been kept constant the 19° line would have been pushed further to the west, and a closer agreement with our observations would have been attained. Considering the circumstances under which Sir F. EVANS's map was drawn, we think the concordance between the general directions of the lines is satisfactory. We are also of opinion that our 17° isogonal is not too far to the west, and that the fact that in the eastern counties the calculated are less than the observed Declinations is due to a real physical cause. M. MOUREAUX's results confirm our own as to the 18° isogonal, and if we suppose that the easterly tendency of the lines which

he assumes in the western parts of the English Channel is continued to the Straits of Dover, his 17° isogonal and our own would be in practical accord

As regards the considerable deviation between the district and calculated terrestrial lines in the North of Scotland, we have no hesitation in adhering to the calculated lines as being probably the more correct The masses of basalt and other igneous rocks which occur in the West of Scotland and North of Ireland produce a very marked effect on the magnetic elements The general formula gives the Declination at the outlying station at Lerwick almost as accurately as the local district equations obtained from Scotland only The three values are —

		°	′
Observed Declination		20	29 7
Calculated by	Local formula .	20	31 3
	General formula	20	33 7

Regarding this station as giving a fixed point it is evident that in drawing the curves to it from the South of Scotland, great weight ought to be given to the general law which is found to be obeyed with accuracy in Great Britain from the English Channel to the Tay, and in the South and East of Ireland The only way of eliminating the effect of regional disturbances of such magnitude as these which exist in Scotland, is by studying the shape of the terrestrial lines in adjacent districts Observations to the north of Scotland would be very valuable for this purpose, but as it would be difficult to obtain them we think that the results of the general formula must be accepted as a close approximation to the truth. We shall return to this subject in the discussion of local disturbances

(2) *The Isoclinal Lines*

The isoclinals were obtained in a precisely similar manner to that above described. The districts were the same, but the positions of the central stations were in some cases slightly different, as at a few places Dips had been observed without Declinations and *vice versâ*.

In District VII. the inclusion of the Channel Isles led to rates of change with latitude and longitude so widely different from those obtained elsewhere that it was thought better to omit them. When this was done the coefficients assumed normal values. We give in the next Table the latitude and longitude of the central stations, and the corresponding values of the Dips, and the rates of change of Inclination per degree of latitude and longitude.

TABLE VII

District	Central Station		$D = \theta_0'$	$x = \frac{d\theta'}{dl}$	$y = \frac{d\theta'}{d\lambda}$
	Lat N	Long W			
I	56° 48′ 0	4° 19′ 0	71° 8′ 0	34′ 1	5′ 8
II	55 27 3	3 41 6	70 19 6	31 9	8 2
III	53 26 7	2 26 0	69 3 0	36 1	7 1
IV	51 47 7	0 17 4	67 45 6	40 8	7 6
V	54 1 4	7 39 3	69 59 8	38 3	9 8
VI	53 26 7	5 42 0	69 24 8	36 2	6 3
VII	51 8 1	3 9 6	67 41 2	38 8	6 6
VIII	52 57 1	8 13 1	69 24 3	38 7	8 5
IX	51 49 5	4 47 4	68 18 2	39 1	6 8

A general formula was next found to embrace the whole country. For this purpose a Table similar to Table VI, pp 238–9, was prepared, and the district isoclinal lines were drawn from it on curve paper, on a purely artificial system, in which all degrees of latitude and longitude were regarded as of equal length When thus drawn the mean directions of the isoclinals were nearly straight lines and practically parallel.

The equation to the 67° isoclinal was

$$l - 49^\circ 92 + 0\ 2\ (\lambda - 4) = 0,$$

where l and λ are the latitude and longitude expressed in degrees and fractions of a degree If then s be the length of the perpendicular on this line from any point, the Dip at the station indicated by that point would (if the lines were equidistant) be given by the equation

$$\theta = 67 + \mathrm{A}s,$$

where A is a constant This condition was not fulfilled. The distance between the lines increased approximately in the proportion of their distance from the 67° line, but a small periodic term was necessary in addition to this correction to produce the desired accuracy.

The following plan was finally adopted :—

If we write

$$p = l - 49{\cdot}92 + 0{\cdot}2\ (\lambda - 4)$$

and

$$q = p - 0\ 1 \sin (20\,p)$$

we get the Dip in degrees from the equation

$$\theta = 67^\circ + \frac{1{\cdot}0083\,q}{1{\cdot}456 + 0\,03\,q}.$$

The Dips given by this formula for the central stations in each district are, in the following Table compared with those given in Table VII, p 245

District	Inclination at Central Station		Difference
	Mean of Values observed in District	Calculated	
	° ′	° ′	′
I	71 8 0	71 10 3	−2 3
II	70 19 6	70 21 2	−1 6
III	69 3 0	69 1 8	+1 2
IV	67 45 6	67 44 5	+0 9
V	69 59 8	69 59 3	+0 5
VI	69 24 8	69 25 3	−0 5
VII	67 41 2	67 41 2	0 0
VIII	69 24 3	69 25 6	−1 3
IX	68 18 2	68 19 7	−1 5

In Plate III we show the broken curves obtained by the district lines and also the terrestrial isoclinals as represented by the formula

The agreement is on the whole satisfactory, but it is possible that a closer approximation to the true terrestrial lines might have been obtained had we made the inclination to the geographical meridian increase rather more rapidly in the west. This would have diminished the discrepancy in the south of Ireland. On the other hand it would have considerably increased it in the north of Ireland and the central districts of Scotland, in which large regional disturbances undoubtedly exist It does not, however, appear to be safe to depart from the rule that the terrestrial lines deduced from any survey should give, as nearly as possible, the mean directions of the true lines in the district under investigation. This end is better attained by our formulæ than if the agreement were closer in the south of Ireland.

Our 67° isoclinal agrees almost exactly with that of M MOUREAUX in the more easterly parts of the English Channel (see fig 2, p. 243). Both just cut Dungeness and Beachy Head. For the longitude of Falmouth, however, our line is 10 or 12 miles to the south of that of M. MOUREAUX

As it is at this point much nearer to the English than the French coast, our result is probably the more trustworthy, and this opinion is confirmed by a study of the map of the French Survey. The isoclinals drawn by M. MOUREAUX upon a map on Mercator's projection are curved in the south of France, the convexity being towards the north. The curvature becomes less as the Dip increases, and the 67° line is represented as quite straight. M. MOUREAUX cannot have much to guide him in drawing this line, and a divergence such as that which exists between his line and ours could easily be introduced by an apparently trifling error in the estimation of the rate of disappearance of the curvature of the lines. Our lines would be slightly concave to the north if drawn upon a map on Mercator's projection.

(3) *Lines of equal Horizontal Force*

These lines were treated in the same way as the isogonals and isoclinals, and in the following Table the constants for the nine districts are tabulated as before As Professor BALFOUR STEWART did not deduce the lines of equal Horizontal Force from Mr WELSH's observations, there was not as much reason in this case as in the others for taking the whole of Scotland as District I. We therefore included in this district only all stations north of lat 56°

District	Central Station		H_0'	$x = \frac{dH}{dl}$	$y = \frac{dH}{d\lambda}$
	Lat N	Long W			
I	57° 26′ 3	4° 30′ 0	1 5580	−0 03930	−0 00575
II	55 27 3	3 41 6	1 6363	−0 03324	−0 00882
III	53 26 7	2 26 0	1 7164	−0 03868	−0 00661
IV	51 47 7	0 17 4	1 7970	−0 04374	−0 00620
V	54 2 9	7 36 5	1 6650	−0 04380	−0 01046
VI	53 29 0	5 43 0	1 6979	−0 03771	−0 00439
VII	50 49 3	2 57 6	1 8212	−0 04134	−0 00598
VIII	52 57 1	8 13 1	1 7053	−0 04237	−0 00628
IX	51 49 5	4 47 4	1 7694	−0 04212	−0 00598

By means of these data a Table like Table VI, pp 238–9, was prepared, and lines of equal Horizontal Force were drawn on curve paper as in the case of the isoclinals (see p. 245) These at once showed that the lines above and below that corresponding to 1 7 units were differently disposed The mean direction of each of the southern lines can be accurately represented by a linear function of the latitude and longitude, their departure from parallelism is not great, and their mean distances are nearly the same. On the other hand, though the lines in District I, *i e*, in the extreme north of Scotland, are parallel to the 1 7 line, which runs from the neighbourhood of Miltown Malbay to that of Scarborough, the intermediate isodynamics make smaller angles with the geographical meridian, and their average distance is greater than in the south. It is, however, difficult to decide what is the direction of the terrestrial lines Too much weight must not be attached to the fact that the lines in the north of Scotland agree with those in the Midlands and south of England, as they are deduced from a single district of irregular form and the seat of great local and regional disturbances There is also reason to suppose from what is known of the lines of equal Horizontal Force on the continent that the distance between them increases towards the east, and those crossing England are so represented on M MOUREAUX's map It may well be, therefore, that the diverging lines are those which agree most closely with the terrestrial lines of equal Horizontal Force.

On the whole then, the risk of introducing fictitious disturbances by attempting to include the whole country under one simple law would be very great.

We decided, therefore, in calculating the lines of equal Horizontal Force, to employ different formulæ for districts to the north and south of the 1·7 line.

Taking the southern district first, the lines may be regarded as straight but, inasmuch as they run across the whole breadth of the kingdom, a very small error in the calculated slope is important. A rather complicated formula is therefore unfortunately necessary

If c be the mean distance, expressed in degrees of latitude and measured along longitude 5° W, between any line and that which corresponds to 1·85 units, we have a relation of the form

$$1\ 85 - \mathrm{H} = ac,$$

where a is a constant.

The 1 85 line cuts longitude 5° W. in latitude 49° 83, and thus the equation to the isodynamic through c is

$$l - 49\ 83 = -(\lambda - 5)\, m + c,$$

where m measures the slope of the line.

Neither a nor m are quite constant, but both are functions of c, so that

$$1/a = 24\ 47\left(1 + \frac{c}{1000}\right);$$

$$m = 157 - \cdot 0019c - \cdot 0155 \sin (50c)$$

Thus, if we wish to find where the line corresponding to H cuts longitude λ, we find c from the equation

$$c = 24\ 47\,(1\cdot 85 - \mathrm{H}) + 0\ 001c^2,$$

where the value of c, used in the small term in c^2, is the approximate value obtained by neglecting it. From this m is found, and l is then known.

If the latitude and longitude are given, and the Horizontal Force is required, we first find c approximately from the formula

$$c' = l - 49\cdot 83 + 0\ 157\,(\lambda - 5).$$

Subtracting from this

$$c'' = (\lambda - 5)\,\{0\cdot 0019\, c' + 0\cdot 0155 \sin (50\, c')\},$$

the difference is c, whence H is found

Taking next the district north of the 1·7 line, there is no particular difficulty in finding similar equations to express the mean direction of the lines with great exactitude. If we write

$$1\ 7 - \mathrm{H} = 0\ 036854\,c,$$
$$l - 53\ 514 = -\,m\,(\lambda - 5) + c,$$
$$m = \ 151 + \ 07 \sin\,(45\,c),$$

all that can be desired in this respect is attained

These equations do not, however, give the forces at the central stations with the accuracy attained in the southern districts Far better results are obtained if we assume that the lines are parallel to the 1 7 line and are at equal distances from each other In this case m is constant, and, in addition to the first of the above equations, we have only

$$l - 53\ 514 = -\ 0\ 151\,(\lambda - 5) + c$$

The values obtained from the central stations by both formulæ are given in the following Table —

District	Horizontal Force at central stations in northern districts deduced from		
	Mean of all stations in district	Formula with m variable	Formula with m constant
I	1 5580	1 5583	1 5582
II	1 6363	1 6390	1 6358
V	1 6650	1 6608	1 6658
VI	1 6979	1 6971	1 6972

We have, therefore, to choose between two formulæ The simpler represents the lines throughout Scotland as practically parallel, both with those which in the south extend right across the kingdom without considerable curvature and with those found by the district equations for the north of Scotland, and also reproduces the values of the Horizontal Force at the central stations with great accuracy. The more complex expression represents the mean directions of the lines in the centre of Scotland and north of Ireland more satisfactorily, though these are obviously affected by some widespread disturbance, but it fails when tried by the test of the reproduction of the forces at the central stations. We think there can be no question that, under these circumstances, the simpler formula, which assigns directions to the lines in harmony with those obtained in other parts of the kingdom, must be that selected ; and we have therefore employed it

In the following Table the values of the Horizontal Forces at the central stations are compared with the calculated values given by the formulæ—

$$1\ 85 - H = c\,(1 - 0\ 001\ c)\ /24\ 47$$

and

$$l - 49\ 83 = c - m\,(\lambda - 5)$$

and

$$m = 0\ 157 - 0\ 0019\ c - 0\ 0155\ \sin\,(50\,c)$$

for values of H > 1 7, and by

$$1\ 7 - H = 0\ 036854\ c$$

and

$$l - 53\ 514 = c - 0\ 151\ (\lambda - 5)$$

for values of H < 1 7

The difference between the observed and calculated results in District VIII (South Ireland), is larger than in the other cases. It must, however, be remembered that it is absolutely very small Thus, if the disturbed stations at Galway (which have been excluded from the calculations) were introduced, the difference would be reduced from 0022 to 0007 As the effect of one station is so great, it is remarkable that the discrepancies are not larger.

District	Horizontal Force at central stations		Difference in terms of 0 0001
	Mean of values observed in district	Calculated	
I	1 5580	1 5582	− 2
II	1 6363	1 6358	+ 5
III	1 7164	1 7180	−16
IV	1 7970	1 7969	+ 1
V	1 6650	1 6658	− 8
VI	1 6979	1 6972	+ 7
VII	1 8212	1 8218	− 6
VIII	1 7053	1 7031	+22
IX	1 7694	1 7698	− 4

We conclude this section with a Table, which gives the observed and calculated elements at every station

In the calculations, graphic methods have been partly used, and, in some cases, this may have led to slight differences (which are, however, so small as to be quite unimportant) between the values given and those which would be deduced directly from the formulæ.

SUMMARY of Decl ations, I ations H and Ve cal Forces, Observed and Cal ated ffe

Scotl

Station		Declination			Inclination			Horizontal Force			Vertical Force			No of Station
No	Name	Observed	Calculated	Difference	Observed.	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	
1	Aberdeen	20° 16′ 3	20° 21′ 0	− 4′ 7	71° 12′·3	71° 7′ 7	+ 4′ 6	1 5734	1 5813	− 0079	4 6232	4 6260	− 0028	1
2	Arinagower (Coll)	23 40 4	22 58 2	+42 2		71 18 8		.	1 5772			.	.	2
3	L Aylort .	23 16 5	22 35 4	+41 1	71 24 0	71 21 5	+ 2 5	1 5663	1 5726	− 0063	4 6542	4 6617	− 0075	3
4	Ayr	21 17 9	21 26 8	− 8 9	70 21 4	70 27 8	− 6 4	1 6345	1 6300	+ 0045	4 5792	4 5937	− 0145	4
5	Ballater .	20 29 5	20 55 3	−25 8	71 15 4	71 10 0	+ 5 4	1 5714	1 5809	− 0095	4 6309	4 6350	− 0041	5
6	Banavie	22 6 7	22 9 5	− 2 8	71 11 4	71 17 0	− 5 6	1 5940	1 5764	+ 0176	4 6797	4 6529	+ 0268	6
7	Banff	21 4 5	20 46 1	+18 4	71 19 0	71 26 7	− 7 7	1 5684	1 5610	+ 0074	4 6381	4 6505	− 0124	7
8	Berwick	19 36 4	19 57 9	−21 5	70 15 9	70 20 2	− 4 3	1 6483	1 6336	+ 0147	4 5947	4 5717	+ 0230	8
9	Boat of Garten	22 7·7	21 26 1	+41 6	71 16 3	71 21 2	− 4 9	1 5786	1 5693	+ 0093	4 6562	4 6506	+ 0056	9
10	L. Boisdale	22 53 3	23 39 2	−45 9	71 39 3	71 40 9	− 1 6	1 5310	1 5530	− 0220	4 6172	4 6908	− 0736	10
11	Bunnahabham .	23 10 3	22 29 0	+41·3	70 43 0	70 52 1	− 9 1	1 6243	1 6063	+ 0180	4 6427	4 6305	+ 0122	11
12	Callernish	23 40 6	23 40 9	− 0 3	72 7 1	72 9 4	− 2 3	1 5236	1 5191	+ 0045	4 7223	4 7192	+ 0031	12
13	Campbelton	22 8 1	22 1 2	+ 6 9	70 34 2	70 33 4	+ 0 8	1 6244	1 6259	− 0015	4 6050	4 6059	− 0009	13
14	Canna (mean)	23 13 0	23 6 5	+ 6 5	72 45 0	71 32 7	+72 3	1 5092	1 5607	− 0515	4 8605	4 6766	+ 1839	14
15	Carstairs	20 52 2	20 56 8	− 4 6	70 15 7	70 29 0	−13 3	1 6448	1 6276	+ 0172	4 5841	4 5920	− 0079	15
16	Crianlarich	21 50 6	21 43 2	+ 7 4	70 52 5	70 58 8	− 6 3	1 5980	1 5961	+ 0019	4 6083	4 6302	− 0219	16
17	Crieff .	21 33·6	21 14 1	+19 5	70 53 6	70 53 2	+ 0 4	1 6079	1 6010	+ 0069	4 6416	4 6199	+ 0217	17
18	Cumbrae	21 37 2	21 42 6	− 5 4	71 2 3	70 40 9	+21 4	1 5911	1 6159	− 0248	4 6310	4 6095	+ 0215	18
19	Dalwhinnie	21 45 5	21 38 6	+ 6 9	71 0 1	71 14 0	−13 9	1 5909	1 5784	+ 0125	4 6207	4 6454	− 0247	19
20	Dumfries	20 47 4	20 43 9	+ 3 5	70 2 6	70 6 2	− 3 6	1 6542	1 6519	+ 0023	4 5555	4 5642	− 0087	20
21	Dundee .	20 44 5	20 42 7	+ 1 8	70 52 2	70 50 5	+ 1 7	1 6002	1 6024	− 0022	4 6132	4 6122	+ 0010	21
22	Edinburgh	20 47 2	20 44 4	+ 2 8	70 38 5	70 35 2	+ 3 3	1 6183	1 6195	− 0012	4 6061	4 5954	+ 0107	22
23	Elgin	20 57 5	21 16 5	−19 0	71 32 0	71 31 2	+ 0 8	1 5577	1 5574	+ 0003	4 6644	4 6599	+ 0045	23
24	L Eriboll .	22 18 1	22 26 3	− 8 2	72 9 4	72 6 2	+ 3 2	1 5198	1 5190	+ 0008	4 7214	4 7039	+ 0175	24
25	Fairlie	.	21 40 4		70 42 8	70 39 3	+ 3 5	1 6172	1 6176	− 0004	4 6222	4 6087	+ 0135	25
26	Fort Augustus	21 45 6	21 59 2	−13 6	71 27 7	71 23 8	+ 3 9	1 5641	1 5680	− 0039	4 6641	4 6584	+ 0057	26
27	Gairloch	22 14 4	22 59 0	−44 6	71 44 3	71 49 9	− 5 6	1 5353	1 5417	− 0064	4 6528	4 6979	− 0451	27
28	Glasgow	21 11 5	21 22 4	−10 9	70 44 7	70 39 6	+ 5 1	1 6064	1 6169	− 0105	4 5988	4 6068	− 0080	28
29	Golspie .	21 30 2	21 48 5	−18 3	71 46 7	71 45 8	+ 0 9	1 5382	1 5417	− 0035	4 6725	4 6790	− 0065	29
30	Hawick	20 16 0	20 22 0	− 6 0	70 7 3	70 14 4	− 7 1	1 6487	1 6416	+ 0071	4 5598	4 5697	− 0099	30
31	L. Inver	22 7 4	22 42 7	−35 3	72 0 2	71 59 6	+ 0 6	1 4990	1 5276	− 0286	4 6143	4 6996	− 0853	31
32	Inverness	21 13 3	21 18 3	− 5 0	71 31 1	71 31 4	− 0 3	1 5642	1 5585	+ 0057	4 6798	4 6642	+ 0156	32

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Scotland (continued

Station.		Declination			Inclination			Horizontal Force			Vertical Force			No of Station
No.	Name.	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	
33	Iona	23° 28′ 6	22° 47′ 7	+ 40′ 9	70° 55′ 8	71° 8′ 7	−12′ 9	1 6185	1 5883	+ 0302	4 6711	4 6509	+ 0202	33
34	Kirkwall . .	21 29 3	21 27 9	+ 1·4	72 12 8	72 11 0	+ 1 8	1 5108	1 5100	+ 0008	4 7093	4 6984	+ 0109	34
35	Kyle Akin .	23 10 4	22 42 4	+ 28 0	71 38 5	71 34 8	+ 3 7	1 5465	1 5577	− 0112	4 6603	4 6771	− 0168	35
36	Lairg .	21 50·3	22 6 3	− 16·0	71 50 3	71 50 1	+ 0 2	1 5356	1 5370	− 0014	4 6811	4 6845	− 0034	36
37	Lerwick . .	20 29 7	20 33 7	− 4 0	72 47 1	72 35 3	+11 8	1 4710	1 4770	− 0060	4 7476	4 7098	+ 0378	37
38	Lochgoilhead .	21 54 2	21 49 5	+ 4 7	70 46 1	70 53 4	− 7 3	1 6021	1 6025	− 0004	4 5924	4 6252	− 0328	38
39	L. Maddy . .	23 18 0	23 34 3	− 16 3	71 52 1	71 54 1	− 2 0	1 5365	1 5372	− 0007	4 6921	4 7035	− 0114	39
40a	Oban . .	22 9 4	22 15·4	− 6 0	70 53 2	71 5 4	−12 2	1 6044	1 5904	+ 0140	4 6297	4 6426	− 0129	40a
40b	„ (Kerrera)	22 11 9	22 16 4	− 4 5	70 48 8	71 5 7	−16 9	1 6103	1 5896	+ 0207	4 6276	4 6416	− 0140	40b
41	Pitlochrie	21 8 3	21 15 4	− 7 1	70 57 4	71 3·2	− 5 8	1 5899	1 5895	+ 0004	4 6061	4 6303	− 0242	41
42	Port Askaig	23 0 7	22 27 6	+ 33 1	70 36 2	70 50 6	−14 4	1 6340	1 6073	+ 0267	4 6409	4 6268	+ 0141	42
43a	Portree . .	25 1 1	23 3 0	+118 1	72 18 4		+36 5	1 5177	1 5500	− 0323				43a
43b	„ .	22 45 6	23 2 5	− 16 9	71 8 6	71 41 9	−33 3	1 5876	1 5500	+ 0376	4 7517	4 6863	+ 0654	43b
43c	„ .	20 14 3	23 2 9	−168 6	72 36 6		+54 7	1 5211	1 5500	− 0289				43c
44	Row (Gairloch) .	21 47 7	21 42 5	+ 5 2	70 51 0	70 47 5	+ 3 5	1 5978	1 6096	− 0118	4 6012	4 6200	− 0188	44
45	Scarnish (Tiree)	24 27 9	23 5 7	+ 82 2	71 19 4	71 16 8	+ 2 6	1 5909	1 5798	+ 0111	4 7063	4 6619	+ 0444	45
46	Soa	23 14 9	22 56 5	+ 18 4	71 59 6	71 34 0	+25 6	1 5072	1 5583	− 0511	4 6368	4 6753	− 0385	46
47	Stirling .	21 28 6	21 13 9	+ 14 7	70 53 3	70 45 3	+ 8 0	1 5945	1 6097	− 0152	4 6015	4 6107	− 0092	47
48a	Stornoway (Aird Point)	24 16 3	23 29 3	+ 47 0	72 10·5	72 8 2	+ 2 3	1 5196	1 5193	+ 0003	4 7259	4 7141	+ 0118	48a
48b	Stornoway (Castle)	24 7 6	23 29 4	+ 38 2	72 9 1	72 8 4	+ 0 7	1 5146	1 5193	− 0047	4 7038	4 7151	− 0113	48b
49	Strachur .	21 48 9	21 56 1	− 7 2	70 42 9	70 54 6	−11 7	1 6095	1 6013	+ 0082	4 5999	4 6268	− 0269	49
50	Stranraer	21 35 0	21 31 9	+ 3 1	70 13 5	70 11 7	+ 1 8	1 6435	1 6481	− 0046	4 5713	4 5765	− 0052	50
51	Stromness	21 27 9	21 41 1	− 13 2	72 11 7	72 12 3	− 0 6	1 5149	1 5093	+ 0056	4 7169	4 7023	+ 0146	51
52	E L Tarbert	22 4 3	22 2 4	+ 1 9	70 46 8	70 46 7	+ 0 1	1 6053	1 6107	− 0054	4 6047	4 6197	− 0150	52
53	Thurso	21 38 4	21 42 9	− 4 5	72 1 1	72 2 4	− 1 3	1 5217	1 5212	+ 0005	4 6883	4 6930	− 0047	53
54	Wick .	21 15 3	21 22 8	− 7 5	72 9 8	71 54 9	+14 9	1 5144	1 5290	− 0146	4 7064	4 6822	+ 0242	54

Station		Declination			Inclination			Horizontal Force			Vertical Force			No of Station
No	Name	Observed	Calculated.	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	
		° ′	° ′	′	° ′	° ′	′							
55	Aberystwith	19 56 5	20 11 2	−14 7	68 34 7	68 35 7	− 1 0	1 7488	1 7506	− 0018	4 4575	4 4653	− 0078	55
56	Alderney	18 2 8	18 22 8	−20 0	66 38 9	66 37 2	+ 1 7	1 8691	1 8735	− 0044	4 3292	4 3335	− 0043	56
57	Alnwick	19 45 0	19 43 9	+ 1 1	70 3 6	70 6 7	− 3 1	1 6511	1 6485	+ 0026	4 5510	4 5560	− 0050	57
58	Alresford	18 9 7	18 18 0	− 8 3	67 22 3	67 23 5	− 1 2	1 8241	1 8221	+ 0020	4 3761	4 3755	+ 0006	58
59	Appleby	20 5 8	19 58 3	+ 7 5	69 44 9	69 42 3	+ 2 6	1 6690	1 6750	− 0060	4 5237	4 5293	− 0056	59
60	Barrow	20 9 3	20 16 1	− 6 8	69 30 6	69 31 7	− 1 1	1 6875	1 6885	− 0010	4 5157	4 5228	− 0071	60
61	Bedford	18 27 4	18 14 5	+12 9	68 7 3	67 58 7	+ 8 6	1 7705	1 7818	− 0113	4 4091	4 4052	+ 0039	61
62	Birkenhead	19 58 3	19 58 9	− 0 6	69 4 3	69 4 8	− 0 5	1 7176	1 7160	+ 0016	4 4912	4 4886	+ 0026	62
63	Birmingham	18 44 0	19 4 7	−20 7	68 21 3	68 21 8	− 0 5	1 7669	1 7605	+ 0064	4 4525	4 4373	+ 0152	63
64	Braintree	17 55 4	17 40 3	+15 1	67 45 4	67 41 2	+ 4 2	1 7942	1 7984	− 0042	4 3871	4 3821	+ 0050	64
65	Brecon	19 38 6	19 41 1	− 2 5	68 15 8	68 14 0	+ 1 8	1 7701	1 7727	− 0026	4 4399	4 4392	+ 0007	65
66a	Bude Haven	19 56 5	19 49 8	+ 6 6	67 44 2	67 39 9	+ 4 3	1 8080	1 8120	− 0040	4 4164	4 4104	+ 0060	66a
66b	″		19 50 0											66b
67	Cambridge	18 5 0	17 57 9	+ 7 1	68 2 4	67 56 5	+ 5 9	1 7784	1 7828	− 0044	4 4105	4 3995	+ 0110	67
68	Cardiff	19 19 7	19 24 6	− 4 9	67 52 3	67 55 0	− 2 7	1 7944	1 7926	+ 0018	4 4129	4 4182	− 0053	68
69	Cardigan	20 25 8	20 22 3	+ 3 5	68 31 3	68 28 6	+ 2 7	1 7535	1 7600	− 0065	4 4565	4 4617	− 0052	69
70	Carlisle	20 25 8	20 18 6	+ 7·2	69 54 0	69 56 8	− 2 8	1 6625	1 6602	+ 0023	4 5430	4 5482	− 0052	70
71	Chesterfield	19 11 9	19 2 4	+ 9 5	68 48 5	68 46 8	+ 1 7	1 7351	1 7321	+ 0030	4 1752	4 4607	+ 0145	71
72	Chichester	18 5 5	18 2 6	+ 2 9	67 11 6	67 10 8	+ 0 8	1 8395	1 8352	+ 0043	4 3745	4 3615	+ 0130	72
73	Clenchwarton	18 10 3	17 59 0	+11 3	68 17 9	68 16 0	+ 1 9	1 7662	1 7611	+ 0051	4 4378	4 4171	+ 0207	73
74	Clifton	19 10 7	19 7 5	+ 3 2	67 48 7	67 49 3	− 0 6	1 7996	1 7974	+ 0022	4 4124	4 4091	+ 0033	74
75	Clovelly	19 53 8	19 49 5	+ 4 3	67 49 9	67 45 4	+ 4 5	1 8000	1 8059	− 0059	4 4177	4 4156	+ 0021	75
76	Coalville	18 41 4	18 52 6	−11 2	68 24 1	68 27 7	− 3 6	1 7534	1 7526	+ 0008	4 4289	4 4398	− 0109	76
77	Colchester	17 55 2	17 29 9	+25 3	67 35 3	67 39 0	− 3 7	1 8012	1 7998	+ 0014	4 3675	4 3774	− 0099	77
78	Cromer	17 35 8	17 31 1	+ 4 7	68 20 0	68 15 1	+ 4 9	1 7603	1 7597	+ 0006	4 4310	4 4102	+ 0208	78
79	Dover	16 57 2	17 7 0	− 9 8	67 8 0	67 5 4	+ 2 6	1 8336	1 8366	− 0030	4 3477	4 3456	+ 0021	79
80	Falmouth	19 53 4	19 49 7	+ 3·7	67 15·0	67 17 5	− 2 5	1 8323	1 8366	− 0043	4 3696	4 3886	− 0190	80
81	Gainsborough	18 36 4	18 43 3	− 6 9	68 49 3	68 47 6	+ 1 7	1 7321	1 7296	+ 0025	4 4705	4 4573	+ 0132	81
82	Giggleswick	19 35 3	19 44 4	− 9 1	69 22 3	69 23 4	− 1 1	1 6962	1 6941	+ 0021	4 5059	4 5047	+ 0012	82
83	Gloucester	19 12 9	19 4 5	+ 8·4	68 4 3	68 2 3	+ 2 0	1 7811	1 7824	− 0013	4 4244	4 4198	+ 0046	83
84	Grantham	18 29 0	18 32 5	− 3 5	68 28 0	68 29 3	− 1 3	1 7527	1 7490	+ 0037	4·4419	4 4369	+ 0050	84
85	Guernsey (L'Erée)	18 32·7	18 29 9	+ 2 8	66 34 1	66 29 9	+ 4 2	1 8746	1 8815	− 0069	4 3254	4 3267	− 0013	85
86	″ (Peter Port)	18 18 4	18 27 7	− 9 3	66 32 4	66 29 4	+ 3 0	1 8796	1 8820	− 0024	4 3311	4 3262	+ 0049	86
87	Harwich	17 18 8	17 18 8	0 0	67 38 4	67 38 2	+ 0 2	1 8031	1 8001	+ 0030	4 3833	4 3754	+ 0079	87

England (continued).

Station No.	Station Name.	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No of Station
		° ′	° ′	′	° ′	° ′	′							
88	Harpenden	18 16·5	18 6·3	+10·2	67 52·4	67 45·4	+ 7·0	1·7916	1·7961	− ·0045	4·4063	4·3917	+ ·0146	88
89	Haslemere .	18 7·7	18 6·0	+ 1·7	67 20·6	67 21·1	− 0·5	1·8282	1·8238	+ ·0044	4·3797	4·3710	+ ·0087	89
90	Holyhead .	20 51·1	20 47·2	+ 3·9	69 23·1	69 12·5	+10·6	1·6958	1·7109	− ·0151	4·5080	4·5059	+ ·0021	90
91	Horsham . .	18 3·3	17 54·6	+ 8·7	67 15·2	67 16·9	− 1·7	1·8309	1·8275	+ ·0034	4·3669	4·3649	+ ·0020	91
92	Hull . .	18 57·8	18 35·8	+22·0	69 3·9	68 57·8	+ 6·1	1·7125	1·7182	− ·0057	4·4764	4·4670	+ ·0094	92
93	Ilfracombe .	19 46·5	19 46·3	+ 0·2	67 53·8	67 51·5	+ 2·3	1·7952	1·7987	− ·0035	4·4203	4·4205	− ·0002	93
94	Jersey (Grouville)	18 36·0	18 10·1	+25·9	66 8·7	66 14·3	− 5·6	1·9053	1·8965	+ ·0088	4·3086	4·3077	+ ·0009	94
95	" (S. Louis)	17 49·7	18 12·0	−22·3	66 13·0	66 14·8	− 1·8	1·8887	1·8961	− ·0074	4·2856	4·3085	− ·0229	95
96	" (S Owen)	18 24·7	18 15·3	+ 9·4	66 15·5	66 16·8	− 1·3	1·8904	1·8942	− ·0038	4·2979	4·3110	− ·0131	96
97	Kenilworth . .	19 1·4	18 52·6	+ 8·8	68 28·8	68 15·2	+13·6	1·7576	1·7668	− ·0092	4·4574	4·4287	+ ·0287	97
98	Kettering	18 36·0	18 27·6	+ 8·4	68 10·7	68 10·7	0·0	1·7656	1·7695	− ·0039	4·4095	4·4187	− ·0092	98
99	Kew	18 16·3	17 59·5	+16·8	67 37·4	67 31·8	+ 5·6	1·8093	1·8110	− ·0017	4·3950	4·3788	+ ·0162	99
100a	King's Lynn	17 57·9	17 57·7	+ 0·2	68 17·8	68 15·9	+ 1·9	1·7656	1·7610	+ ·0046	4·4359	4·4164	+ ·0195	100a
100b	" (Gaywood) .	18 1·7	17 56·6	+ 5·1	68 16·8	68 15·4	+ 1·4	1·7646	1·7615	+ ·0031	4·4298	4·4159	+ ·0139	100b
101	King's Sutton .	18 51·8	18 37·5	+14·3	68 6·4	68 0·5	+ 5·9	1·7778	1·7819	− ·0041	4·4240	4·4119	+ ·0121	101
102	Lampeter .	18 55·3	20 4·8	− 9·5	68 25·5	68 25·0	+ 0·5	1·7559	1·7624	− ·0065	4·4406	4·4543	− ·0137	102
103	Leeds .	19 8·9	19 17·4	− 8·5	69 10·8	69 10·3	+ 0·5	1·7082	1·7067	+ ·0015	4·4922	4·4863	+ ·0059	103
104	Leicester	18 23·6	18 43·1	−19·5	68 24·0	68 22·1	+ 1·9	1·7538	1·7581	− ·0043	4·4296	4·4323	− ·0027	104
105	Lincoln	18 18·9	18 33·3	−14·4	68 43·0	68 39·3	+ 3·7	1·7395	1·7385	+ ·0010	4·4654	4·4482	+ ·0172	105
106	Llandudno	20 51·5	20 22·2	+29·3	69 12·0	69 7·5	+ 4·5	1·7084	1·7149	− ·0065	4·4974	4·4962	+ ·0012	106
107	Llangollen	20 8·4	19 54·3	+14·1	68 49·4	68 50·1	− 0·7	1·7331	1·7326	+ ·0005	4·4736	4·4745	− ·0009	107
108	Llanidloes	19 53·8	19 55·1	− 1·3	68 33·8	68 33·5	+ 0·3	1·7501	1·7517	− ·0016	4·4573	4·4595	− ·0022	108
109	Loughborough	18 18·7	18 49·0	−30·3	68 27·7	68 28·5	− 0·8	1·7531	1·7513	+ ·0018	4·4419	4·4397	+ ·0022	109
110	Lowestoft	17 24·0	17 12·2	+11·8	68 0·4	67 54·6	+ 5·8	1·7797	1·7807	− ·0010	4·4064	4·3872	+ ·0192	110
111	Mabelthorpe	18 16·5	18 9·9	+ 6·6	68 42·7	68 38·4	+ 4·3	1·7370	1·7373	− ·0003	4·4578	4·4415	+ ·0163	111
	Malvern—													
112a	Colwall Green	19 3·6	19 11·7	− 8·1	..	68 10·5	.. .	1·7627	1·7737	− ·0110				112a
112b	Great Malvern	19 33·0	19 10·5	+22·5	68 14·4	68 11·5	+ 2·9	1·7687	1·7722	− ·0035	4·4310	4·4289	+ ·0021	112b
112c	Malvern Wells	19 22·4	19 10·1	+12·3		68 10·7		1·7682	1·7733	− ·0051			.	112c
112d	Mathon	18 46·5	19 11·9	−25·4	.	68 11·7		1·7655	1·7722	− ·0067				112d
	Manchester—													
113	Old Trafford	19 16·7	19 34·2	−17·5	69 3·9	69 1·4	+ 2·5	1·7125	1·7183	− ·0058	4·4764	4·4814	− ·0050	113
114	Manton .	18 21·7	18 30·0	− 8·3	68 17·1	68 19·0	− 1·9	1·7661	1·7605	+ ·0056	4·4347	4·4268	+ ·0079	114
115	March .	18 2·8	18 4·7	− 1·9	68 10·3	68 9·8	+ 0·5	1·7719	1·7684	+ ·0035	4·4237	4·4126	+ ·0111	115

England (continued)

Station No.	Station Name.	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No of Station
		° ′	° ′	′	° ′	° ′	′							
116*a*	Melton Mowbray	19 10.9	18 37.7	+33.2	68 27.8	68 25.0	+ 2.8	1.7620	1.7546	+.0074	4.4648	4.4343	+.0305	116*a*
116*b*	" "	19 4.2	18 38.2	+26.0	68 36.0	68 25.6	+10.4	1.7415	1.7539	−.0124	4.4138	4.4358	+.0080	116*b*
117	Milford Haven	20 8.8	20 21.9	−13.1	68 9.9	68 16.6	− 6.7	1.7808	1.7738	+.0070	4.4445	4.4517	−.0072	117
118	Newark	18 46.2	18 40.7	+ 5.5	68 33.4	68 36.6	− 3.2	1.7464	1.7416	+.0048	4.4464	4.4458	+.0006	118
119	Newcastle . .	19 30.3	19 35.6	− 5.3	69 49.5	69 51.2	− 1.7	1.6665	1.6641	+.0024	4.5356	4.5360	−.0004	119
120	Northampton .	18 41·7	18 29.7	+12.0	68 9.4	68 5.2	+ 4.2	1·7667	1.7759	−.0092	4.4074	4.4143	−.0069	120
121	Nottingham .	18 44.9	18 49.7	− 4.8	68 37.6	68 34.5	+ 3.1	1.7470	1.7448	+.0022	4.4639	4.4460	+.0179	121
122	Oxford	18 33.7	18 34.7	− 1.0	67 57.5	67 50.4	+ 7.1	1.7890	1.7927	−.0037	4.4187	4.4018	+.0169	122
123	Peterborough	18 21.9	18 15.8	+ 6.1	68 14.8	68 13.7	+ 1.1	1.7692	1.7651	+.0041	4.4338	4.4188	+.0150	123
124	Plymouth	19 31.6	19 28.8	+ 2.8	67 14.7	67 18.9	− 4.2	1.8309	1.8334	−.0025	4.3651	4.3861	−.0210	124
125	Port Erin	20 55.4	21 7.4	−12.0	69 48.1	69 41.3	+ 6.8	1.6678	1.6800	−.0122	4.5333	4.5388	−.0055	125
126	Preston	19 52.3	19 52.8	− 0.5	69 14.7	69 13.6	+ 1.1	1.7053	1.7051	+.0002	4.4999	4.4950	+.0049	126
127	Purfleet .	17 54.5	17 43.1	+11.4	67 30.9	67 28.3	+ 2.6	1.8134	1.8136	−.0002	4.3812	4.3723	+.0089	127
128	Pwllheli	20 41.9	20 31.7	+10.2	68 50.9	68 55.8	− 4.9	1.7407	1.7291	+.0116	4.4991	4.4877	+.0114	128
129	Ramsey	20 54.9	20 58.9	− 4.0	69 55.0	69 46.9	+ 8.1	1.6617	1.6737	−.0120	4.5449	4.5445	+.0004	129
130	Ranmore .	18 8.9	17 43.9	+25.0	67 20.0	67 23.6	− 3.6	1.8261	1.8202	+.0059	4.3725	4.3714	+.0011	130
131*a*	Reading (Caversham)	18 15.2	18 19.3	− 4.1	67 40.7	67 37.1	+ 3.6	1·8110	1.8067	+.0043	4.4109	4.3874	+.0235	131*a*
131*b*	Reading (Caversham)	18 11·6	18 19.3	− 7.7	.			1.8141	1.8067	+.0071				131*b*
132*a*	Redcar . .	19 5.6	19 7.9	− 2.6	69 31.5	69 32.5	− 1.0	1.6847	1.6828	+.0019	4.5119	4.5108	+.0011	132*a*
132*b*	" .		19 8.6											132*b*
133	Ryde .	18 1.6	18 12.1	−10.5	67 7.8	67 9.4	− 1.6	1.8391	1.8375	+.0016	4.3602	4.3620	−.0018	133
134	St. Cyres (Exeter)	19 28.6	19 21.6	+ 7.0	67 26.2	67 30.5	− 4.3	1.8260	1.8198	+.0062	4.3946	4.3951	−.0005	134
135	St Leonards .	17 24.8	17 25.5	− 0.7	66 58.9	67 0.9	− 2.0	1.8437	1.8433	+.0004	4.3396	4.3457	−.0061	135
136	Salisbury .	18 23.9	18 36.5	−12.6	67 25·6	67 28.6	− 3.0	1.8242	1.8179	+.0063	4.3881	4.3838	+.0043	136
137	Scarborough	18 48.3	18 43.0	+ 5.3	69 15.6	69 16.6	− 1.0	1.7017	1.6978	+.0039	4.4939	4.4876	+.0063	137
138	Shrewsbury	19 41.2	19 36.1	+ 5.1	68 36.4	68 37.2	− 0.8	1.7342	1.7457	−.0115	4.4266	4.4585	−.0319	138
139	Southend .	17 44.4	17 29.9	+14.5	67 30.8	67 27.0	+ 3.8	1.8112	1.8138	−.0026	4.3755	4.3681	+.0074	139
140	Spalding	17 51.6	18 15.3	−23.7	68 23.1	68 20.8	+ 2.3	1.7512	1.7570	−.0058	4.4197	4.4245	−.0048	140
141	Stoke-on-Trent	19 22.7	19 23.3	− 0.6	68 43.6	68 42.7	+ 0.9	1.7401	1·7384	+.0017	4.4693	4.4608	+.0085	141
142	Sutton Bridge	17 54.1	18 4.1	−10.0	68 21.1	68 17.4	+ 3.7	1.7620	1.7600	+.0020	4.4395	4.4195	+.0200	142
143	Swansea	19 45.6	19 51.1	− 5.5	67 59.7	68 5.7	− 6.0	1.7931	1.7830	+.0101	4.4370	4.4340	+.0030	143
144	Swindon .	..	18 44.9		67 51.4	67 47.3	+ 4.1	1.7930	1.7975	−.0045	4.4061	4.4023	+.0038	144
145	Taunton . .	19 10.7	19 12.3	− 1.6	67 32.7	67 36.0	− 3.3	1.8154	1.8129	+.0025	4.3925	4.3984	−.0059	145

England (continued).

Station		Declination			Inclination.			Horizontal Force			Vertical Force			No of Station
No	Name	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	
		° ′	° ′	′	° ′	° ′	′							
146	Thetford .	17 41 2	17 42·6	− 1 4	68 1 4	67 59 9	+ 1 5	1 7791	1 7776	+ 0015	4 4085	4 3991	+ 0094	146
147	Thirsk .	19 21 7	19 15 0	+ 6 7	69 28 3	69 22 7	+ 5 6	1 6912	1 6931	− 0019	4 5165	4 4992	+ 0173	147
148	Tilney	17 58·1	18 0 3	− 2 2	68 20 7	68 14 5	+ 6 2	1 7655	1 7628	+ 0027	4 4468	4 4157	+ 0311	148
149	Tunbridge Wells	17 41 3	17 37 2	+ 4 1	67 10 8	67 14 1	− 3 3	1·8297	1 8292	+ 0005	4 3485	4 3589	−·0104	149
150	Wallingford .	18 21 6	18 25 7	− 4 1	67 48 4	67 43 4	+ 5 0	1 7986	1 8001	− 0015	4 4087	4 3942	+ 0145	150
151	Weymouth. .	18 46 7	18 45 9	+ 0 8	67 11 7	67 14 9	− 3 2	1 8329	1 8342	− 0013	4 3593	4 3737	− 0144	151
152	Wheelock	.	.		68 49 0	68 49 7	− 0 7	.	.					152
153	Whitehaven	20 41 6	20 35 2	+ 6 4	69 47 6	69 48 9	− 1 3	1 6727	1 6699	+ 0028	4 5447	4 5423	+ 0024	153
154	Windsor .	18 29 9	18 19 1	+10 8	67 38 8	67 35 4	+ 3 4	1 8084	1 8079	+ 0005	4 3977	4 3841	+ 0136	154
155	Wisbech	18 5 6	18 4 7	+ 0 9	68 19 0	68 14 6	+ 4 4	1 7653	1 7631	+ 0022	4 4398	4 4171	+ 0227	155
156	Worthing	17 59 0	17 51 1	+ 7 9	67 6 4	67 6 7	− 0 3	1 8402	1 8388	+ 0014	4 3578	4 3554	+ 0024	156
G	Greenwich	17 56 3	17 50 4	+ 5 9	67 28 6	67 29 9	− 1 3	1 8141	1 8124	+ 0017	4 3746	4 3751	− 0005	G
S	Stonyhurst	19 42 6	19 46 7	− 4 1	69 11 4	69 16 5	− 5 1	1 7002	1 7024	− 0022	4 4734	4 4957	− 0223	S

NOTE —In the case of a few places at which several sets of observations have been made, the numbers given above differ slightly from the means of those in the Table on pp. 270–273 The reason of this is that the disturbances given above were calculated from the mean values of the elements at the various sub-stations, and are thus not quite the same as the means of the disturbances deduced for each sub-station from the elements proper to it

Station		Declination			Inclination			Horizontal Force			Vertical Force			No of Station
No.	Name.	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	
		° ′	° ′	′	° ′	° ′	′							
157	Armagh	22 16·5	22 16·2	+ 0·3	69 57·6	70 3·1	− 5·5	1·6625	1·6600	+ ·0025	4·5578	4·5736	− ·0158	157
158	Athlone	22 26·7	22 38·3	−11·6	69 40·0	69 40·8	− 0·8	1·6852	1·6865	− ·0013	4·5476	4·5544	− ·0068	158
159	Bagnalstown	21 55·0	21 47·8	+ 7·2	69 5·1	69 7·4	− 2·3	1·7208	1·7218	− ·0010	4·5028	4·5141	− ·0113	159
160	Ballina	23 26·9	23 34·9	− 8·0	70 25·8	70 12·9	+12·9	1·6323	1·6544	− ·0221	4·5917	4·5991	− ·0074	160
161	Ballywilliam	21 37·3	21 38·5	− 1·2	69 5·6	68 57·7	+ 7·9	1·7222	1·7326	− ·0104	4·5084	4·5039	+ ·0045	161
162	Bangor	21 44·4	21 48·9	− 4·5	70 1·3	70 7·8	− 6·5	1·6598	1·6534	+ ·0064	4·5657	4·5750	− ·0093	162
163	Bantry	22 40·3	22 36·6	+ 3·7	68 46·0	68 48·7	− 2·7	1·7529	1·7495	+ ·0034	4·5115	4·5127	− ·0012	163
164	Carrick-on-Shannon	23 3·1	22 55·3	+ 7·8	69 53·3	69 59·6	− 6·3	1·6700	1·6668	+ ·0032	4·5607	4·5778	− ·0171	164
165	Castlereagh	23 11·1	23 3·6	+ 7·5	69 56·3	69 55·9	+ 0·4	1·6716	1·6713	+ ·0003	4·5773	4·5748	+ ·0025	165
166	Cavan	22 37·8	22 31·9	+ 5·9	69 57·3	69 55·9	+ 1·4	1·6629	1·6691	− ·0062	4·5577	4·5688	− ·0111	166
167	Charleville	22 30·8	22 31·8	− 1·0	69 5·3	69 7·3	− 2·0	1·7226	1·7255	− ·0029	4·5083	4·5235	− ·0152	167
168	Clifden	24 20·7	23 48·6	+32·1	70 4·8	69 57·6	+ 7·2	1·6631	1·6726	− ·0095	4·5891	4·5854	+ ·0037	168
169	Coleraine	22 36·9	22 33·1	+ 3·8	70 47·7	70 30·3	+17·4	1·6085	1·6310	− ·0225	4·6176	4·6070	+ ·0106	169
170	Cookstown Junction	21 32·8	22 11·4	−38·6	69 34·5	70 14·7	−40·2	1·6830	1·6474	+ ·0356	4·5194	4·5871	− ·0677	170
171	Cork	22 18·1	22 13·3	+ 4·8	68 46·4	68 49·5	− 3·1	1·7506	1·7460	+ ·0046	4·5071	4·5068	+ ·0003	171
172	Donegal	23 20·1	23 12·7	+ 7·4	70 15·3	70 24·0	− 8·7	1·6449	1·6407	+ ·0042	4·5827	4·6077	− ·0250	172
173	Drogheda	21 54·7	21 52·9	+ 1·8	69 36·3	69 39·5	− 3·2	1·6895	1·6852	+ ·0043	4·5441	4·5455	− ·0014	173
174	Dublin	21 40·8	21 40·9	− 0·1	69 15·7	69 25·6	− 9·9	1·7087	1·6985	+ ·0102	4·5128	4·5254	− ·0126	174
175a	Enniskillen	23 5·3	22 50·5	+15·2	70 14·2	70 10·7	+ 3·5	1·6489	1·6543	− ·0054	4·5816	4·5894	− ·0078	175a
175b	"		22 50·1					1·6465		− ·0078				175b
176a	Galway	24 6·3	23 10·1	+56·2	69 53·0	69 43·1	+ 9·9	1·6583	1·6856	− ·0273	4·5274	4·5612	− ·0338	176a
176b	"	23 29·8	23 10·2	+19·6	69 41·6	69 43·6	− 2·0	1·6867		+ ·0011	4·5581	4·5632	− ·0051	176b
177	Gort	22 50·5	22 56·8	− 6·3	69 31·7	69 34·2	− 2·5	1·7013	1·6954	+ ·0059	4·5571	4·5515	+ ·0056	177
178	Greenore	22 14·5	21 51·8	+22·7	69 42·2	69 48·8	− 6·6	1·6833	1·6756	+ ·0077	4·5514	4·5574	− ·0060	178
179	Kells	22 7·0	22 9·8	− 2·8	69 38·7	69 43·2	− 4·5	1·6925	1·6818	+ ·0107	4·5619	4·5513	+ ·0106	179
180	Kildare	22 0·4	21 57·8	+ 2·6	69 17·2	69 23·6	− 6·4	1·7073	1·7021	+ ·0052	4·5150	4·5268	− ·0118	180
181	Kilkenny	21 58·7	21 55·9	+ 2·8	69 5·0	69 7·9	− 2·9	1·7258	1·7218	+ ·0040	4·5154	4·5158	− ·0004	181
182	Killarney	22 55·8	22 50·1	+ 5·7	68 56·5	69 3·3	− 6·8	1·7405	1·7325	+ ·0080	4·5203	4·5257	− ·0054	182
183	Kilrush	23 11·4	23 5·5	+ 5·9	69 23·2	69 23·4	− 0·2	1·7090	1·7088	+ ·0002	4·5435	4·5439	− ·0004	183
184	Leenane	23 36·2	23 39·7	− 3·5	70 8·0	69 58·7	+ 9·3	1·6512	1·6707	− ·0195	4·5697	4·5848	− ·0151	184
185	Limerick	23 36·6	22 39·7	− 3·1	69 8·8	69 18·0	− 9·2	1·7235	1·7131	+ ·0104	4·5244	4·5333	− ·0089	185
186	Lisdoonvarna	22 58·5	23 10·5	−12·0	69 31·8	69 36·0	− 4·2	1·7042	1·6941	+ ·0101	4·5653	4·5553	+ ·0100	186
187	Lismore	22 5·5	22 2·8	+ 2·7	68 48·6	68 54·3	− 5·7	1·7427	1·7395	+ ·0032	4·4952	4·5087	− ·0135	187

Ireland—continued.

Station.		Declination			Inclination			Horizontal Force			Vertical Force			No of Station
No.	Name.	Observed.	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	
		° ′	° ′	′	° ′	° ′	′							
188	Londonderry	22 50.5	22 52.5	− 2.0	70 26.9	70 31.1	− 4.2	1.6335	1.6313	+ .0022	4.5997	4.6113	− .0116	188
189	Oughterard	23 40.6	23 22.6	+18.0	69 56.7	69 50.5	+ 6.2	1.6762	1.6789	− .0027	4.5916	4.5733	+ .0183	189
190	*Parsonstown .	22 27.0	22 27.5	− 0.5	69 30.3	69 28.6	+ 1.7	1.6989	1.6997	− .0008	4.5452	4.5404	+ .0048	190
191	Sligo .	23 4.6	23 16.2	−11.6	70 17.8	70 13.6	+ 4.2	1.6430	1.6522	− .0092	4.5880	4.5959	− .0079	191
192	Strabane .	22 46.9	22 54.7	− 7.8	70 23.5	70 25.7	− 2.2	1.6352	1.6377	− .0025	4.5901	4.6064	− .0163	192
193	Tipperary	22 22.6	22 19.4	+ 3.2	69 4.9	69 8.2	− 3.3	1.7272	1.7236	+ .0036	4.5188	4.5219	− .0031	193
194	Tralee .	...	23 2.5		69 9.4	69 12.2	− 2.8	.	1.7227	.			..	194
195	Valentia .	23 16.0	23 8.5	+ 7.5	68 54.7	69 3.6	− 8.9	1.7448	1.7336	+ .0112	4.5245	4.5298	− .0053	195
196	Waterfoot	22 15.1	22 9.5	+ 5.6	70 29.0	70 23.6	+ 5.4	1.6293	1.6374	− .0081	4.5968	4.5967	+ .0001	196
197	Waterford . .	21 27.9	21 43.1	−15.2	68 53.7	68 53.9	− 0.2	1.7329	1.7374	− .0045	4.4897	4.5015	− .0118	197
198	Westport	23 15.1	23 38.4	−23.3	70 17.9	70 4.4	+13.5	1.6509	1.6642	− .0133	4.6103	4.5907	+ .0196	198
199	Wexford .	21 18.1	21 23.7	− 5.6	68 56.2	68 51.6	+ 4.6	1.7324	1.7389	− .0065	4.4982	4.4967	+ .0015	199
200	Wicklow	21 21.4	21 26.4	− 5.0	69 9.9	69 11.3	− 1.4	1.7126	1.7151	− .0025	4.5002	4.5119	− .0117	200

Supplementary Stations

Station		Declination			Inclination.			Horizontal Force			Vertical Force			No of Station
No	Name	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	Observed	Calculated	Difference	
201	Chepstow				67 57.5	67 56.6	+0.9							201
202	Goodrich				68 9.0	68 5.1	+3.9							202
203	Hereford				68 15.6	68 12.6	+3.0							203
204	Ross .				68 6.9	68 7.1	−0.2							204
205	Tintern				68 1.3	67 59.1	+2.2							205

Local and Regional Disturbances

We now come to the consideration of the local and regional disturbances which exist in many parts of the United Kingdom and which we have investigated much more fully than has hitherto been done

As the problem is one of difficulty, it is necessary that every step should be carefully considered It may be attacked in three ways, which are not independent, but each of which is attended with special advantages and disadvantages, and which, when combined, afford in many cases the means of arriving at definite conclusions

(1) If the true isomagnetic curves are drawn as accurately as possible, without any attempt to smooth the irregularities, they present in disturbed districts distorted forms which enable us to judge of the nature and magnitude of the disturbance The great advantage of this method is, that it is independent of calculation It is not affected by errors possibly introduced by the method of determining the terrestrial curves The conclusions arrived at are based directly upon the observations

The objections that may be raised to it are, that it can only be used with effect in the case of considerable disturbances, and that it may tend to exaggerate the importance of those which, though of great local intensity, are of small range A curve may be carried many miles from its true position, in order to pass through a station, the disturbance at which dies out within a very short distance

When the stations are as numerous as ours, and when the curves are drawn with a due regard to the possibility of isolated maxima and minima, we do not think that the risk of error on this account is as great as it might at first sight appear, but in so far as it exists, it may be checked by the second method

(2) If the disturbances of the elements, *i e*, the differences between the observed values and those calculated from the general equations given above, are plotted down on a map, it is found that they are not scattered haphazard, but that certain districts exhibit definite peculiarities, such as that the observed value is always too large or too small From a study of such maps deductions can be made as to the nature of the disturbing forces

This method is open to the objection that the peculiarities in question may not correspond to physical realities, but may be due to the inadequacy of the formulæ The calculated rate of increase of the declination with longitude, for instance, may be made a little too rapid in one part of the country, and a little too slow elsewhere. The observed declinations may, therefore, appear too large in the one district and too small in another, and thus a mere mathematical error may create a fictitious attractive or repulsive force.

To this objection, it may be answered that the observed disturbances are too large and too irregular to be thus explained Taking, for instance, the English and Welsh stations which lie between long. 3 W. and long. 5 W., the mean disturbances of the Declinations for the groups indicated, are as follows :—

Four most northerly stations .	14′ 4 W
Intermediate group of eight stations	6 0 E.
Six most southerly stations .	3 0 W

A glance at the isogonals (Plate II) in this part of the kingdom, is sufficient to show that it is impossible to believe that the real terrestrial curve not only crosses and recrosses that which we have drawn between Plymouth and Holyhead (which is quite possible), but that the amplitude of the oscillation amounts to more than 20′ of Declination, or to about 45′ of longitude. If this were so, the terrestrial would be nearly as sinuous as the true curves (*cf* Plate V)

It is also to be observed, that even if there is a slight tendency of the kind supposed, it will be partly corrected by the first method. The objection to that is, that too much stress may be laid on the peculiarities of individual stations—to this, that too great weight may be given to the characteristics of districts.

If the two methods point to the same conclusion the two criticisms are mutually destructive

A more serious objection is, that the indications are, at times, somewhat ambiguous. An increase in the Declination may be due to a small force acting at right angles to the magnetic meridian, or to a large one acting nearly parallel to it. The value of the disturbance of the Horizontal Force will often decide to which of these the effect is due, but nevertheless, it is not easy to deduce definite conclusions from a mere inspection of the maps. This consideration has, therefore, led us to supplement this method by another

(3.) In this third method, we calculate the magnitude and direction of the disturbing force at each station. If δ be the observed, and δ_c the calculated value of the Declination, if a similar notation be used for the other elements, and if N, W, and Z be the northerly, westerly, and vertical components of the disturbing force, we have

$$N = H_c \cos \delta_c - H \cos \delta,$$
$$W = H_c \sin \delta_c - H \sin \delta,$$
$$Z = H_c \tan \theta_c - H \tan \theta,$$

whence the magnitude and direction of the disturbing force are known. Obviously the danger to be feared, in this case, is that the differences with which we deal are too small to give trustworthy results. The calculation must, however, lead to conclusions compatible with those to which the study of the disturbances points, and even if the results are only approximate, they are valuable as indicating, in a way which is more easily interpreted, the direction and magnitude of the forces under investigation.

Having thus described the general course which we propose to adopt, we shall now consider the methods more in detail. After that we shall apply them to certain districts, of which we have made a special study, and then to the whole area of the survey.

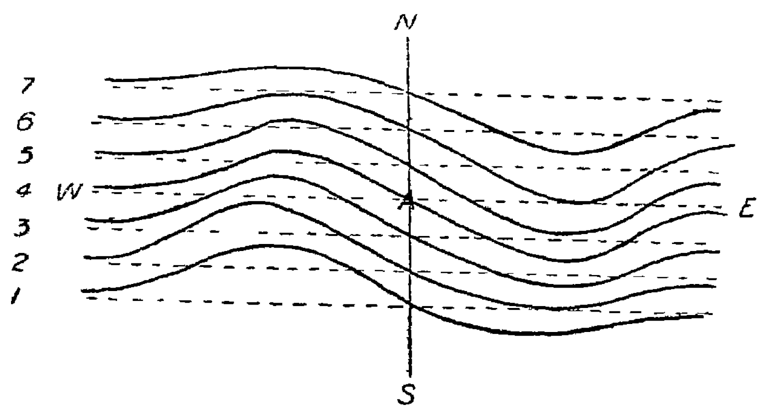
In attempting to discover the order which underlies the apparent irregularity of the disturbances, it is necessary to proceed on a working hypothesis, and we shall

postulate only the possibility of the existence of points or surfaces which exert magnetic forces They will be called attractive or repulsive according as they attract or repel a north-seeking pole If the existence or apparent existence of such centres is established the cause of the phenomenon will be a proper subject for enquiry

The distortion produced by a symmetrical mass of magnetic matter on the isogonals depends upon the angle which these curves make with the magnetic meridian

If we are dealing with a sufficiently small area, both the isogonals and the magnetic meridians may be represented by straight lines If these lines are mutually perpendicular, and if the declination is westerly and increases with the latitude, the forms of the true isogonals in the neighbourhood of a symmetrical attracting mass, the centre of which is below the surface at A, will be of the type shown in Fig 3, in which NS is the magnetic meridian through A

Fig 3



- - - - Terrestrial Isogonals running approximately east and west

——— True Isogonals produced by centre of attraction at A

In Japan the distribution of the isogonals and meridians approximates to this simple arrangement, and in the neighbourhood of the Fossa Magna according to Dr. NAUMANN ('Die Erscheinungen des Erdmagnetismus,' Stuttgart, 1887), a very remarkable disturbance of this kind is produced

If the isogonals and meridians are very nearly coincident, and if the Declination increases with the longitude, the effect on the isogonals of a weak attractive centre will be of the type shown in Fig 4 If, however, the disturbing force is sufficiently powerful to make the Declination, at some point to the east of the centre, greater than its undisturbed value at the centre, the form of the lines must approximate to that shown in Fig. 5 The Declination will increase rapidly with the longitude, attain a maximum value, fall to its normal value at points on the magnetic meridian which pass through the centre—on each side of which the attracting matter is supposed to be symmetrically situated—then fall to a minimum, and finally resume its normal rate of increase

If, retaining all the other assumptions, we suppose that the isogonals and meridians are inclined at an angle of less than 90°, the distorted curves will assume forms similar to those in Fig. 6. It will be noticed that if the undisturbed isogonals be

rotated about A in the limiting cases when they coincide with AE or AN, the distorted curves will assume the forms shown in Figures 3, and 4, or 5

Fig 4

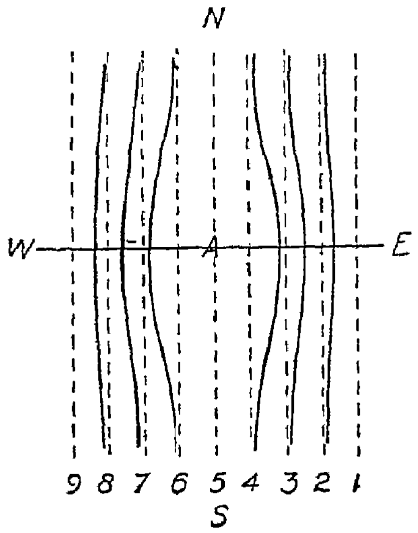
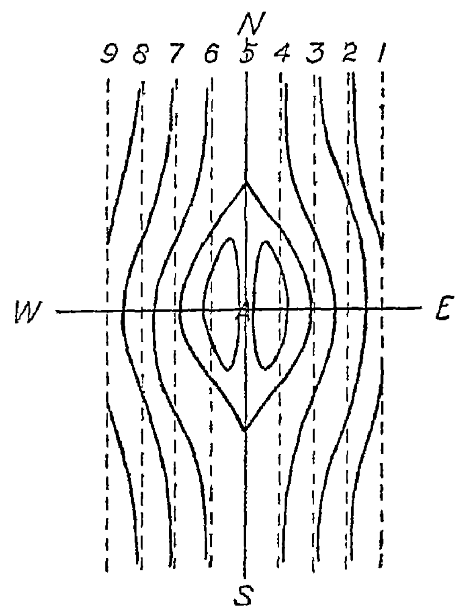
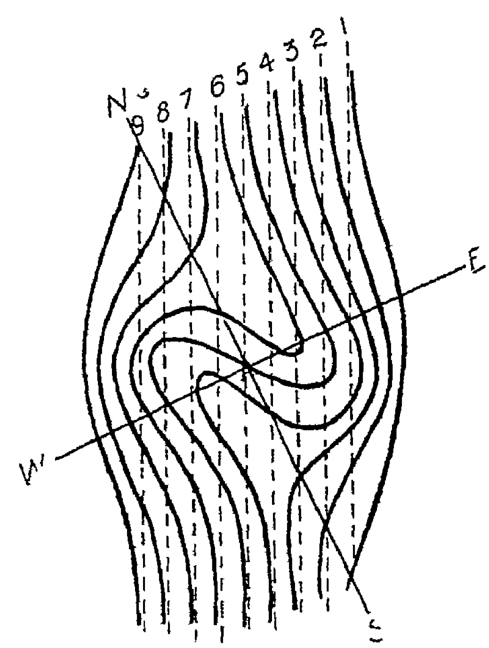


Fig 5



Terrestrial Isogonals running approximately north and south
True Isogonals produced by (Fig 4) a weak, and (Fig 5) a strong centre of attraction at A

Fig 6



- - - - Terrestrial Isogonals approximately parallel
——— True Isogonals produced by a centre of attraction at A

It is, of course, quite possible that a single closed curve may sometimes be formed An underground attracting surface, rising up steeply on the eastern side and falling away very gently to the west, might cause a maximum declination on the east without any corresponding minimum on the west, but in general the normal complement of a closed isogonal is another of the same type

The curves shown in figs 3 to 6 will not be realised in practice, but their main

characteristics may be reproduced, and a knowledge of their forms is very useful in interpreting maps on which the true isomagnetic curves are drawn

These facts may also be illustrated by means of diagrams of another kind Thus if, in figs. 7 and 8, the slope of the line AD represents the normal rate of increase of the Declination with longitude, the effect of an attractive centre below A will be represented by the curves shown, which correspond to the two cases of a weak and strong attraction respectively. If, however, the force is repulsive the curves assume the forms shown in figs 9 and 10. In this case there may be two maxima and minima, following each other in order on opposite sides of the centre of repulsion The isogonals will be drawn together in its neighbourhood, and there may be two systems of loops, one on each side of the centre

Figs 7 to 10

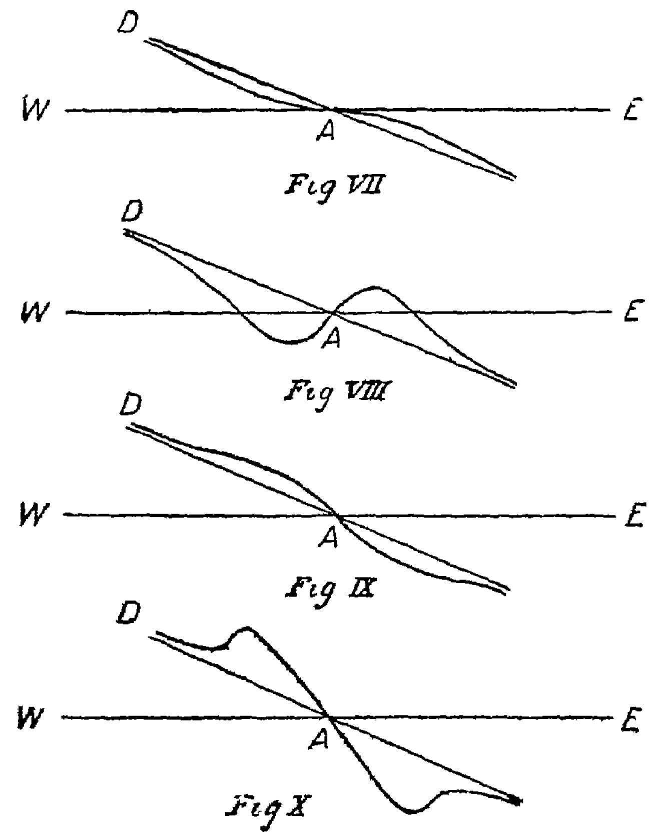


Fig VII

Fig VIII

Fig IX

Fig X

A centre of attraction will also be indicated by a convergence of the lines of equal Horizontal Force and Dip in its neighbourhood.

The observed will be greater than the calculated Force to the south, and smaller to north, of the centre. The Dip, on the other hand, will be less and greater than its calculated values at stations to the south and north respectively of a point at which the directions of the normal and disturbing Forces coincide.

The phenomena are more complicated if the effect of the disturbance is to increase the value of the element on that side of the centre of attraction on which it would be normally the greater, instead of increasing it (as in the case of the Declinations) on that side (the east) on which it is normally the less.

Thus, if we represent a northward movement by progress along the line WE (fig. 10) from W to E, the effect of a strong centre of attraction would be to check

the ordinary decrease of the Horizontal Force with latitude If the centre were sufficiently powerful, the decrease might be converted into an increase, and since immediately over the centre the value must be normal, a minimum and maximum must follow in order Another minimum and maximum may occur on the further side The effect of a centre of attraction on the lines of equal Horizontal Force is thus the same as that of a centre of repulsion on the isogonals In the case of the Dip, as in that of the Declination, there can only be two critical points on opposite sides of the point defined above

The Vertical Force will increase with latitude at more than the normal rate to the south of a centre of attraction, and at less than the normal rate to the north of it. The lines of equal Vertical Force will thus be drawn southwards in the neighbourhood of such a centre

If the Vertical Force attains a maximum value it will be at a point above the centre, and it must be followed by a minimum—at which, however, the Vertical Force will be greater than its calculated value.

It is evident from this discussion that there are a number of signs of a centre of attraction which may not all coexist, and which will be complicated in actual practice by irregularities in the distribution of the attracting masses, but which may nevertheless be of considerable practical help in a survey of local magnetic disturbances. (*Cf* LAMONT: 'Erdmagnetismus in Nord-Deutschland,' 1859, p 21)

We have dwelt on them, not because they present any difficulty, or even because they are altogether novel. Mr BENNETT BROUGH, A.R.S M, has given a full account of the methods employed by the Swedish mining engineers in exploring for iron ore. ("Use of the Magnetic Needle," &c 'Journal of the Iron and Steel Institute,' No 1, 1887, pp. 289–303.) They are accustomed to map out the neighbourhood of a mass of ironstone with a magnet, in order to determine its exact position. We believe, however, that the systematic use of the forms of the isomagnetic curves has been largely overlooked in the case of surveys comparable with that described in this paper

In Plates V. to VIII, the values of the Declinations, Horizontal Forces, Dips, and Vertical Forces determined at the various stations are entered and true magnetic curves are drawn These curves have not been chosen for equal differences of the values of the element to which they refer, but those have been selected which exhibit the most marked peculiarities. In this way attention is best drawn to disturbed stations and districts which can then be studied in accordance with the plan above suggested

Turning next to the investigation of the disturbances or differences between the observed and calculated values of the elements, we note that apart from the assurance they afford that the peculiarities studied are common to a district and do not depend only on a single station, they often supply additional information to that which can be gained from the true isomagnetics.

Thus, in the case of the Horizontal Force, a maximum followed by a minimum may be either to the north or the south of the centre of attraction. In the former case,

however, both the maximum and minimum values are less, while in the latter case they are greater than the normal values at the stations at which they occur, and if one maximum and minimum only are formed, the disturbances at once decide whether they are to the north or south of a centre of attraction

The lines which separate regions of positive disturbance in which the observed is greater than the calculated value from those in which it is negative are also of importance Thus, in the case of the Horizontal Force, if such a line is approximately perpendicular to the magnetic meridian, and if in passing over it from south to north we leave a region in which the disturbance is positive and enter one in which it is negative, we either pass over a centre or line of attraction or pass from the range of the influence of one centre of repulsion into that of another

If the centre is not at a great depth below the surface, and if the magnetic matter is not widespread at its minimum depth, it will cause a sudden reversal in the sign of the disturbances which will, however, be large near the centre on both sides. On the other hand, passage from the region of influence of one centre to that of another will, if the distance between them is considerable, be marked by a transition from small positive to small negative disturbances, or *vice versâ.*

Similar remarks may be made with respect to the disturbances of the Declination, but they are most easy to interpret when the line which separates a positive from a negative region runs approximately north and south If such a line meets another which separates a southern region of positive from a northern region of negative Horizontal Force disturbance, all the disturbing forces in the neighbourhood tend towards the point of intersection It will be convenient to speak of such a point as a *peak*, and to call a line which divides regions of positive from those of negative disturbance of the Declination or Horizontal Force, so as to indicate attraction towards it, a *ridge line.* In like manner, a line which separates the regions of influence of two attractive centres may be called a *valley line*

In the choice of these terms we are, no doubt, influenced by our views of the facts to be hereafter set forth, but they are convenient, quite apart from any theory of the cause of local magnetic forces It will be proved beyond doubt that in some cases these forces emanate from matter below the surface of the earth If this is so, an increment in their intensity must be due either to a closer approximation to the matter or to an increase in its magnetisation. We have no magnetic test to discrimate between the two, and therefore, without prejudice, adopt a nomenclature which is perhaps most consistent with the first hypothesis. If we wished to keep absolutely free from all expression of an opinion as to the causes of the phenomena, we might have called a peak (to which the lines of magnetic disturbing force converge) a magnetic sink, and so on, and for the present the terms we suggest may be taken as indicating merely points and lines from and to which such lines of force run.

The disturbances of the Declination and Horizontal Force, together with the ridge and valley lines are shown in Plates IX. and X.

The absolute value of the disturbance of the Vertical Force is especially uncertain In the cases of the Declination and Horizontal Force a centre of attraction affects stations on opposite sides with disturbances of opposite signs, to that the true mean value for the district can be found, and the true disturbance at each station deduced. It is, however, possible that the disturbance of the Vertical Force may always be of the same sign Thus, if the centres of force are rocks magnetised by induction, the north-seeking pole of a magnet would most frequently be attracted downwards in the northern hemisphere as the north-seeking poles of the rock magnets would be deeply buried in the earth This hypothesis is that which experience has justified in Sweden If this is so, the mean value of the Dip or Vertical Force in a district will be greater than its undisturbed value, and negative values do not necessarily indicate an upward force It is therefore safer only to use the disturbances of the Vertical Force as a means of indicating relative maxima and minima

If, however, centres of repulsion exist, they might be detected by the observation that a large negative (upward) disturbance of the Vertical Force was accompanied with a sudden reversal of the disturbing Horizontal Force in neighbouring stations If the Horizontal Forces are small, the negative disturbing Vertical Force is more probably to be interpreted as indicating a downward attraction of less than average magnitude.

On turning to Plate XI., in which the disturbances of the Vertical Force are plotted down, it will be observed that there is, on the whole, an excess of positive values in the south-east, and that large negative values are more common in the north and west It is obvious that in Sc and the disturbances are much greater than in England, and very large positive val s of vertical disturbance occur in the western isles Indeed, if we include the enormous value obtained at Canna, the average for the whole of Scotland is positive Leaving out, however, this very abnormal station, it can hardly be doubted that negative values are more common in the north-west. If this result could be trusted as corresponding to physical fact, it might indicate that the country as a whole is magnetised in the direction of the magnetic meridian We cannot, however, draw such a conclusion, more especially as it can be shown that the observed effect may probably be a result of our ignorance of the datum lines from which the disturbances of the Dips and Vertical Forces ought to be measured

We have taken the Dip, as given by direct observation, as one of our fundamental elements, and in drawing the terrestrial curves which satisfy the conditions that the mean value of the Dip for each district when attributed to the central station is accurately reproduced by the formula, we have been compelled to ignore the possibility of an unbalanced downward Vertical Force acting at every station. If we suppose that in consequence of this the calculated Dips ought, all over the country, to be diminished by a positive quantity, we get for the corresponding decrement in Vertical Force

$$dV = H \sec^2 \theta \, d\theta.$$

If we take the values of H and θ at Wick and St Leonards and write for $d\theta$ the circular measure of 1′ multiplied by x, the unknown number of minutes by which the calculated Dip has been taken too large, we get at

$$\text{Wick} \quad . \quad . \quad . \quad . \quad dV = 0\,0048x,$$
$$\text{St Leonards} \quad . \quad . \quad . \quad dV = 0\,0036x.$$

Hence in the north the error will be greater than in the south, that is, the excess of the calculated over the true undisturbed Vertical Force is greatest in the north Consequently the difference between the observed and calculated Vertical Force will be more frequently negative in the north, the excess of the error amounting to 0 0012 metric unit for every minute by which the calculated Dip is too large

If, then, apart from local attractions the country, as a whole, attracts the needle so that the Dip is everywhere 10′ greater than it would be if the British Isles were replaced by sea, the error in the calculated Vertical Forces deduced from calculated Dips, obtained on the assumption that no such defect exists, would be 0 0120 greater in the north of Scotland than in the Channel, so that negative disturbances would largely predominate in the north

As from the geological character of Scotland it is probable that the error in the calculated Dip would not be constant, but would be greater than in England, the validity of this explanation is even more probable than this calculation indicates

It is not, therefore, safe to draw conclusions based on the relation between the Vertical disturbing Forces at distant stations

In the calculation of the disturbing forces we are not aiming at, and could not attain to, results which would give more than a general idea of their direction and magnitude. It may be possible to conclude with certainty that an attractive mass exists near certain stations, even if the directions of the forces are wrong by 10° or 15° and their magnitudes are inaccurate by 50 per cent

To show that a degree of accuracy of this kind is attained we collect here the results at stations where we have taken a full set of observations at two or more places or on different occasions There are of course localities where it would be hopeless to get a good result from a single observation Such places are Canna and Portree At these we should never have observed except for the sake of investigating the disturbances which we knew were very great, and we therefore do not now take them into account It is sufficient if the agreement between different observations is satisfactory at stations which we regarded at the time of observation as normal.

At Stornoway we made four sets of measurements, the first in 1884 in the Castle Grounds, the other three in 1885 and 1888 on Ard Point, about a mile distant from the former station. The ground was not good, as when we observed simultaneously about 50 yards apart on Ard Point the declinations differed by 18′

At Loch Aylort we observed in 1884 and 1888 as nearly as possible on the same station At Oban we observed on the mainland in 1888, and on the island of Kerrera

which is about a mile distant, in 1884 and 1885 In the Sound of Islay we had two stations about four miles apart, viz, Port Askaig in 1884 and Bunnahabhain in 1888 All these places, though typical Scotch stations, were on ground which elsewhere would be considered but indifferently good for our purpose At three other places more favourably situated we also observed twice, viz, at Stranraer in 1884 and 1888, at Reading in 1886 and 1888, and at Bude Haven in 1886 At the two former the stations were as nearly as possible the same on the two occasions, at Bude Haven they were less than half a mile apart

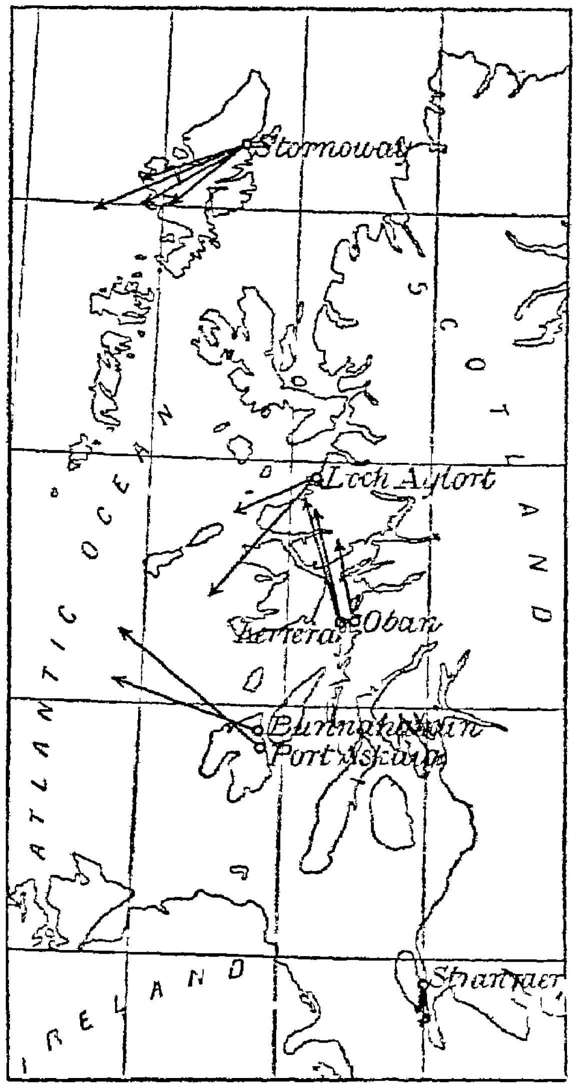
The results are summed up in the following Table. The resultant Horizontal disturbing Force is indicated by F and the angle which its direction makes with the geographical meridian by ϕ. The latter is taken as positive on the western side, due north being represented by 0°.

Station	Rock	Date	F	φ	Z
				°	
Stornoway—					
(1) Castle Grounds	Gneiss	1884	0175	129 6	− 0113
(2) Aird Point		1885	0179	109 2	Dip not observed
		1888 (T)	0198	119 7	+ 0046
		1888 (R)	0278	113 5	+ 0120
Loch Aylort	Gneiss	1884	0267	141 1	− 0326
		1888	0139	112 3	+ 0178
Oban—					
(1) Mainland	Trap	1888	0143	10 9	− 0129
(2) Kerrera		1884	0196	16 9	− 0132
		1885	0219	16 2	− 0124
Sound of Islay—					
(1) Port Askaig	Primary	1884	0309	53 1	+ 0141
(2) Bunnahabhain	Limestone	1888	0265	69 9	+ 0122
Stranraer	Clay, Slate	1884	0038	178 5	− 0072
		1888 (R)	0065	181 8	− 0034
		„ (T)	0062	175 4	
Reading	Clay	1886	0049	− 8 3	+ 0235
		1888	0084	−10 2	Dip not observed
Bude Haven	Shale	1886	0065	143 1	+ 0091
			0048	180 0	+ 0030

The annexed small map (fig. 11) illustrates these numbers by showing the direction and magnitudes of the disturbing forces as determined on two occasions at stations in Scotland.

An inspection of this map and of the table justifies the statement that the magnitude and direction of the disturbing forces can be determined with an accuracy sufficient to enable us to draw conclusions from groups of stations even if it would not always be safe to argue from one.

Fig 11



If the station is on good ground we may place more confidence in the results Stranraer is interesting as being the only place where we both took complete sets of observations simultaneously on good ground. The results are in very close accord But those obtained elsewhere leave no doubt as to the order of the magnitude of the disturbing force nor as to its direction to within (in unfavourable cases) 15° or 20°.

The difference in the signs of the Vertical Forces obtained at Loch Aylort and Stornoway in different years may in part be due to the uncertainty of the secular correction for the Dip, which appears to be very abnormal, especially at Loch Aylort (see p. 86) In this case we should be driven to the conclusion that a real change in the local Force had taken place It is noticeable that the Vertical Force disturbance was apparently (algebraically) greater at Stornoway, Loch Aylort, Kerrera, and Stranraer on our second visits to these places

The following Table contains the particulars as to the disturbing force at every station in accordance with the notation described above On Plate XIII., the directions and magnitudes of the Horizontal disturbing Forces are shown, and regions of positive and negative Vertical Force disturbance are indicated, the former being shaded.

TABLE of Disturbing Forces.

No of Station	Name of Station	F	φ	Z
			°	
1	Aberdeen	82	−144	− 28
2	Arinagower	.	. .	.
3	Loch Aylort	267	+141	− 326
		139	+112	+ 178
4	Ayr	61	− 22	− 145
5	Ballater	151	−108	− 41
6	Banavie	176	+ 18	+ 268
7	Banff	112	+ 70	− 124
8	Berwick	179	− 15	+ 230
9	Boat of Garten	212	+ 86	+ 56
10	L Boisdale	301	−114	− 736
11	Bunnahabhain	265	+ 70	+ 122
12	Callernish	45	+ 21	+ 31
13	Campbelton	36	+136	− 9
14	Canna	737	−112	+1839
15	Carstairs	173	+ 14	− 79
16	Crianlarich	39	+ 83	− 219
17	Crieff	114	+ 74	+ 217
18	Cumbrae	222	+174	+ 215
19	Dalwhinnie	127	+ 36	− 247
20	Dumfries	29	+ 56	− 87
21	Dundee	23	+180	+ 10
22	Edinburgh	17	+152	+ 107
23	Elgin	86	− 67	+ 45
24	L Eriboll	37	− 56	+ 175
25	Fairlie		.	+ 135
26	Fort Augustus	74	−101	+ 57
27	Gairloch	210	− 85	− 451
28	Glasgow	117	−133	− 80
29	Golspie	89	− 92	− 65
30	Hawick	76	− 1	− 99
31	L Inver	326	−129	− 853
32	Inverness	61	0	+ 156
33	Iona	326	+ 59	+ 202
34	Kirkwall	27	− 39	+ 109
35	Kyle Akin	140	+154	− 168
36	Lairg	73	− 79	− 34
37	Lerwick	62	−144	+ 378
38	Lochgoilhead	23	+122	− 328
39	Loch Maddy	73	− 72	− 114
40a	Oban	143	+ 11	− 129
40b	Oban (Kerrera)	196	+ 17	− 132
		219	+ 16	− 124
41	Pitlochrie	33	− 63	− 242
42	Port Askaig	309	+ 53	+ 141
43	Portree	619	+145	+ 712
		384	+ 11	− 378
		806	− 89	+1705
44	Row (Gairloch)	121	−170	− 188
45	Scarnish	394	+ 98	+ 444
46	Soa	518	−166	− 385
47	Stirling	166	+177	− 92
48a	Stornoway (Ard Point)	179	+109	. .
		198	+120	+ 46
		278	+114	+ 120
48b	Stornoway (Castle)	175	+130	− 113
49	Strachur	88	− 1	− 269

TABLE of Disturbing Forces—*continued*

No of Station	Name of Station	F	φ	Z
50	Stranraer	{38 65 62	+178 −178 +175	− 72 − 34
51	Stromness	80	− 25	+146
52	E Loch Tarbert	54	−167	−150
53	Thurso	20	− 53	− 47
54	Wick	150	−146	+242
55	Aberystwith	77	− 83	− 78
56	Alderney	117	− 94	− 43
57	Alnwick	29	+ 28	− 50
58	Alresford	48	− 47	+ 6
59	Appleby	69	+168	− 56
60	Barrow	35	− 85	− 71
61	Bedford	132	+168	+ 39
62	Birkenhead	15	+ 8	+ 26
63	Birmingham	124	− 40	+152
64	Braintree	91	+136	+ 50
65	Brecon	28	−135	+ 7
66	Bude Haven	{65 48	+143 +180	+ 91 + 30
67	Cambridge	57	+158	+110
68	Cardiff	32	− 35	− 53
69	Cardigan	67	−175	− 52
70	Carlisle	41	+ 77	− 52
71	Chesterfield	56	+ 78	+145
72	Chichester	46	+ 37	+130
73	Clenchwarton	77	+ 67	+207
74	Clifton	27	+ 57	+ 33
75	Clovelly	63	+179	+ 21
76	Coalville	57	− 63	−109
77	Colchester	133	+102	− 99
78	Cromer	25	+ 92	+208
79	Dover	60	−102	+ 21
80	Falmouth	47	+175	−190
81	Gainsborough	42	− 36	+132
82	Giggleswick	49	− 45	+ 12
83	Gloucester	46	+126	+ 46
84	Grantham	41	− 7	+ 50
85	Guernsey, L'Erée	101	−134	− 13
86	„ Peter Port	56	− 96	+ 49
87	Harwich	30	+ 17	+ 79
88	Harpenden	70	+148	+146
89	Haslemere	44	+ 30	+ 87
90	Holyhead	153	−166	+ 21
91	Horsham	57	+ 73	+ 20
92	Hull	124	+136	+ 94
93	Ilfracombe	36	−162	− 2
94	Jersey, Grouville	168	+ 77	+ 9
95	„ S Louis	144	−103	−229
96	„ S. Owen	65	+144	−131
97	Kenilworth	103	+173	+287
98	Kettering	59	+150	− 92
99	Kew	90	+119	+162
100*a*	King's Lynn	46	+ 19	+195
100*b*	„ „ (Gaywood)	40	+ 60	+139
101	King's Sutton	85	+138	+121
102	Lampeter	81	−123	−137

TABLE of Disturbing Forces—*continued*

No of Station	Name of Station	F	ϕ	Z
			°	
103	Leeds	45	− 51	+ 59
104	Leicester	108	− 95	− 27
105	Lincoln	89	− 48	+172
106	Llandudno	159	+134	+ 12
107	Llangollen	71	+107	− 9
108	Llanidloes	20	−140	− 22
109	Loughborough	155	− 65	+ 22
110	Lowestoft	61	+116	+192
111	Mablethorpe	34	+113	+163
112*a*	Malvern, Colwall	119	−140	
112*b*	„ Great Malvern	121	+127	+ 21
112*c*	„ Wells	81	+149	.
112*d*	„ Mathon	147	− 98	
113	Manchester	105	−104	− 50
114	Manton	70	− 18	+ 79
115	March	36	+ 2	+111
116*a*	Melton Mowbray	185	+ 85	+305
116*b*	„ „	182	+152	+ 80
117	Milford Haven	98	− 23	− 72
118	Newark	56	+ 49	+ 6
119	Newcastle	35	− 27	− 4
120	Northampton	111	+165	− 69
121	Nottingham	33	− 27	+179
122	Oxford	38	−153	+169
123	Peterborough	52	+ 56	+150
124	Plymouth	30	+168	−210
125	Port Erin	135	−134	− 55
126	Preston	4	− 34	+ 49
127	Purfleet	60	+111	+ 89
128	Pwllheli	125	+ 45	+114
129	Ramsey	121	−150	+ 4
130	Ranmore	85	+ 65	+ 11
131*a*	Reading	49	− 8	+235
131*b*	„ (Caversham)	84	− 10	.
132	Redcar	23	− 15	+ 11
133	Ryde	59	− 56	− 18
134	St Cyres (Exeter)	72	+ 50	− 5
135	St Leonards	6	− 31	− 61
136	Salisbury	91	− 29	+ 43
137	Scarborough	47	+ 52	+ 63
138	Shrewsbury	118	−173	−319
139	Southend	81	+126	+ 74
140	Spalding	134	− 98	− 48
141	Stoke-on-Trent	17	+ 10	+ 85
142	Sutton Bridge	55	− 51	+200
143	Swansea	105	+ 4	+ 30
144	Swindon	. .		+ 38
145	Taunton	26	0	− 59
146	Thetford	17	− 7	+ 94
147	Thirsk	38	+138	+173
148	Tilney	29	− 4	+311
149	Tunbridge Wells	22	+ 95	−104
150	Wallingford	26	−108	+145
151	Weymouth	13	180	−144
152	Wheelock	. .	. .	. .
153	Whitehaven	42	+ 69	+ 24
154	Windsor	58	+104	+136

TABLE of Disturbing Forces—*continued*

No of Station	Name of Station	F	ρ	Z
155	Wisbech	22	+ 30	+227
156	Worthing .	44	+ 90	+ 24
G	Greenwich	36	+ 79	− 5
S	Stonyhurst	22	− 93	−223
157	Armagh	25	+ 27	−158
158	Athlone	59	− 81	− 68
159	Bagnalstown	38	+131	−113
160	Ballina	224	−146	− 74
161	Ballywilliam	104	−154	+ 45
162	Bangor	68	+ 3	− 93
163	Bantry	39	+ 52	− 12
164	Carrick-on-Shannon	49	+ 72	−171
165	Castlereagh	37	+107	+ 25
166	Cavan	68	+177	−111
167	Charleville	31	−147	−152
168	Clifden . .	183	+145	+ 37
169	Coleraine	226	−162	+106
170	Cookstown Junction	402	− 6	−677
171	Cork	52	+ 49	+ 3
172	Donegal	56	+ 63	−250
173	Drogheda	44	+ 33	− 14
174	Dublin	103	+ 21	−126
175	Enniskillen	107	+160	− 78
176*a*	Galway	386	+159	−338
176*b*	,,	97	+106	− 51
177	Gort	67	− 5	+ 56
178	Greenore	134	+ 78	− 60
179	Kells	108	+ 14	+106
180	Kildare	53	+ 36	−118
181	Kilkenny	43	+ 41	− 4
182	Killarney	85	+ 42	− 54
183	Kilrush .	29	+110	− 4
184	Leenane	195	−151	−151
185	Limerick	105	+ 14	− 89
186	Lisdoonvarna	117	− 7	+100
187	Lismore	36	+ 46	−135
188	Londonderry	24	0	−116
189	Oughterard	92	+130	+183
190	Parsonstown	8	−140	+ 48
191	Sligo	107	−126	− 79
192	Strabane	45	−102	−163
193	Tipperary	38	+ 47	− 31
194	Tralee .	..		.
195	Valentia .	118	+ 42	− 53
196	Waterfoot	85	−176	+ 1
197	Waterford	18	− 98	−118
198	Westport	174	−116	+196
199	Wexford .	70	−134	+ 15
200	Wicklow	36	−114	−117

Surveys of Selected Districts

Having described and, as we hope, justified the methods by which we propose to investigate local and regional disturbances, we now proceed to discuss their application to districts of which we have made a special study The results we shall arrive at will help us in the further elucidation of the magnetic state of the whole country

The Malvern Hills

The general nature and direction of the magnetization of igneous rocks is a problem on which comparatively few observations have been made It is, indeed, known that when examined in detail they present great irregularities, and Commander CREAK, F R S, has shown ('Roy Soc Proc,' vol 40, 1886, p 83) that when islands disturb the magnetic needle in the northern hemisphere they attract, and in the southern repel the north-seeking pole of a magnet This is what would be expected if they were the upper extremities of magnetic masses magnetized by the Earth's induction

In like manner in Sweden, where (as has already been stated) the method of searching for iron ore by means of the magnet has been carried to considerable perfection, the assumption made, and justified by experience, is that the upper parts of the beds of ironstone attract the north-seeking pole

Observations somewhat similar to those of Commander CREAK can be carried out on land in cases where igneous rocks rise in the midst of sedimentary deposits, and such observations are specially interesting in cases, such as that of the Malvern Hills, in which the axis of the magnetic mass runs north and south

If its depth is considerable with respect to its length, and if it is magnetized by induction, we should expect the upper visible parts to attract (in the northern hemisphere) the north-seeking pole. If, on the other hand, it is possible to conceive of a shallow mass of magnetic rock surrounded by non-magnetic matter, the northern end might repel the north-seeking pole. Finally, if the mass was itself magnetized independently of the present inductive action of the Earth, as is certainly the case with small masses of highly magnetized lodestone, its effect on a compass needle could only be determined by experiment

With the view, then, of making a beginning towards the study of these questions in the United Kingdom, we determined the polarity of the northern end of the Malvern Hills.

This range consists of syenite and granite The mass of igneous rocks runs due north and south for about eight miles, and is at the broadest part but little more than half a mile wide. On its eastern side is a great fault which extends many miles north and south of the range itself. Two stations, Great Malvern (112*b*) and Malvern Wells (112*c*), were taken on the eastern side of the hills They were both on the Red Marl, and distant about a mile and a quarter from the centre of the range Two

corresponding stations were also taken on the western side at about the same distance, and both on the Old Red Sandstone Of these Mathon (112*d*) was 0 3 mile to the north of the latitude of Great Malvern, and the line joining them cuts the range at about a mile from its northern end, passing close to its highest point Colwall (112*a*) is about three-quarters of a mile to the south of the latitude of Malvern Wells, and the line joining them cuts the range about 2½ miles south of the northern termination of the igneous rocks

The Declination and Force were determined at all these stations Time did not allow of the Dip being taken elsewhere than at Great Malvern

The results are given in the following Table --

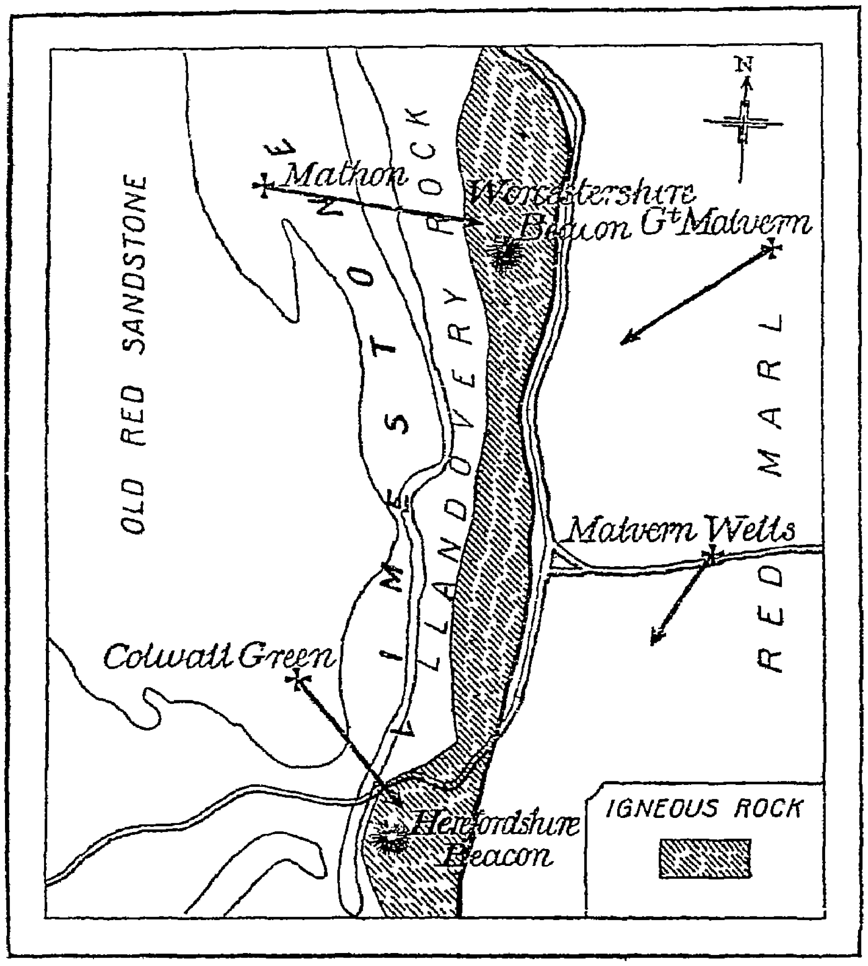
		Declination	Force
Eastern Stations	Great Malvern	19° 33′ 0	1 7687
	Malvern Wells	19 22 4	1 7682
Western Stations	Mathon	18 46 5	1 7655
	Colwall Green	19 3 6	1 7627

From these the disturbing forces were deduced, the notation being the same as that used on p 268

	F	φ
Great Malvern	0121	126 6
Malvern Wells	·0081	148 7
Mathon	0147	− 97 8
Colwall Green.	0119	−140 4

The accompanying Map shows the direction of these forces and their relative magnitudes. The Worcestershire Beacon (1440 ft) is the highest point on the range ; to the south of this the height diminishes, and then increases again to the Herefordshire Beacon. The directions of the disturbing forces tend towards these hills, and the results are, we think, only compatible with the view that the Malverns attract the north-seeking pole of the magnet.

Fig 12



Disturbing Forces near the Malvern Hills

The Island of Canna

The attraction exerted by the Malverns having been demonstrated, it is convenient to discuss in the next place a locality where the disturbances are enormously greater

Popular tradition has long attributed to the basaltic rocks of the Island of Canna the power of deviating the needle through very large angles The compasses of passing ships are supposed to be affected by the eastern extremity of the island, on which stands Compass Hill

Magnetic observations were made on the Island by Sir FREDERICK EVANS ('Phil. Trans,' 1872, vol. 162, p. 325), but, we have, we believe, been able to add considerably to what was already known of its magnetic properties.

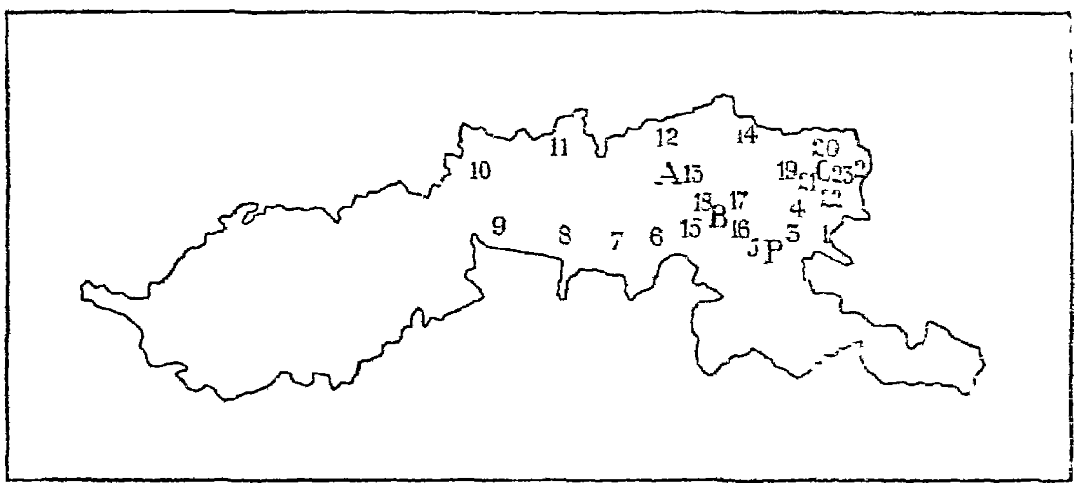
The island, which is about fifteen miles south-west of Skye, is about five miles long, its greatest length lying nearly due east and west It is divided into two approximately equal portions by a neck of comparatively low elevation. The highest ground is in the eastern part where it rises to a height of 724 feet, and our observations have been confined to this portion. The cliffs on the north side are here some hundreds of feet in height and fall sheer into the sea. On the south side several small valleys lead to the shore, but the hills rise very steeply about a quarter of a mile inland, and the tops of several consist of irregular masses of basaltic columns. On Compass Hill,

which is the most easterly point on the island, the longer axis of the mass runs north and south. Its height on the inland side is from 20 to 40 feet, while it falls steeply towards the sea. On another hill which rises behind Kaill, the residence of R. THOM, Esq., the columnar mass is small and more clearly defined. The north or landward side is 15 or 20 feet high, the south side is a steep cliff. We roughly estimate the length at about 50 yards. In fig. 13 the position of this hill is indicated by B, that of the highest point on the island by A, and that of Compass Hill by C.

In 1884 we determined the magnetic elements by means of the Kew Magnetometer No. 60, and the Dip Circle No. 74, at the position marked P. The differences between the observed and calculated values are given in the following table:—

Date 1884			
	Observed	Calculated	Difference
Declination	21° 8'.4	23° 6'.5	−1° 58'.1
Dip	72 45.0	71 32.7	1 12.3
Horizontal Force	1.5092	1.5607	−.0515

Fig. 13.



Stations on Canna.

Although these observations show that the station was highly disturbed, the effect on the Declination is not so great as to be detected, except by a careful observation. In 1888 we observed by means of a small azimuth compass at 23 stations which are indicated on the map, and determined at each the bearings of a number of distant points. The observations were made on August 16 and 17, bright sunny days with a northerly wind, on which the atmosphere was very transparent. We were therefore able to take the bearings, not only of prominent headlands on Skye and Rum, but also of some distant objects, such as the Ushinish Lighthouse on N. Uist in the Hebrides.

The azimuth compass was a small instrument which did not admit of great accuracy, but, as will be seen in the sequel, it was sufficient for the purpose we had in view.

We carried a chart with us, and marked the position of each station while on the spot On the average, about four bearings were taken at each station, and at no station was the number less than two

The bearings of the objects selected were afterwards taken from the chart, and, by comparing these with the observations, the Declination was determined The agreement between the individual observations at each station was in general only moderately good, but, even in cases where only two observations were taken, we think the means are accurate to about half a degree

Throwing out thirteen stations which were evidently highly disturbed, we took the mean of the Declinations at the other ten as giving the mean Declination for the easterly half of the island, which is 22°·8.

By subtracting this from the Declinations found at the stations, we obtained the disturbance of the Declination at each The results are given in the following Table —

Station	Disturbance	Station	Disturbance
I	0°·4 W	XIII	0°·4 E
II	1·6 E	XIV	6·0 W
III	0·9 W	XV (W)	1·9 W
IV	3·3 W	XVI (S)	10·4 E
V	1·1 W	XVII (E)	3·6 E
VI	1·6 E	XVIII (N)	11·5 W
VII	1·5 W	XIX	5·8 E
VIII	1·3 W	XX (N)	9·9 W
IX	8·1 W	XXI (W)	25·8 E
X	0·6 E	XXII (S)	10·3 E
XI	5·8 E	XXIII (E)	23·8 E
XII	0·3 E		

Stations XV to XVIII. inclusive were taken round the basaltic mass on the summit of the hill behind Kaill, and on the sides indicated by the letters which follow the numbers.

Stations XX. to XXIII. were in like manner taken round the summit of Compass Hill. In both these cases the compass was generally within a foot or two of the basaltic columns. The observations, therefore, show that these are powerfully magnetic. The disturbing force at Station XXI. was nearly half that due to the horizontal intensity of the earth's magnetic field. On the other hand, it is evident that their influence diminishes very rapidly with the distance. Station VI was only a few hundred yards from the hill behind Kaill, yet the compass was not affected by more than 1°·6.

Stations I., II., III., and IV. are grouped round the southern half of Compass Hill, at distances between 200 and 500 yards from the summit, but the largest disturbance of the Declination is 3°·3 W.

Stations XXIII. and II. are situated one above the other on the eastern side of Compass Hill, the horizontal distance between the two being not more than 80 yards,

yet the disturbance diminishes from 23° 8 E at Station XXIII, near the top of the hill, to 1° 6 E at Station II, near its base

These conclusions are completely borne out by observations made on the ' Coventina," in which we visited the island in 1884 and 1888

On leaving Canna for Loch Boisdale in 1884 we sailed as close as possible to the northern face of the island, and took frequently the compass bearings of points on Skye. We were unable to detect the smallest deviation of the needle

In 1888 observations were made under still more favourable circumstances We approached the island from the north, and, when about three miles distant, the yacht was directed towards a mark on Rum, by which its course could be kept without reference to the compass We were then sailing magnetic S $\frac{1}{4}$ E, in the most favourable direction for the effect of Compass Hill (if any) to be detected We passed it within 200 yards of the shore but observed no deviation of the compass, and we are quite certain that, if there was any, it was less than one-eighth of a point, *i e*, less than 1° 5.

The net result of our observations is that the basaltic cliffs of Canna are powerfully magnetic, and may deviate the needle of a compass placed near them by about two points, *i e*, about 23°, but that the effect diminishes very rapidly with the distance, and is inappreciable on a ship's compass 200 yards from the base of the hill, to which tradition ascribes, and in which we have ourselves detected, the most powerful magnetic properties

We have adopted 22° 8 as the mean value of the Declination at the less disturbed stations in Canna in August, 1888, this leads to 23° 13′ for January 1, 1886, which is only 6′ in excess of the calculated value This is interesting, inasmuch as Plate IX shows that neighbouring stations have Declination disturbances of opposite signs, and indicates that a line of no regional disturbance runs near to Canna

Thus the four stations, Kyle Akin (No 35), Soa (No 46), Canna (No 14), and Loch Boisdale (No. 10), lie very nearly in a straight line, and the disturbances of the Declinations vary continuously, being 28′ 0, 18′ 4, 6′ 0, and — 45′ 9, which proves that Canna lies near an attractive centre or ridge This is in harmony with the fact that the disturbance of the Vertical Force is positive, and is enormously great amounting to 0 1839, or about 0 04 of the whole Vertical Force

A comparison of the results obtained at Malvern and Canna points very clearly to the otherwise probable conclusion that far-reaching effects are to be expected quite as much from the great mass and uniform magnetization of rocks as from their being highly magnetized The basalt of Canna is far more susceptible and more magnetic than the Malvern syenite, but we doubt if a mile and a quarter from Compass Hill it would produce an effect on the Declination needle at all equal to that which we have shown is due at that distance to the Malverns

The Eastern and South-Eastern Counties

A problem of considerable interest has to be considered in connection with the Eastern and South-eastern counties. In the greater part of this district, the surface soil is such that it certainly can produce no marked effect upon the magnet, yet in this apparently "good ground," we have found magnetic disturbances of very wide range

It is well known that the Declinations obtained at the Kew and Greenwich Observatories differ more widely than the difference of longitude will explain The difference is not so great as appears at first sight, as the Kew results as published are not corrected for diurnal variation.

The published mean Declinations at these two stations in 1886 are, Greenwich 17° 54′ 5, and Kew 18° 16′ 9, of which the Kew result must be diminished by about 6′ This gives a difference of 16′, which exceeds that corresponding to the difference of longitude by 10′. We are not aware that any attempt has hitherto been made to connect this discrepancy between the Declinations at these two important observatories with any regional disturbance in their neighbourhood

The fact which first led us to believe that they lie within the area of such a disturbance was that there is not only a small decrease in the Declination between Kew and Reading (instead of an increase as the difference of longitude requires), but that there is also a very small difference between Worthing and Ryde.

The Declination at Reading should be about 20′ greater than that at Kew, but the observation made there in 1886 proved that it is 1′ less.

The Declination was again determined in 1888 at the same spot near Reading, with the result that the Declination, when reduced to epoch, came out 3′ 6 lower than before, thus increasing the discrepancy between the actual and calculated differences between the two stations.

In like manner, the difference of longitude between Ryde and Worthing is equivalent to a change of 22′ in the Declination, whereas the observed value at Ryde was only 2′ 6 higher than that at the more easterly station

In order to investigate these differences more fully, we determined to run a chain of stations along the valley of the Thames, to observe at another series half-way between the Thames and the Channel, and to interpolate a station between Worthing and Ryde.

The result is shown in the annexed diagram (fig 14). Horizontal lengths indicate the longitudes of stations and vertical lines the Declination.

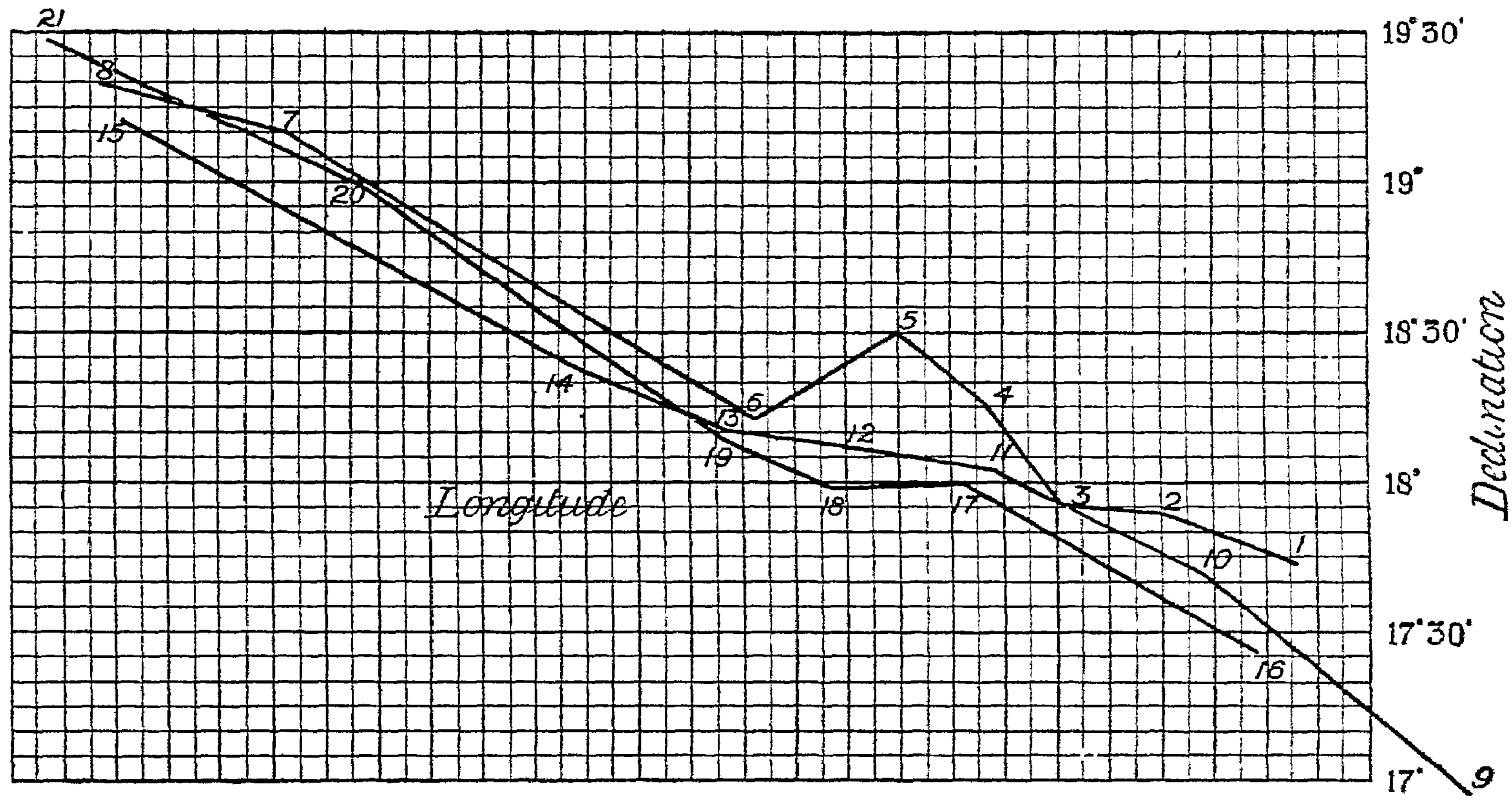
Three groups of stations, such that all places included in each are of nearly the same latitude are taken, thus forming three lines crossing the whole of the south of England. In the cases of Weymouth and Ryde, which lie considerably to the south of the line which nearly passes through the other stations corrections of 14′ and 7′ respectively have been added. Chichester is to the north of the line and its Declination has also been corrected by 7′.

It will be noticed that all three exhibit an anomaly on or immediately to the south

of a line joining Greenwich and Reading. In the valley of the Thames the Declination attains a maximum value near Windsor and a well marked minimum near Reading. To the south of this district there is a remarkable slackening in the rate of increase of the Declination.

An inspection of this figure is sufficient to prove that the anomaly in the Declination difference of Greenwich and Kew is not due to any accidental peculiarity of the position of either observatory, but is the result of a regional disturbance extending at least from that part of the valley of the Thames which lies between Greenwich and Reading to the south coast.

Fig. 14.



Declinations at—

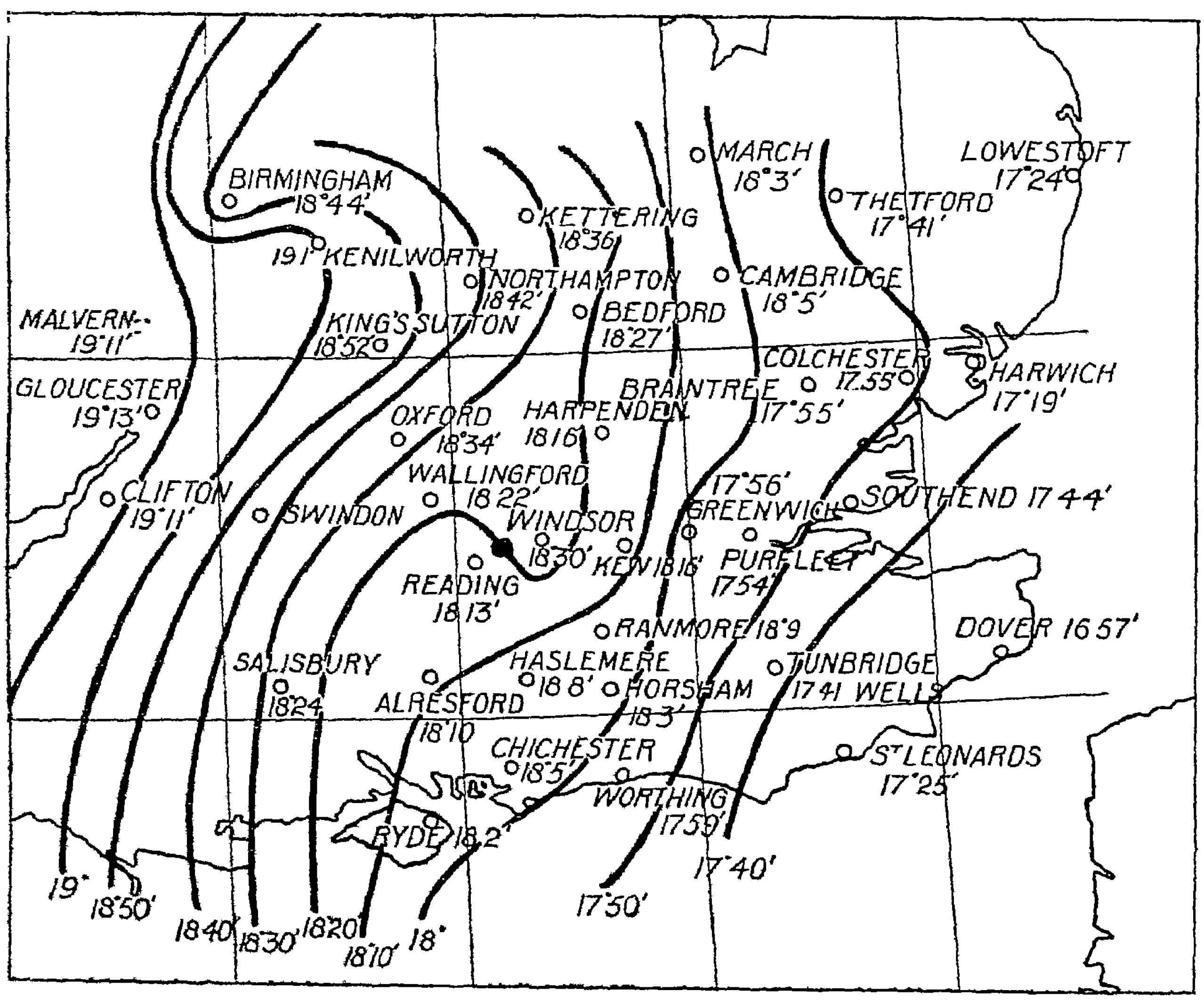
1 Southend
2 Purfleet
3 Greenwich
4 Kew
5. Windsor
6 Reading
7 Clifton
8 Cardiff
9. Dover
10 Tunbridge Wells
11 Horsham
12 Haslemere
13 Alresford
14. Salisbury
15 Taunton
16 St Leonards
17 Worthing
18 Chichester
19 Ryde
20 Weymouth
21 Exeter

The curves in the diagram when compared with figs. 7 and 8 (p. 263) are seen to be such as would be produced if a centre of force attracting the north-seeking pole of a magnet were situated near Windsor. Immediately over such a centre the value of the Declination would be normal, while it would be too great and too small at stations to the east and west respectively. If the centre were relatively weak the increase in the Declination with longitude instead of being represented by the slope of a straight line would be given by a curve of the same type as that in fig. 7, if it were strong we should have a curve like that in fig. 8.

Fig 15 shows the true isogonals in this part of the country and proves that the form they assume corresponds to the last case

The existence of a widespread disturbance is proved not only by the remarkable bend in the 18° 10′ isogonal between Kew and Reading, but also by several other stations Thus the Declination at Harpenden is the same as that of Kew (18° 16′), though it is about thirty miles to the north of it and should therefore be 9′ greater Again, Ranmore, Haslemere and Alresford, with Declinations of 18° 9′, 18° 8′ and 18° 10′ prove that the 18° 10′ line has really the great inclination to the meridian which the outer curves in fig 6, p 262, show They are further supported by Horsham, Chichester, and Ryde, with Declinations of 18° 3′, 18° 5′ and 18° 2′ respectively

The isoclinals and lines of equal Horizontal Force do not show any equally striking peculiarities in this part of the country. The latter, however, run a little too far north to the south of the Thames (see Plate VII) This is as it should be, for to the south of an attracting centre the Horizontal Force will be abnormally great

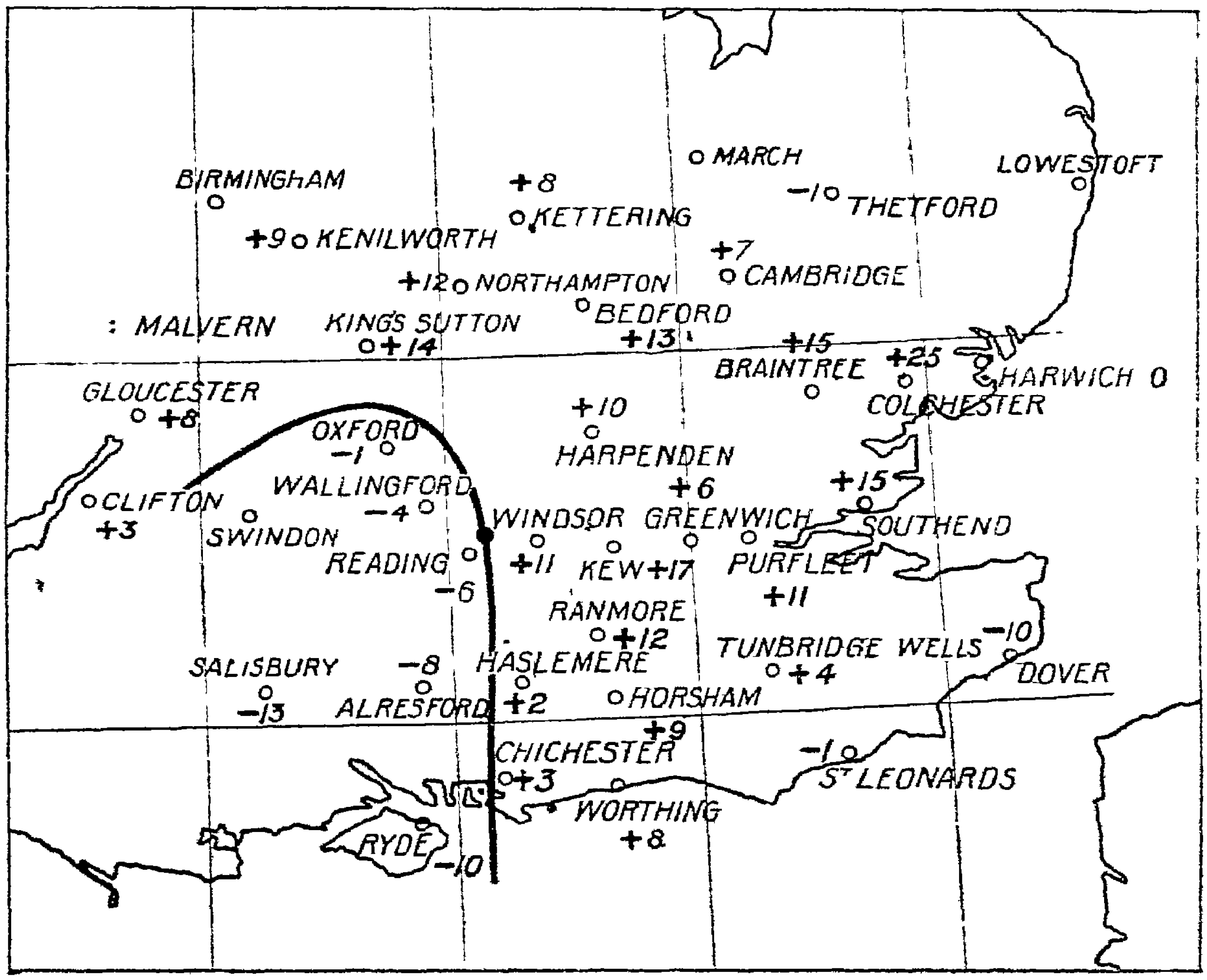


Isogonals in South-Eastern England.

The point at which the terrestrial and true isogonals would intersect if the district were not otherwise disturbed is marked by a dot, which will hereafter be called the *focus* of the Reading disturbance.

Let us now turn to the calculated disturbances of the elements In the accompanying map (fig 16) the figures represent the differences between the observed and calculated values of the Declination expressed in minutes of arc, and taken as positive when the needle is turned to the west There is a sharply marked boundary between the regions of positive and negative disturbance which passes through the focus To the west of it the needle is deflected to the east and *vice versa*

Fig 16



Declination disturbances in minutes of arc

+ Indicates that the observed westerly Declination is greater than the calculated value

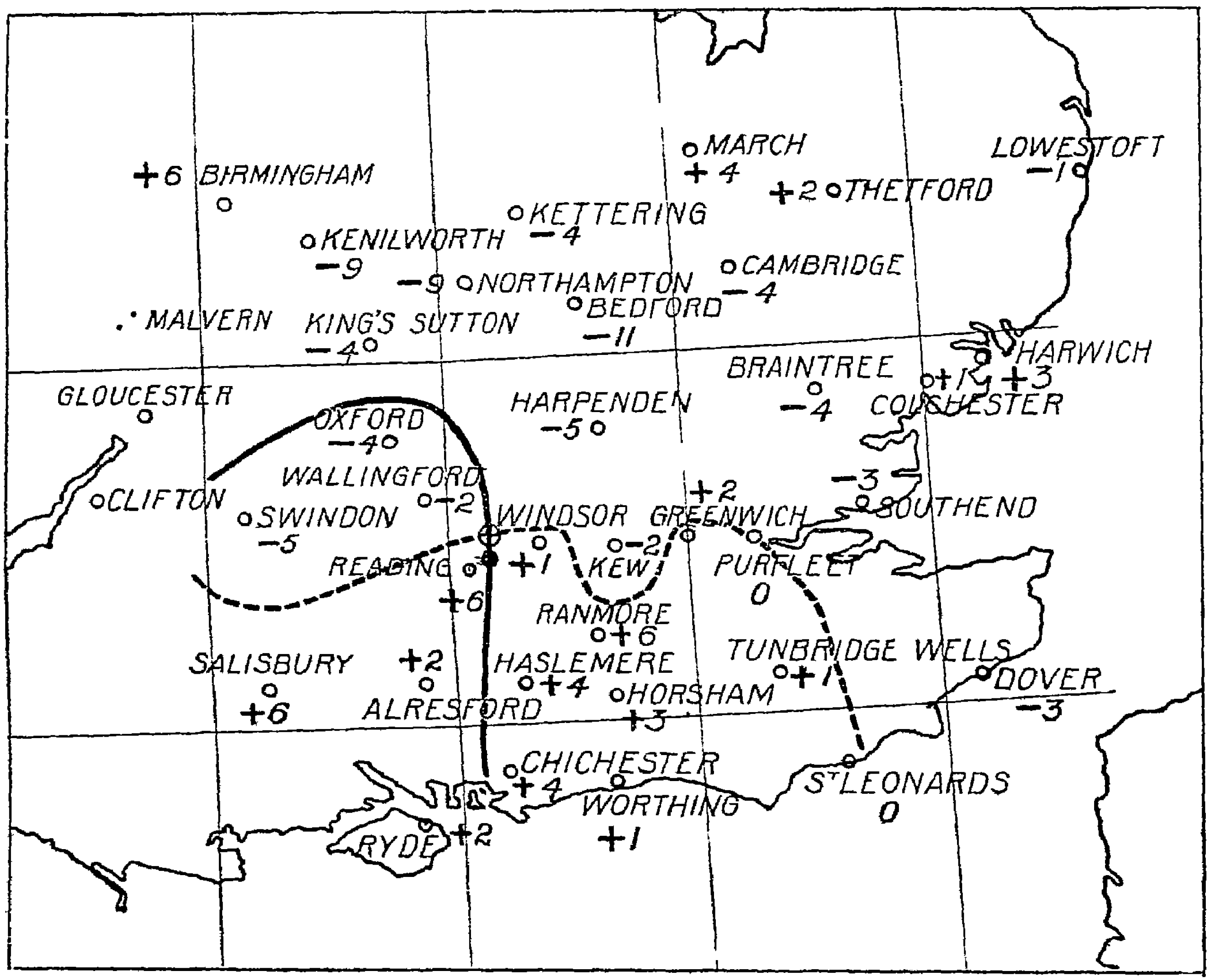
——— Curve of no disturbance

To the south of the focus the needle is oppositely deflected on each side of a line which runs nearly due north and south. To the north the effect on the Declination needle dies out

If the cause of the disturbance were a mass of "magnetic matter" below the surface of the Earth symmetrical with respect to the isogonals, the line which divided easterly from westerly disturbances of the Declination would intersect that which divided positive from negative disturbances of the Horizontal Force over the focus or centre. The next map (fig. 17) shows that this condition is very nearly fulfilled, the two points being only 7 or 8 miles apart Lastly, the maximum disturbance of the

Vertical Force should occur at the same point, and fig 18 shows that we find it to be at Reading, which is our nearest station to the two points found as above described All three elements then combine to indicate that the centre of the Thames Valley disturbance lies between Reading and Windsor, and is a few miles to the north and east of Reading

Fig 17



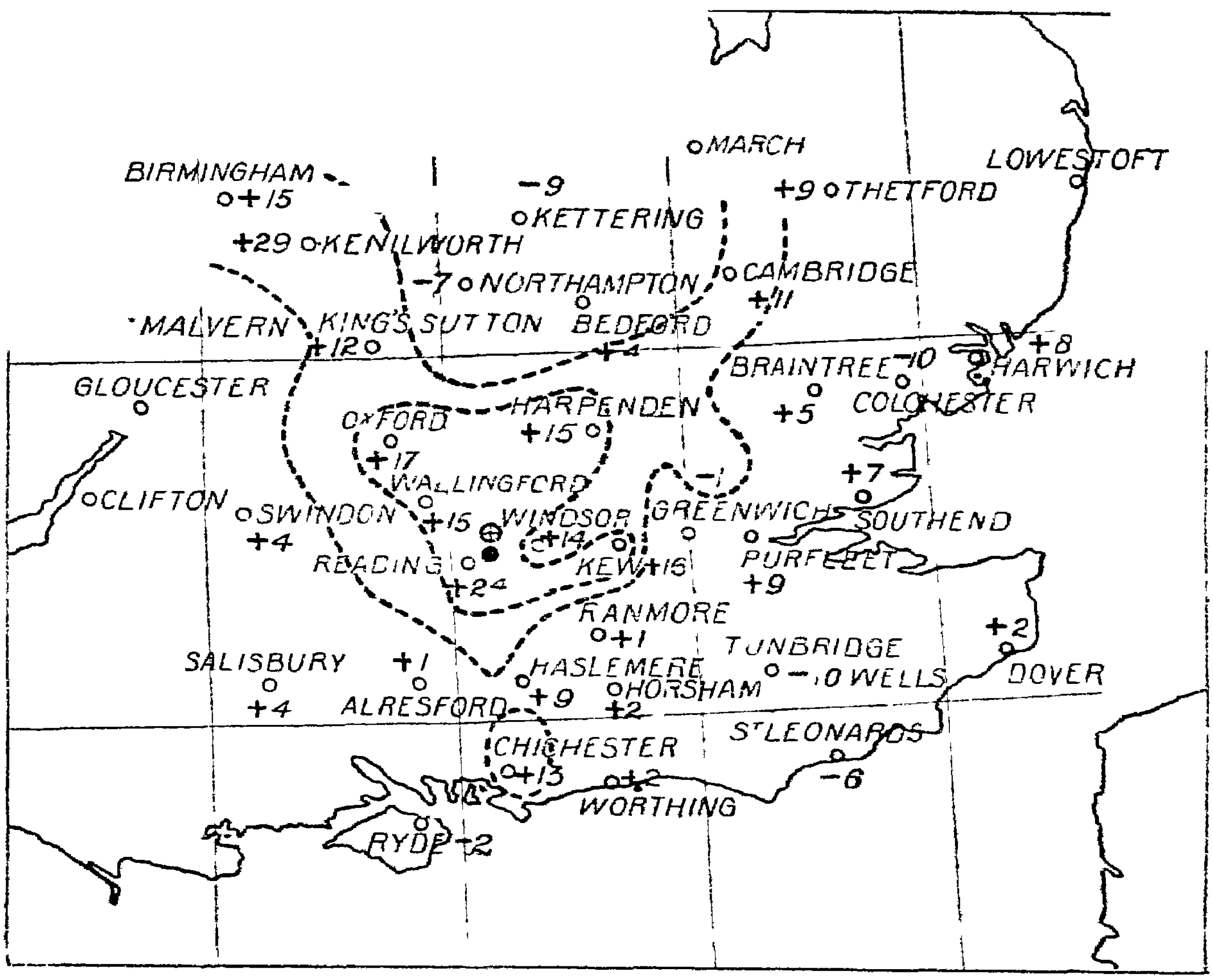
Horizontal Force disturbances in terms of 0 001 metric unit
+ Indicates that the observed is greater than the calculated value of H
——— Locus of no declination disturbance
- - - - Locus of no horizontal force disturbance

The Map of Vertical Force disturbances, however, teaches us a good deal more If we draw contour lines to enclose all stations at which the Vertical disturbing Force is greater than 0·010 and 0·015 metric unit respectively, the first comprises two independent curves, the one embracing a large area around the focus, and the other surrounding Chichester.

If the cause of the phenomenon were an underground mass of igneous rock we might picture it as a sub-terrestrial mountain of which the peak is near the focus. The slope would be most rapid towards the south-east Two ridges would run north-east and north-west towards Oxford and Cambridge, a third, less lofty, to Chichester, and a fourth, nearly due east, terminating very abruptly at Kew If all this were so, the

disturbing forces at stations in the neighbourhood ought to be directed towards the lofty central mass Close to the peak it would itself form the centre of attraction At stations near to, but not over, outlying ridges, the needle might be deflected towards them Immediately over a ridge the direction of the disturbance would change rapidly, and thus give an idea of instability The Horizontal Forces would be least over the peak, would increase up to a certain distance, and would finally die out.

Fig 18



Vertical Force disturbances in terms of 0 001 metric unit
\+ Indicates that the observed is larger than the calculated value
\- - - - Contour lines of equal Vertical Force disturbance

In fig. 19 we have depicted the disturbing Horizontal Forces They are drawn in the proper directions, and to a scale on which 0 9 mm corresponds to 0 001 metric units. They fulfil the above conditions exactly, and we think leave no doubt that in the south-east of England over an area of 10,000 square miles the lines of magnetic disturbing force tend to a centre which lies near to and probably between Twyford and Henley-on-Thames. We have treated this district in great detail, because we rely upon these results to prove that the methods of calculation and deduction adopted are satisfactory, at all events in districts where the surface rock or soil is non-magnetic If the results attained elsewhere present greater difficulties

it must be due to the greater complications introduced by the interference of local with regional disturbances

Fig 19

Disturbing Horizontal Magnetic Forces in South-Eastern England

The maps which have been used to illustrate our discussion of the Reading disturbance give indications of other minor centres, and in particular the isogonals are considerably distorted in the north. It appears that if we draw a line through King's Lynn (No 100), Spalding (No. 140), Melton Mowbray (No 116), Loughborough (No 109), Birmingham (No 63), and Malvern (No 112), the district through which it passes is the seat of local disturbances, which, though individually less widespread than that already discussed, are nevertheless of considerable intensity

We will now investigate several points in this neighbourhood It will not be necessary to do this in the same detail as before The places to be considered lie so near to the borders of the district already studied, that methods which are so consistent in the one cannot be subject to any important error in the other.

The Wash.

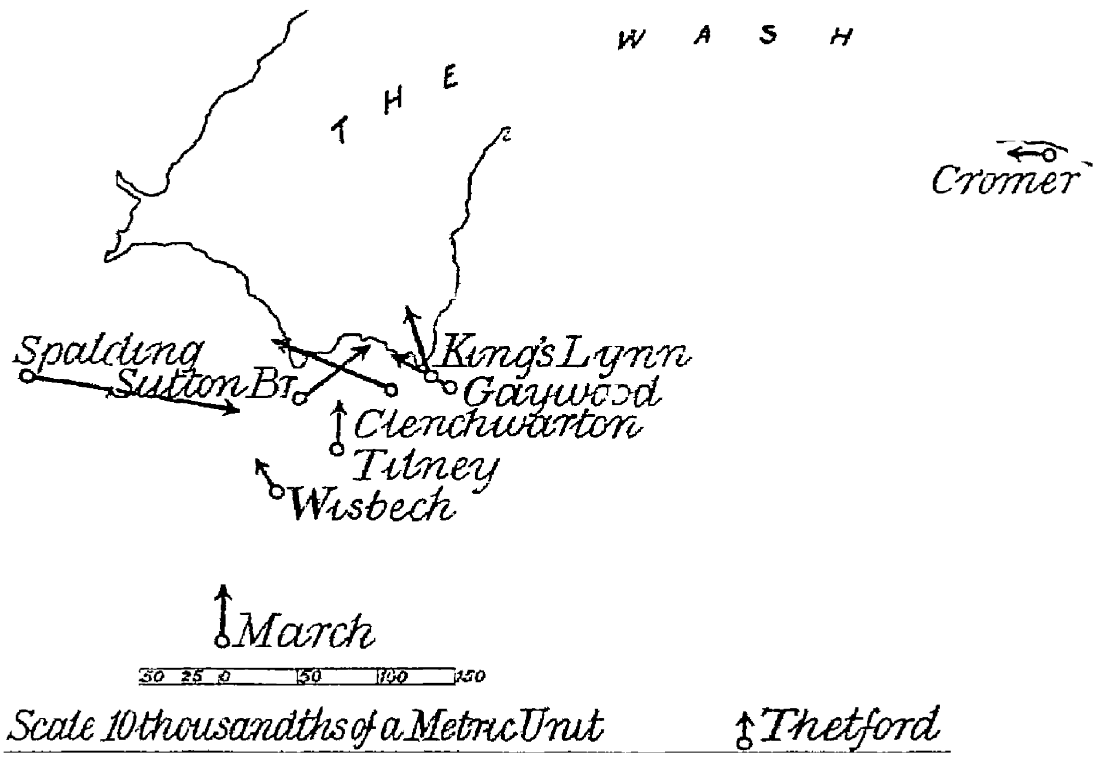
We have discovered a remarkable disturbance in the neighbourhood of the Wash. No more unlikely region could *primâ facie* have been suggested, but the evidence for its existence is conclusive.

Our attention was first called to it by the fact that the Declination found in 1888 at Spalding is less than that observed at King's Lynn in 1886. Allowing for the difference of longitude, the Declination at Spalding ought to be about 16′ greater than at King's Lynn The two values found and reduced to epoch were

King's Lynn, 1886 .	17° 57′ 9,
Spalding, 1888	17 51 6

Thus, if the King's Lynn value is normal, that at Spalding is 22′ too small

Fig 20



Disturbing Horizontal Magnetic Forces near the Wash

A special survey was therefore made of the district with the following results.

A chain of stations was run along the edge of the Wash. The measurements at King's Lynn were repeated near to, but not on the same site, and observations were made at Clenchwarton and Sutton Bridge between King's Lynn and Spalding The latitudes of these stations did not differ by more than 2′, the longitudes and Declinations were as follows —

Stations	Longitude.	Declination Jan 1, 1886
King's Lynn (Gaywood 1888)	0° 26′ 0 E.	18° 1′ 7
King's Lynn (1886)	0 24 3 E	17 57 9
Clenchwarton (1888)	0 21 3 E	18 10 3
Sutton Bridge (1888)	0 11 8 E	17 54 1
Spalding (1888) .	0 8 6 W	17 51 6

These results make it certain that near the south of the Wash the Declination diminishes instead of increasing (as at normal stations) with the longitude

The accompanying map shows the directions and magnitudes of the disturbing forces in this district They indicate a centre of attraction to the north of the line which joins Spalding and King's Lynn

The Leicestershire District

Another series of local disturbances exists to the west of that which has just been described The facts which first attracted our attention to it were that at two pairs of stations, viz , Birmingham and Northampton, Leicester and Peterborough, the observed Declination at the more westerly was only about 2′ greater than that at the more easterly station, though the calculated differences were as much as 35′ in the first case, and 27′ in the second case

The observations indicated that in this district the isogonal lines run nearly east and west instead of nearly north and south, and we proceeded to investigate their forms more closely. We thought that the anomaly was probably connected with the fact that in Charnwood Forest, which is not very distant from Leicester, igneous rocks appear upon the surface, and observations were made round this district, though we were always careful that our station should be on what was apparently good observing ground.

We have thus confirmed the existence of a great easterly trend in the isogonal lines, and though the magnetic state of the district appears to be complicated, and to require further investigation, we have also established several facts which will probably prove to be of fundamental importance in the solution of the problems connected with it

Three of the most interesting stations are Coalville (No 76), Loughborough (No. 109), and Melton Mowbray (No. 116) Loughborough and Coalville are both on the Red Marl, with alluvium near to the streams. Between them lies Charnwood Forest, in which are masses of porphyry, greenstone, and syenite.

We should, perhaps, expect from the analogy of the Malverns that at these two stations the needle would be attracted towards the crystalline rocks. This does not appear to be the case, or, as is more probable, the stations are too far distant to be affected At Loughborough, which is the more easterly station, the disturbance of the Declination is 30′ 3 *towards the east,* while at Coalville it is in the same direction, but only to the extent of 11′ 2

About 12 or 13 miles further to the east is Melton Mowbray, situated on Lower Lias clay with argillaceous limestone at its base, yet a series of observations made here on April 22, 1888, gave a Declination disturbance of + 32′, *i e.*, towards the west. Unfortunately, the observation for the geographical meridian could only be made near noon, as the sun was invisible during the rest of the day; but so remarkable did the result appear that the place was revisited on April 30. Another

station on the other side of the town, about a mile and a half distant from the first, was chosen, and the new observation was made under favourable conditions at about 5 45 P M The result indicated a Declination disturbance of $+26'$, which was in close accord with that previously obtained At first sight then, it appears that the peculiarities of the district might be explained by the hypothesis that a centre of force, powerful relatively to Charnwood Forest, exists somewhere between Melton Mowbray and Loughborough. The disturbances of the Declination at these two stations are in opposite directions and of nearly equal amounts This view does not, however, correspond with the directions of the disturbing forces obtained at Melton Mowbray If it were correct they should both have acted in nearly parallel directions towards the west As a matter of fact, at the first station the disturbing force acts nearly due west, while at the second it is only 11° from south

This indicates that the disturbance at Melton Mowbray is of a more local character, and cannot, in the manner suggested, be brought into relation with the oppositely directed disturbances at Coalville and Loughborough

If, however, we turn from the Horizontal to the Vertical disturbing Forces we find that the peculiarities of these various stations may be connected, and that a magnetic map of the district can be drawn which may furnish the first rough outlines to which details may hereafter be added without rendering them substantially incorrect

This explanation is based upon the fact that whereas in this part of England the disturbances of the Vertical Forces are for the most part positive, at four consecutive stations in this neighbourhood they are negative

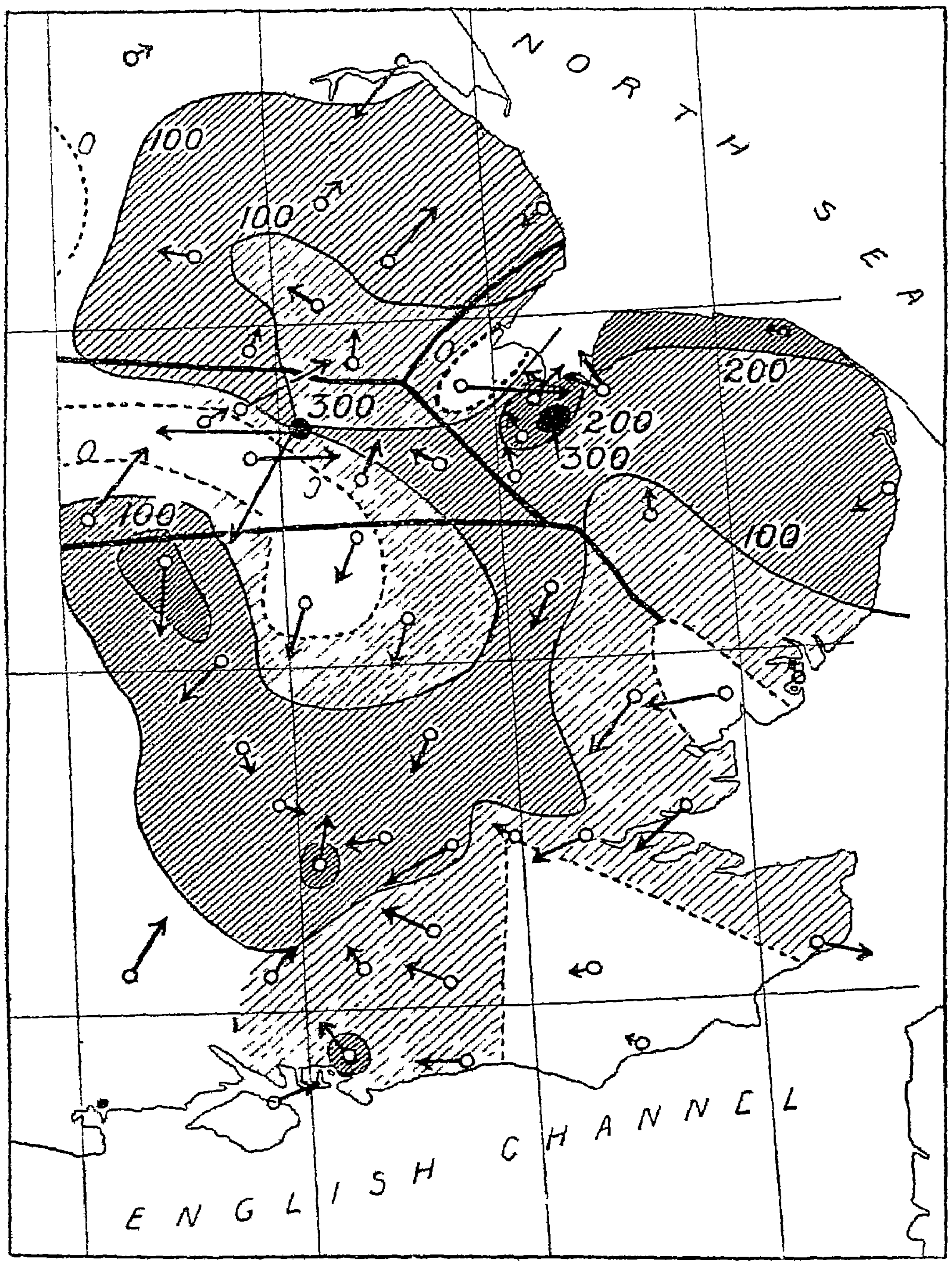
These stations are Coalville, Leicester, Kettering, and Northampton Whether the low Vertical Forces are due to the presence of a repulsive centre, or to a deep cleft or valley in the attracting mass, we cannot tell Indeed, the latter hypothesis might account for a repulsion if the mass of attracting matter which we have supposed to culminate in the Reading peak terminated abruptly on its northern edge For, if it were magnetised by induction it is possible that some of the lines of force might escape upwards to the surface, diminish the Vertical Force, and urge the north pole northwards

Contour lines drawn as in the accompanying map (fig 21), indicate the possibility of a very sudden rise in the magnetic matter from a line drawn through Coalville, Leicester, and Kettering, to another which passes through Chesterfield, Nottingham, Melton Mowbray, and Peterborough. This view is supported by the fact that a ridge line, on passage across which the disturbance of the Declination changes sign, runs from near Chesterfield to near Melton Mowbray.

If this view were correct, Melton Mowbray would have to be regarded as near the summit of an extremely steep peak, as the Vertical Force changed from 0·0305 to 0 0080 in the small distance between the two stations. The direction of the Horizontal Forces would indicate a point a little to the west of both stations as the actual peak

Fig 21 is an attempt to realise the magnetic constitution of this part of England from this point of view In studying it we must remember that the direction of the Horizontal Force at a station where the Vertical Force is a maximum or minimum must be indeterminate in the sense that it cannot be deduced from the Vertical Forces

Fig 21



Contour lines of Vertical Disturbing Force and Horizontal Disturbing Forces in South-Eastern England Each darker tint corresponds to an increment of 0 0100 metric unit in the Vertical Disturbing Force.

at neighbouring stations. All that we know when the Vertical Force is a maximum is, that a peak is probably in the neighbourhood, but on which side of the station we do not know, unless, as in the case of the Reading disturbance, other stations indicate it.

The whole district may apparently be divided into four parts of which the boundaries are indicated on the map by heavy lines

The southernmost is the region of the Reading disturbance The tendency of the Horizontal Forces to act towards regions of high Vertical Force is unmistakeable At King's Sutton the direction of the resultant appears to be affected by the region of high Vertical Force to the north, near Kenilworth At Worthing and Ryde the Horizontal Forces point direct to the Chichester peak At Purfleet and Southend the directions of the Forces are more southerly than the distribution of the Vertical Forces would have led us to suspect

The most easterly district is that of the Wash disturbance Near King's Lynn the Vertical Force is great It is greatest at Tilney which is the central station The Forces at neighbouring stations converge to a point to the north of this, and the Horizontal Force at Tilney itself is directed northward It is therefore likely that here, as in the case of Reading the true peak, though near, is not at the spot at which we found the greatest Vertical Force. It probably lies to the north of it

The central district is that of the Leicestershire disturbance Here the phenomena are more complicated, and we wish it to be distinctly understood that we think it probable that Melton Mowbray does not occupy the position of unique importance which our observations allot to it Nevertheless we must point out that the hypothesis that a narrow ridge of attracting matter runs somewhat in the position we have assigned to it, is remarkably supported by the direction of the Horizontal Forces at Coalville, Loughborough, Leicester, and Manton, which would all be explained on this hypothesis At Melton Mowbray the directions are, of course, indeterminate by means of the Vertical Force, and it is quite possible that the phenomena observed there may be due to some relatively small dyke, and not to an uprising of a part of a widespread mass of igneous rock. Until this district is more fully surveyed we are justified in adopting the view represented on the map, which is consistent with all the known facts

There is, however, one station in the district, viz, Birmingham, which is not in harmony with the rest, as the direction of the Horizontal disturbing Force is toward the region of minimum Vertical Force Perhaps this indicates that this region is not connected, as we have supposed, with the larger region of low Vertical Force to the west, but that the two are severed by a district of high Vertical Force running from Birmingham northward Future investigation can alone decide this question.

The most northerly of the four districts is almost outside the region of our special surveys. We only introduce it to show that there is a large region of high Vertical Force to the north of Melton Mowbray, and that, therefore, there is nothing anomalous in the northerly directions of the Horizontal Forces at Nottingham and Grantham. If it be true that in the Leicestershire disturbance the attracting matter lies within narrow limits, it is quite possible that at these stations the predominant influence may be that of the larger northern mass.

On the whole, then, if we take the region bounded by the sea, by lat 53° and long 2° W, which includes about 50 stations, we think the Horizontal Forces unmistakeably tend to act toward the region of great Vertical Force. There is one striking exception at Birmingham, and one or two more doubtful ones on the lower reaches of the Thames, and the rule must be construed subject to the obvious condition that a true maximum of Vertical Force, though probably near to, is not necessarily at the station at which we happen to have found the largest among the Vertical Forces we have measured

Subject to these exceptions and to this proviso the rule holds good

General Results of the Investigation of the Local and Regional Disturbances

Having described the results obtained in districts to which we have devoted special attention we now proceed to apply the same methods to the whole area of the survey. In adopting this course we are fully aware that the number of our stations is not sufficient to enable us to speak with any certainty as to the details of the magnetic peculiarities of the districts we are about to discuss, and it is quite possible that we may have arrived at some conclusions which must hereafter be modified It appears to us that even under these conditions our work is much more likely to be useful if we give what only professes to be a first rough sketch map of the magnetic forces in play in the country than if we leave our successors to get what hints they can from observations which we ourselves have made no attempt to collate Even, therefore, if our conclusions were much less certain than we believe them to be we think it would be better to state them

Fortunately, however, we are able to take up a much stronger position than this. Our conclusions may be tested, (1) by the agreement of the results of the various methods of attacking the problem, (2) by the agreement of our results in Scotland with those which can be deduced from Mr Welsh's survey, (3) by the establishment of relations between the magnetic phenomena at stations scattered over wide areas, and (4) by the establishment of a connexion between the magnetic and geological characteristics of various districts. In all these particulars we venture to assert that they will bear investigation, and we cannot but believe that we have detected the main directions of the lines of disturbing magnetic force.

The method we adopt is as follows :—

We draw the ridge and valley lines (see p. 265) which mark the centres and the boundaries of districts which are under the influence of a dominant locus of attraction We take each district bounded by two valley lines, and study it by means of the true isomagnetics of the disturbances and disturbing forces, and lastly, we discuss the relations between its magnetic and geological characteristics

The following are *a priori* probable consequences of the hypothesis that each district, bounded by two valley lines, is subject to an attraction tending towards the centre —

(1.) Since the regional forces are weak near a valley line it is in such a position that we should expect the effects of such local forces as might exist to predominate, and discrepancies to occur more frequently than elsewhere

(2.) The centre of a district, in the neighbourhood of the ridge line, should be a region of relatively high Vertical Force It will be remembered that the ridge line is drawn by means of the disturbances of the Declination and Horizontal Force only, and, therefore, an agreement between its position and that of a region of high Vertical Force is an independent confirmation of the accuracy of the theory that the forces in play in the district form a connected system, that is they are regional and not merely local

(3.) The directions of the Horizontal Forces at points of maximum or minimum Vertical Force may be indeterminate in the sense that they cannot be deduced from the distribution of the Vertical Forces This point has already been insisted on

(4.) In crossing a valley line both sets of attractions in the region which it separates tend to produce a maximum and minimum of the Horizontal Force (see fig 10, p 263) If the southerly attraction were alone in play both of these would have values less than the normal If the northerly acted alone both values would be greater than the normal Hence, since the maximum will be to the north of the minimum it is probable that in producing it the effect of the northern attraction will predominate and the values will be greater than the normal, while in the case of the minimum they will probably be less.

Comparison with Mr. WELSH'S *Survey of Scotland*

No more severe test of the physical reality of the disturbing forces deduced by us from our observations can be applied than by the enquiry whether similar methods applied to Mr. WELSH'S survey lead to similar results. In part, the severity of the test lies in the fact that the methods cannot be precisely similar. As England was so imperfectly surveyed in 1857, we are compelled to assume a linear function to express the isomagnetics in Scotland at that date The isogonals and isoclinals were calculated by BALFOUR STEWART, who used the geographical mile system of co-ordinates Unfortunately the third element selected by him for calculation was the Total Force, which throws but little light upon problems such as those we are now investigating. We have, therefore, calculated the lines of equal Horizontal Force, using, as in our other calculations, the differences between the latitudes and longitudes and those of the central station as co-ordinates, and by combining these with BALFOUR STEWART'S calculated values we have found the disturbing forces at the various stations

Again, Mr. WELSH omitted the determination of one or more of the elements

at many stations He apparently had chiefly the terrestrial isomagnetics in view, and did not consider that for the discovery of the local peculiarities of a station, it is essential that all three elements should be determined at it In part this omission was due to misadventure A number of Declination observations had to be rejected as the mirror was found to have been out of adjustment If, therefore, the calculated magnitudes and directions of the disturbing forces depend to any great extent on the distribution of the stations or on the method of reduction, conclusions drawn from WELSH's survey and our own could not be in harmony

In the case of the Declination comparison is possible, not only with the survey of 1857, but with the observations collected by Sir F EVANS in 1872 We have measured the distances of the stations from the isogonals given by him ('Phil Trans,' 1872, vol 162, p 319), and have deduced the disturbances In the accompanying map (fig 22) we have combined the results of all three surveys

The direction of a short line drawn through the stations, shows whether the north pole of the needle was deflected to the east or west Observations made by Mr WELSH are indicated by a dotted line, those due to the naval officers by two short lines with a dot between them, and our own by a continuous line

A discrepancy between any two of the surveys is thus indicated by a cross

Lines of no disturbance are drawn, separating districts in which the regional disturbance is of opposite signs, and passing, when possible, through stations at which different results have been obtained in different surveys

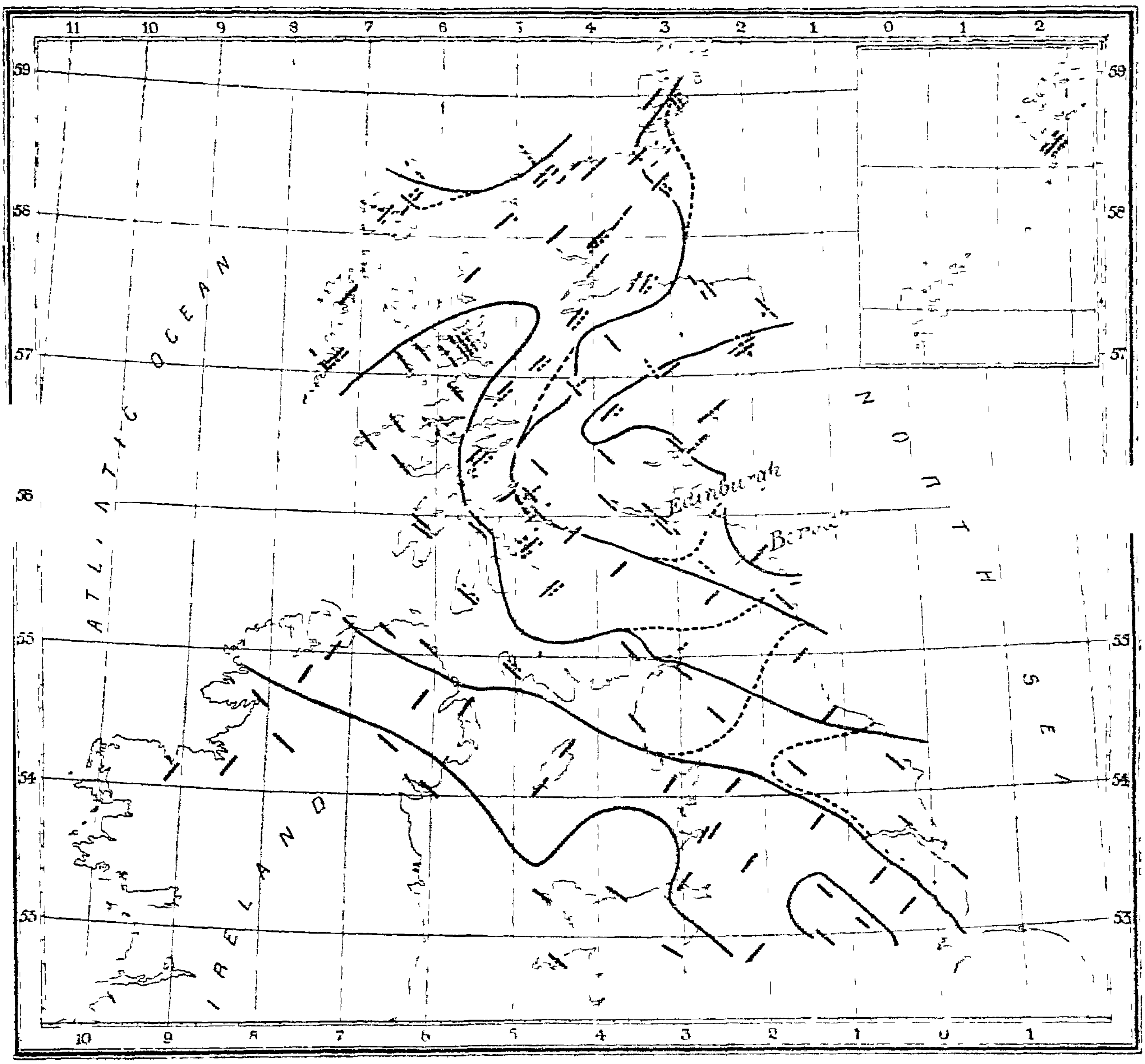
Without insisting on the accuracy of the details of these curves, we think that a study of the map can only lead to the conclusion that the districts which they bound are affected by some common cause which produces a similar effect upon the needle. It can hardly be doubted for instance that the magnet is deviated in opposite directions in the neighbourhood of Elgin and Banff All three surveys agree as to this fact. Again, the evidence is very strong that on the mainland to the north of the Caledonian Canal, except, perhaps, near Cape Wrath and the Pentland Firth, the needle is deflected by regional forces to the east, while in the Islands on the West Coast and the Mull of Cantyre it is deflected to the west Of course it would be possible to find in these districts places, such as Canna or the Cuchullin Hills in Skye, where large deviations in both directions could be obtained within a few yards, but the great majority of survey stations are chosen too carefully for the local error to become so overwhelmingly predominant We have omitted Portree because the ground there is known to be unfavourable to our purpose The differences between the three results obtained are too great to make a mean value trustworthy. At Canna on the other hand, where it will be remembered we observed at twenty-three places, the mean value cleared of the larger disturbances agrees with the results obtained at neighbouring stations.

The following facts are also important.

Seven stations were common to the surveys of 1857, 1872, and 1886. One of

these, Lerwick, we disregard, as it lies so far from the rest of the district. In five out of the remaining six the disturbance of the middle epoch as deduced from the isogonals given by Sir F. Evans is intermediate to those which were obtained in 1857 and 1886. At Oban there is an enormous discrepancy, which is not supported by

Fig. 22.



Directions of Declination Disturbances.

the observations made in 1872 and by ourselves on the neighbouring island of Kerrera. There can be little doubt that the 1872 observation was subject to a powerful local disturbance. The coincidence of the results in the other cases can hardly be accidental. The figures are given in the following Table, Kerrera being added for the sake of comparison with Oban:—

DECLINATION Distuibances

		1872	1886
		′	′
Abeideen	+ 5 1	− 4 0	− 6 7
Edinbuigh	+ 19 8	+ 10 0	+ 4 2
Kyle Akin	+ 38 2	+ 27 0	+ 24 9
Oban	− 11 1	+ 55 0	− 8 3
Kerierı		− 5 0	− 6 8
Thurso	+ 13 8	+ 7 0	− 5 2
Wick	+ 11 6	+ 7 0	− 8 5

It is difficult to offei any satisfactoiy suggestion as to the causes of the appaiently iegular change It may be due only to the fact that the thiee methods of deducing the disturbances by calculations are different, oi it may be due to real changes in the local forces Such alterations aie not impossible since if the iocks were magnetised by induction on the earth's field the diiection and intensity of the induced magetisation would be subject to seculai change, but the diffeiences are too gieat to be explained thus

But whether local changes have taken place or not, there can be no doubt that the general distribution of the Declination disturbances in Scotland is the same now as it was in 1857.

In Map 22 the dotted lines are the cuives of no distuibance which would have been diawn from our own observations alone The differences intioduced by the addition of WELSH's and EVANS's suiveys are quite unimportant in the Noith In the South they simplify the lines instead of introducing complications As the disturbances at Edinbuigh and Beiwick are of opposite signs, we had to caiiy a line of no disturbance between them Mr WELSH, however, had two stations a little to the West of Berwick, viz, Makerstoun and Melrose, and at these the disturbance is of the same sign as at Edinburgh

The introduction of these justifies us in treating Berwick as an isolated station, and the simple system represented by the continuous lines iesults We have adopted these as the true lines of no Declination disturbance in this pait of the Kingdom

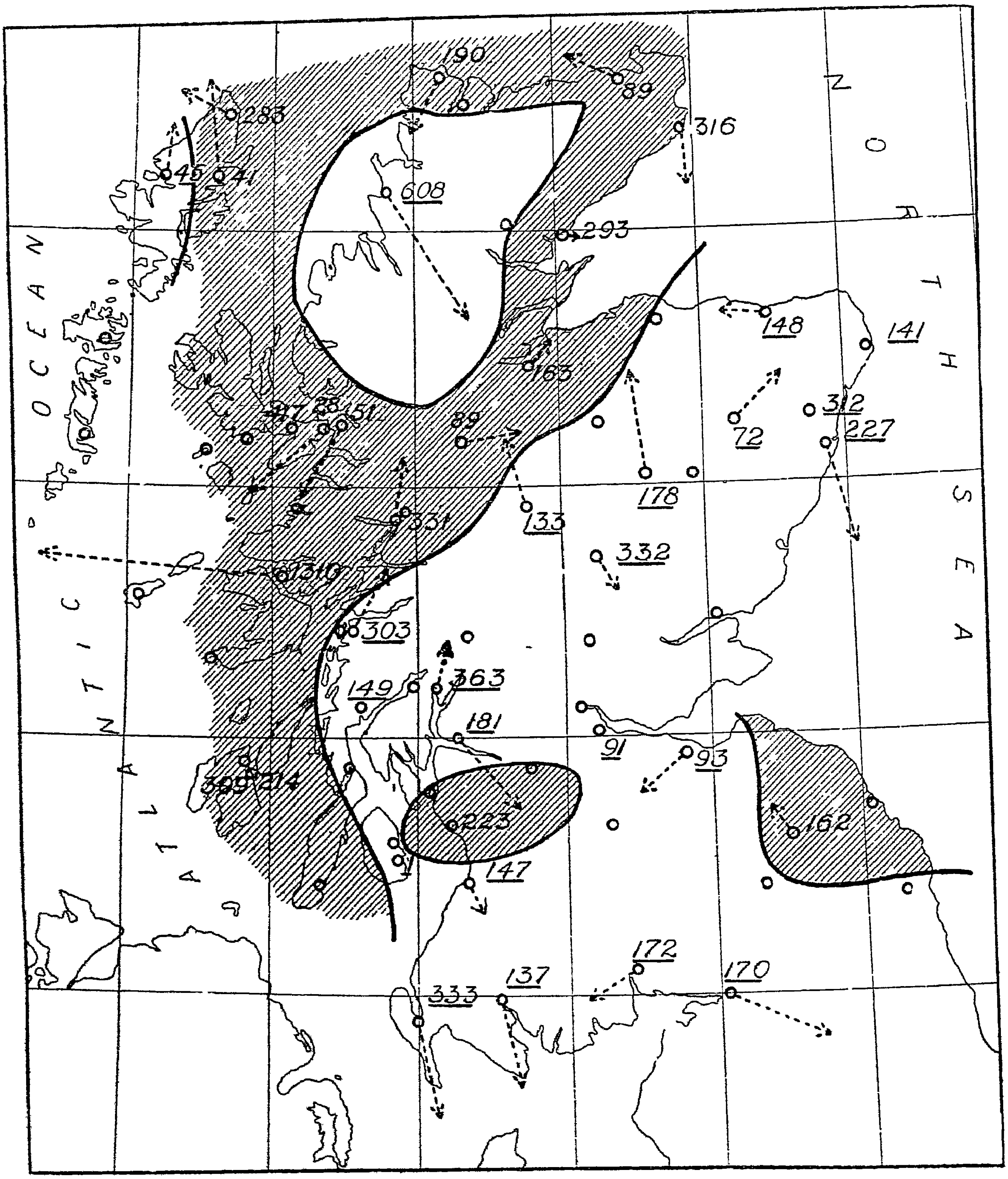
As Sir FREDERICK EVANS did not add any Dips or Hoiizontal Forces to the list of those already known, we proceed at once to the direct compaiison of the disturbing forces deduced from our own and Mr WELSH's surveys. The magnitudes and directions of these calculated by us as above described are as follows —

DISTURBING Forces in Scotland from Mr. WELSH's Survey in 1857.

Station	Latitude	Longitude		φ	Z
Aberdeen	57° 9'	2° 05'	163		227
Alford	57 14	2 45	109	− 43	− 72
Ardrishaig	56 1	5 27	43	+151	− 149
Ardrossan	55 39	4 47			+ 223
Ayr	55 28	4 38	59	−150	− 147
Balmacara	57 17	5 39	127	+152	+ 51
Banff	57 39	2 31	77	+ 87	− 148
Braemar	57 1	3 25	184	+ 9	− 178
Bridgend	55 48	6 16			+ 309
Broadford	57 15	5 51			+ 417
Callernish	58 10	6 44	99	− 6	− 45
Corpach	56 51	5 8	100	− 3	+ 331
Cross	58 29	6 17	90	+ 67	+ 283
Dalwhinnie	56 56	4 17	107	+ 19	− 133
Dumfries	55 5	3 36	96	+123	− 172
Durness	58 34	4 44	112	+152	+ 190
Edinburgh	55 58	3 11	94	+131	− 93
Fort Augustus	57 9	4 40	80	− 87	+ 89
Glenmorven .	56 38	5 58	410	+ 86	+1310
Golspie	57 58	3 58	7	− 90	+ 293
Gretna	55 1	3 3	172	−113	− 170
Helensburgh	56 2	4 43	149	−139	− 181
L Inver .	58 10	5 12	248	−148	− 608
Inverness	57 28	4 11	49	− 39	+ 163
Kintore	57 15	2 23	.	.	− 312
Kirkwall	58 59	2 58	153	+ 45	+ 250
Kyle Akin	57 16	5 44	174	+134	+ 28
Lamlash .	55 31	5 5			− 1
Larbert	56 2	3 49			− 91
Lerwick	60 9	1 8	100	− 6	+ 238
Lochgoilhead	56 10	4 54	66	− 10	− 363
Makerstoun	55 35	2 31	65	+ 37	+ 162
Newton Stewart	54 56	4 28	155	−168	− 137
Oban	56 27	5 26	120	− 31	− 303
Peterhead	57 31	1 46	71	+174	− 141
Pitlochrie	56 42	3 43	58	−150	− 332
Port Askaig	55 52	6 8		.	+ 214
Stornoway	58 15	6 23	152	+ 7	+ 41
Stranraer	54 54	5 2	175	−166	− 333
Thurso	58 35	3 32		+ 65	+ 89
Wick	58 25		90	+172	+ 316

These values are shown in fig. 23 The arrows represent the Horizontal Forces in magnitude and direction The numbers are the Vertical disturbing Forces in terms of 0 0001 metric unit. When negative they are underlined. Lines of no Vertical Force disturbance are also drawn The shaded parts are regions of positive Vertical Force disturbance. A comparison of this with figs 25 and 26 will suffice to show that there is a close agreement between the two Thus, in the case of the Vertical Forces, we both find regions of high Vertical Force along the lines of the Caledonian Canal and the Western Isles, and on the East and West Coasts of South Scotland.

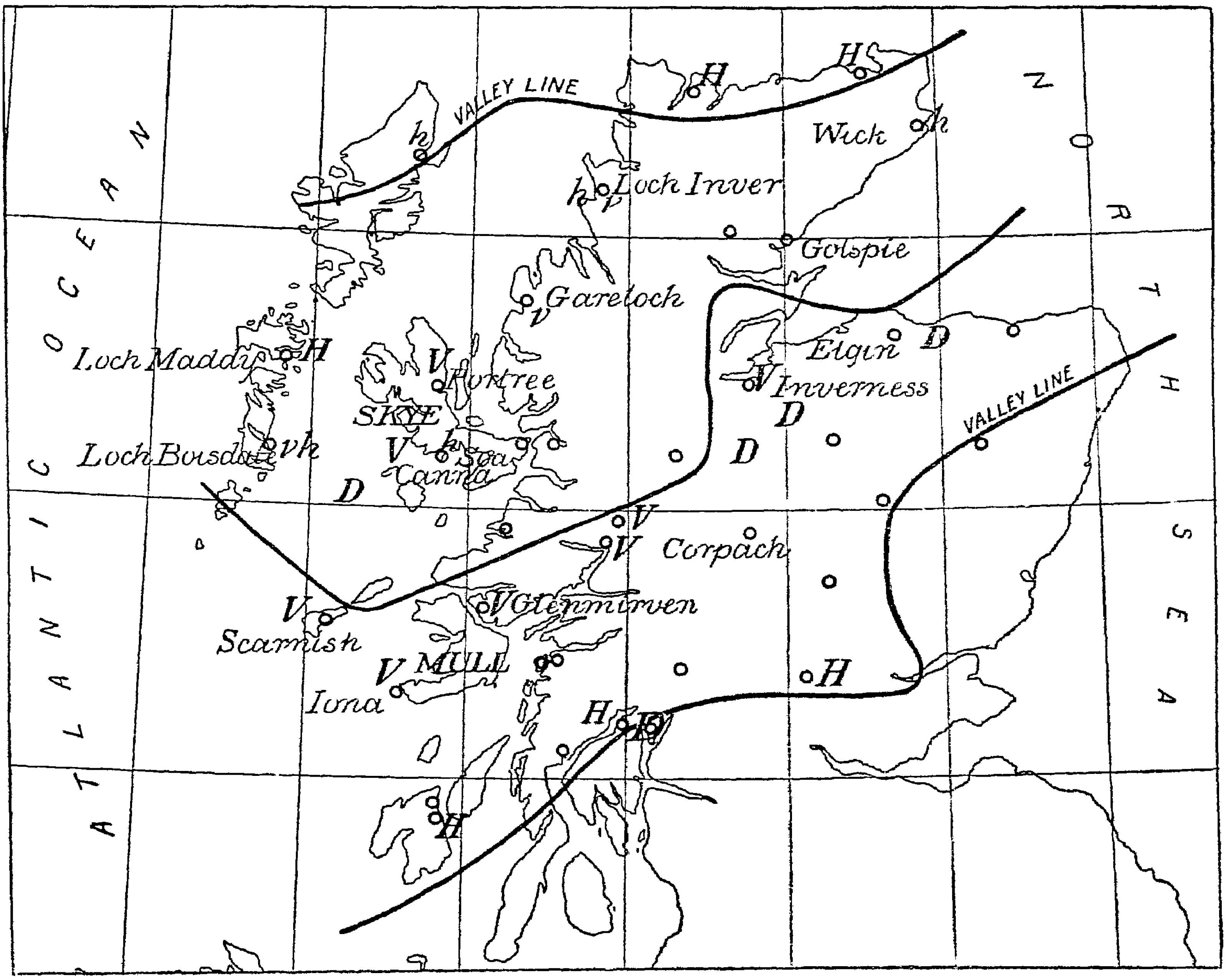
Fig. 23



Disturbance map of Scotland, from Mr. Welsh's survey, 1857–58

There are a few discrepancies in the direction of the Horizontal Forces, but the general agreement and occasional differences will be better discussed when we deal, as we are now about to do, with separate districts We propose to treat of the results of the two surveys simultaneously, and we think we shall succeed in showing that, although it would have been impossible to draw our conclusions from Mr WELSH s less numerous stations, they are strongly confirmed by the deductions which we have made from his survey

Fig 24.



Highland District.

The Highland District.

In fig. 24 we have collected all the information which the isomagnetics in Plates V. to VIII. afford of this district. Those places only are named to which we actually refer. Where the more westerly of two stations has the smaller Declination, the isogonals are

distorted (Plate V), and indicate a centre of attraction between the two All such points are marked D

Loops in the Vertical Force isomagnetics indicate maxima and minima of Vertical Force, which are marked V and v respectively As no observations could be made to the west of the islands in the south-west of Scotland, it is impossible to prove formally that the Vertical Forces in this district are maxima The values at Scarinish and Iona are, however, the same as those at Wick and Golspie respectively, which are nearly two degrees farther north We have, therefore, felt justified in marking them as maxima, thereby indicating that they are very large. Canna and Portree are so disturbed that but little reliance can be placed upon observations taken there, but they both give maximum values of the Vertical Force (Plate VIII) At Corpach and Glenmorven we find that Mr WELSH's observation gave maximum values of the Vertical Force. At two of the most northern stations the Horizontal Force is a minimum, at three of the most southern it is a maximum (Plate VIII) They are marked with h and H respectively.

From this map we can form an opinion as to the magnetic constitution of the district Points of high Vertical Force, and centres to which the needle is attracted, cluster thickly along a line which passes from Elgin to Inverness, thence along the line of the Caledonian Canal to Corpach, and so to Mull

Another similar line runs north through Skye, and indications of a region of low Vertical Force, which separates the two, are given in the minimum values found at Gairloch and Loch Inver There is a subsidiary centre of attraction near Strachur

The view that a general attraction is exerted toward the centre of the district is supported by the occurrence of maximum and minimum values of the Horizontal Force in the south and north respectively. The minima at Loch Boisdale and Soa and the maximum at Loch Maddy (which necessarily follows) indicate a strong subsidiary centre of attraction to the south of these places BALFOUR STEWART, arguing from WELSH's results, placed such a centre to the south of Mull, but as we shall show, it is probably to the west of that island ('Brit Assoc. Report,' 1859, p 190).

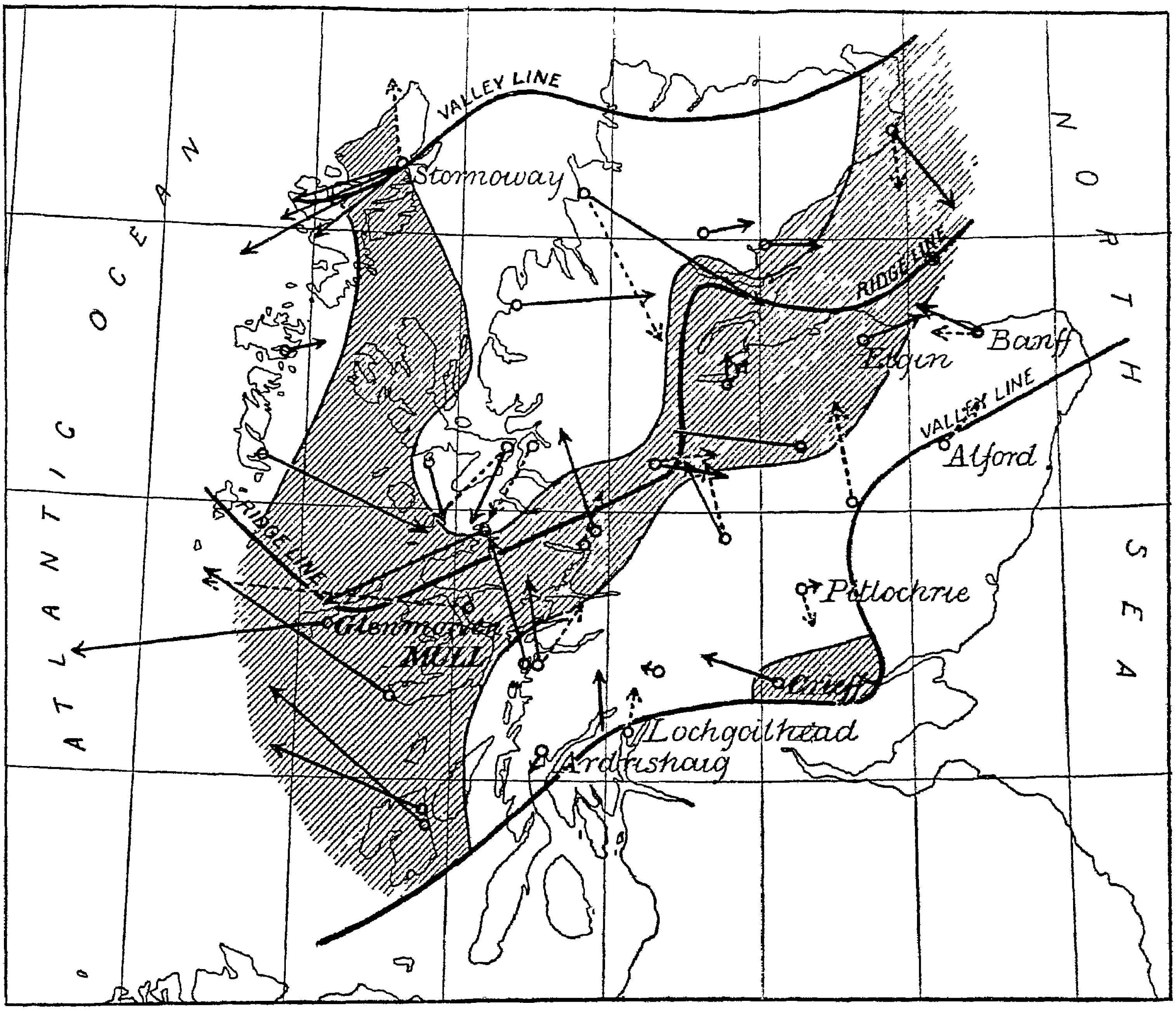
It must be remembered that all these conclusions follow from the mere inspection of the results of the observation with no more calculation than is necessary to reduce them to the same epoch. In the next map (fig. 25) we show the results of the calculations carried as far as possible, *i e.*, to the point of deducing the disturbing forces

The shaded parts are the regions of positive (*i.e*, of great) Vertical Force. The boundaries are fixed by the very rough method of assuming that between neighbouring stations the rate of change of the disturbing force is uniform. The arrows represent the Horizontal disturbing Forces in magnitude and direction. The dotted arrows represent the disturbing forces calculated from Mr. WELSH's survey.

There are only two or three discrepancies, and they all occur, as we anticipated, near valley lines. The most remarkable is Stornoway. There can be no doubt as to the

accuracy of our results obtained on three different occasions and at two stations, and we can only suppose either that there has been a real change or that Mr WELSH's station was subject to some very great local disturbance Lochgoilhead, which our observations place just over the valley line in the next district is, according to WELSH, just within this At Pitlochrie there is a large angle between the two forces It is doubtful whether the valley line is here correctly drawn. The region of high Vertical

Fig 25



Highland District

Force near Crieff seems to belong to the next district At Crieff itself the disturbance of the Vertical Force is a maximum, and there is probably a peak in the neighbourhood. The direction of the Horizontal Force cannot therefore be deduced from the Vertical Forces, and the fact that it happens to act northwards has perhaps led to its being wrongly included in this district. If the valley line runs direct from Alford to Lochgoilhead it would pass close to Pitlochrie. Our result would thus make its

relations uncertain as the direction of the Horizontal Force would lie nearly along the valley line The direction deduced from Welsh's observation would place it in the next district The same remark applies, though more doubtfully, to Ardrishaig

If, however, these three or four border stations be put aside, the map leads to a consistent view of the magnetic state of the district, which is in exact accord with that previously arrived at The ridge line lies in the region of greatest Vertical Force disturbance The Horizontal Forces tend towards that region, and at points within it are, on the whole, directed to the ridge line The discrepancies between our results and those of Mr Welsh are hardly, if at all, greater than those between our own, when repeated It is remarkable that the south-western stations indicate a centre of attraction out at sea, but our results are completely confirmed by Welsh's at Glenmorven. We should certainly have expected *a priori* that Mull, which is highly basaltic, would be a centre of attraction

Between Elgin and Banff there is a strong local centre Welsh observed the Horizontal Force at Banff only, but the Declination obtained by him at Elgin is less than that at Banff, the difference being about 10′, as against 7′ given by our survey. The two sets of observations are thus in agreement

The Scotch Coal-field District

This district is bounded on the north by the valley line which forms the southern boundary of that which has just been discussed

The southern boundary is inclined to the magnetic meridian at an angle which does not give any special advantage to either the Horizontal Force or the Declination as a means of determining its position We have therefore taken the Declination line corrected (as described on p 296) by Welsh's observations at Makerstoun and Melrose

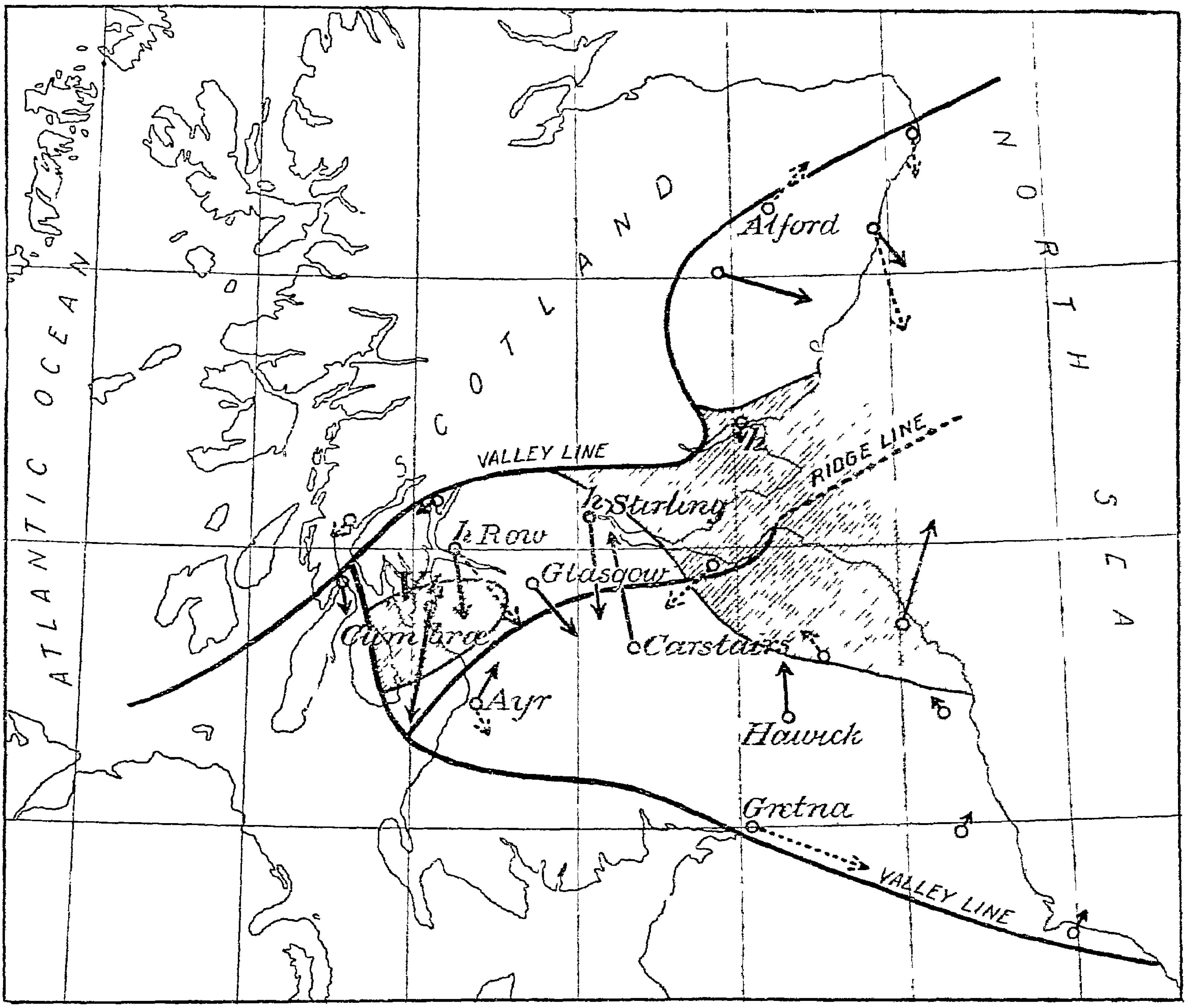
There is not a single abnormal station in this district, and the results of Welsh's observations fit in very well with ours. At Ayr, indeed, there is a very considerable discrepancy, and the forces at Alford and Gretna are larger than we should have expected if they are regarded as the resultants of the attractions of the two regions, close to the boundaries of which these stations are situated

All the four stations at which the Horizontal Force is a minimum, viz. (taking them in order from the west), Cumbrae, Row, Stirling and Dundee, are situated close to the northern boundary of the Scotch Coal-field, in which there are large masses of basalt

The ridge line runs through the middle of this basaltic district and passes through or near two regions in which the disturbance of the Vertical Force is positive (*i.e.*, in which the force is great) to the east and west respectively. At Cumbrae the Vertical Force is a maximum, and the existence of a region of high Vertical Force in this neighbourhood is remarkably confirmed by the fact that Welsh's observations give a positive disturbance at Ardrossan a few miles further south, though at the neighbouring stations both to the south and north it is negative (fig. 23). On the whole then

there can be no doubt that in this district the ridge line passes through the region of greatest Vertical Force disturbance It obviously does so in the east In the west the region of positive disturbance is defined by a single station only, and its boundaries are uncertain If we included WELSH's observations at Ardrossan they would be pushed further south In the central region there are clear indications of a maximum disturbance, though the largest values are negative Thus at Row and Stirling the

Fig 26



Scotch Coal-field District

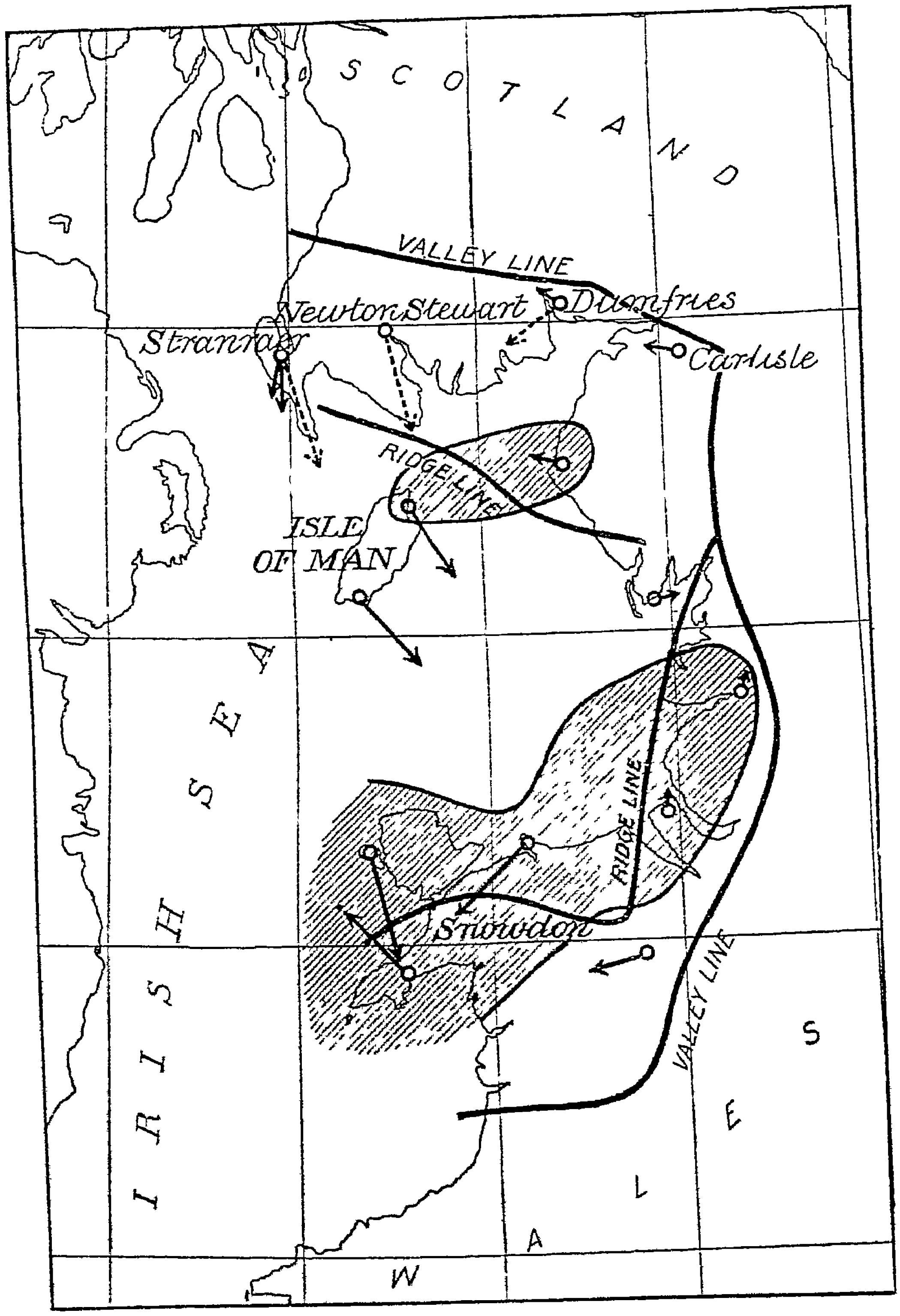
Vertical Force disturbances are −·0188 and −·0092; at Glasgow and Carstairs −·0080 and −·0079, and at Ayr and Hawick −·0145 and −·0099. The two central stations which are nearest to the ridge line have algebraically the largest values.

The convergence of the Horizontal disturbing Forces towards the ridge line is unmistakeable The isogonals (Plate V.) give evidence of a subsidiary centre of attraction near Lochgoilhead.

North-Western England, North Wales, and Galloway.

The stations in this group are bounded on the west by the northern part of the Irish Sea They are few in number, and so large a portion of the intervening spaces is covered with water that we cannot hope to unravel the intricacies of the district

Fig. 27.

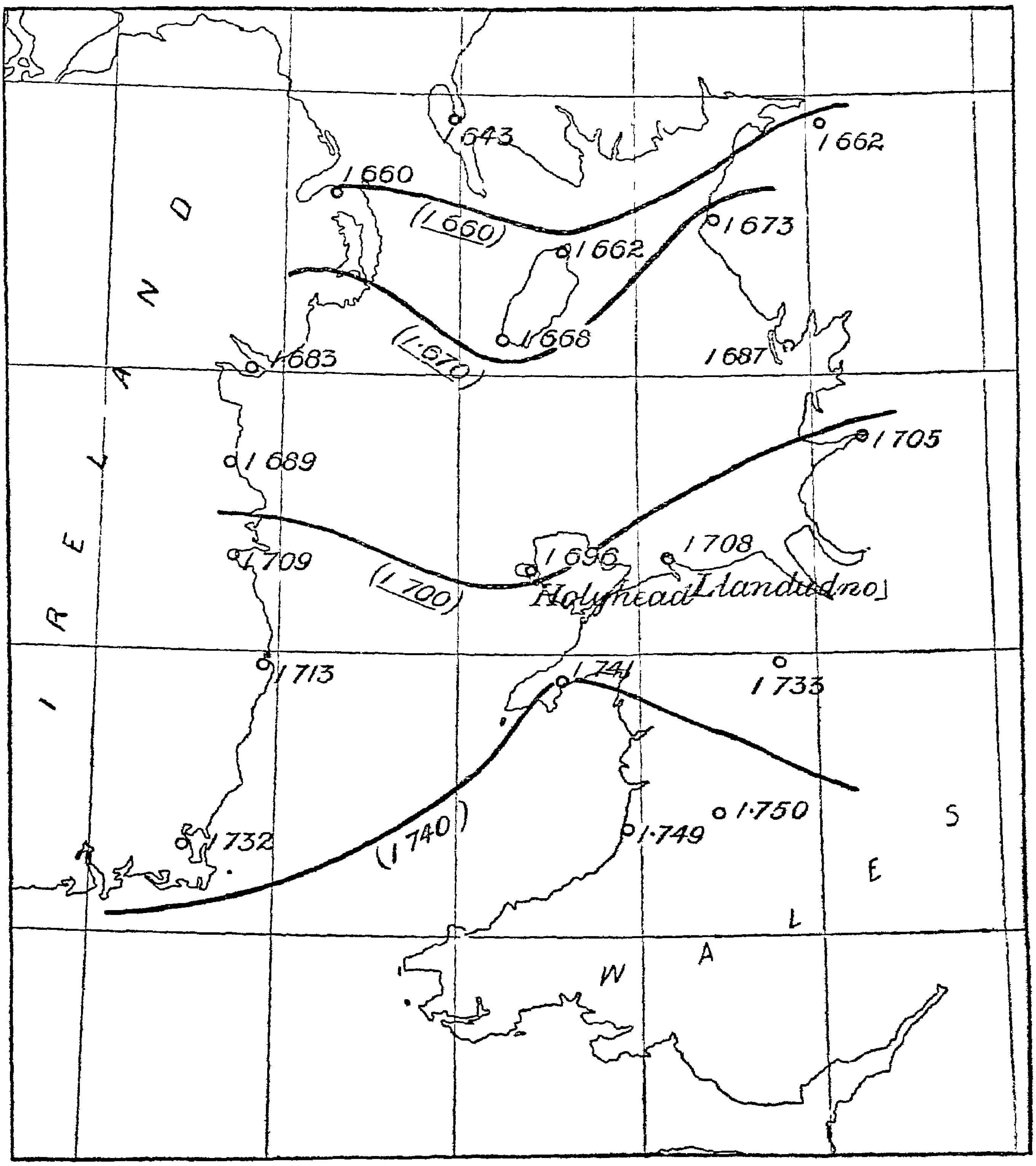


North-West England, North Wales, and Galloway

The valley line which bounds it on the east has been drawn through the centre of the region of low Vertical Force which exists there (see Plate XI). The southern part is obtained from the Horizontal Force disturbances (see Plate X.).

At Dumfries and Carlisle the disturbing forces are small, and coincide in direction with the valley line, *i.e.*, they fairly represent the resultants of two attractions exerted on each side of it. At Stranraer, and in the Isle of Man our disturbing Horizontal Forces act towards the south, and this result is confirmed by those

Fig. 28



North-West England, North Wales, and Galloway

deduced from WELSH'S observations at Stranraer and Newton Stewart. They are much larger than ours, and at Dumfries the southward tendency is more marked. It must be remembered, however, that the calculations by which these results have been deduced from the 1857 survey have been conducted on the assumption that the

Scotch isomagnetics are linear functions of the coordinates which determine the geographical position, and that the errors involved in the assumption would chiefly affect stations such as these, which are on the border of the district

If it is necessary to prove that these results are independent of the method of calculation, Fig 28 is sufficient for the purpose

The lines of equal Horizontal Force converge in a most remarkable way towards North Wales, which is one of the most certain indications of a centre of attraction Again the Declinations at Holyhead and Llandudno differ only by 0′ 4, while the difference of longitude corresponds to 25′. This again is strong evidence of the existence of a centre of attraction between them

If, as appears probable, there are two regions of high Vertical Force in the district, it is possible that the northern one may be connected rather with the Cumberland Lakes than with North Wales. However this may be, and while fully admitting that it requires further study, we think that this district appears to obey the rules which hold good elsewhere

The North-Eastern District

This district is bounded on the north and west by valley lines which have been already described On the south we have drawn a line which passes through the centres of two regions of negative Vertical Force disturbance

The southern portion lies within the limits of our special surveys and has been so fully discussed that nothing need be added here. The northern part presents a feature of peculiar interest

The results as directly observed do not furnish such clear indications as in a district where the presence of crystalline rocks is more obvious The isogonals are, however, bent in the peculiar manner which indicates centres of attraction. Thus the 19° 6′ line (Fig 29) points to such a centre to the south of Thirsk, the 18° 48′ isogonal points to another between Hull and Gainsborough As both these lines depend in part upon the same stations, it is satisfactory to find that they are completely confirmed by the quite independent 18° 16′ line which runs nearly due east and west between Lincoln and Mablethorpe

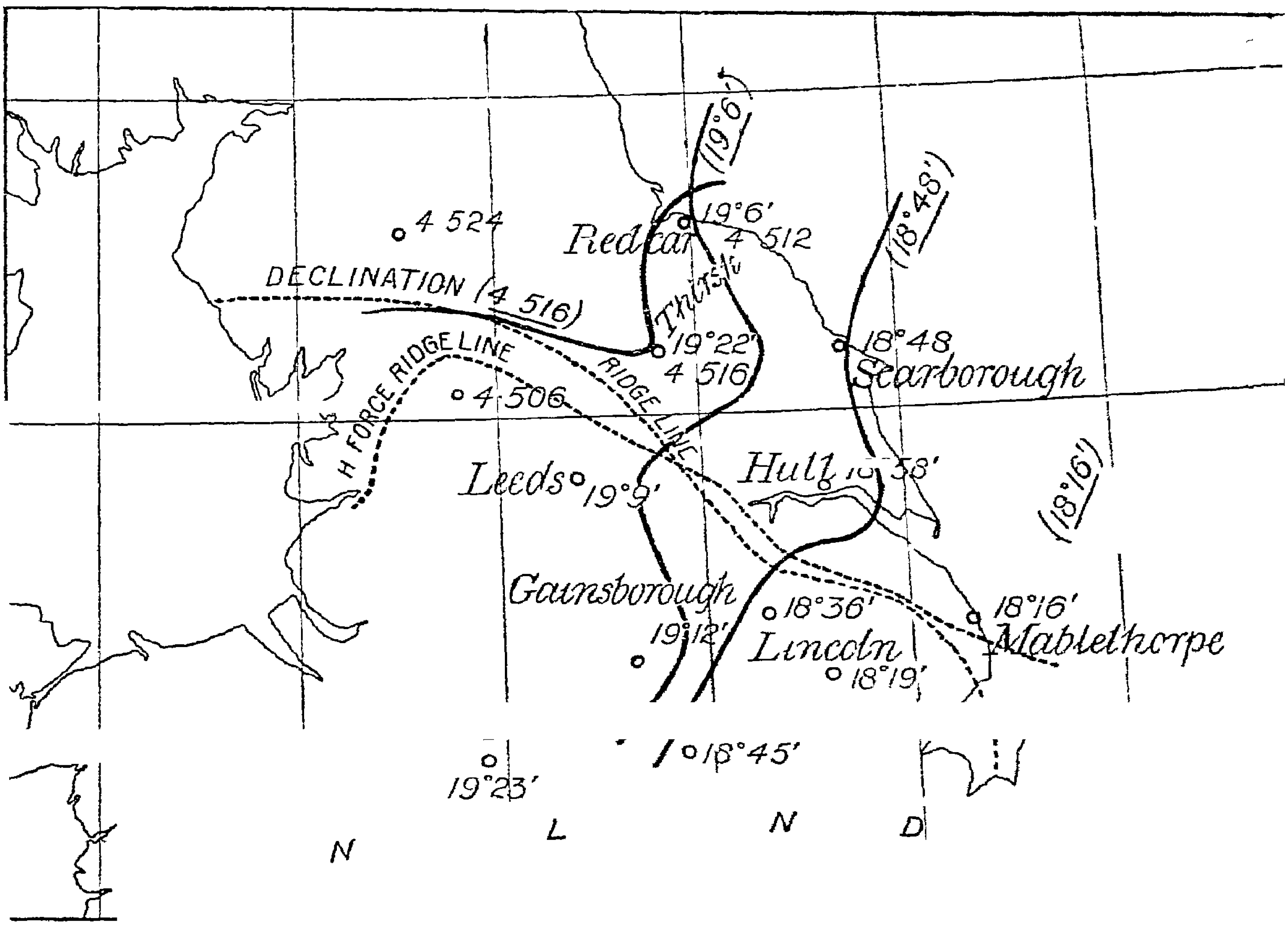
The Vertical Force isomagnetic which corresponds to 4·516 metric units bends southward to Thirsk in a way that is again an indication of a centre of attraction There are thus signs of an attractive region lying between the following pairs of stations, viz, Mablethorpe and Lincoln, Hull and Gainsborough, Thirsk and Leeds.

On turning to the disturbances (Plates IX and X), we find that both the Declination and the Horizontal Force give nearly coincident ridge lines running along the line just indicated. The declination line passes northward to the Cumberland Lakes and southward to the Wash. The Horizonal Force line turns south amid the Yorkshire Hills and runs toward North Wales.

As we should expect from the close agreement of the Declination and Horizontal Force ridge lines the disturbing forces in this district are very easy to interpret In no place in the kingdom is a locus of attraction more clearly indicated At Appleby, Thirsk, Hull, and Mablethorpe the disturbing forces act in a south or south-easterly direction, at Giggleswick, Leeds, Gainsborough and Lincoln they point north-east.

A well-marked ridge line thus runs from the Lincolnshire Wolds through Yorkshire and the limestone district of Westmoreland to the Cumberland Lakes

Fig 29



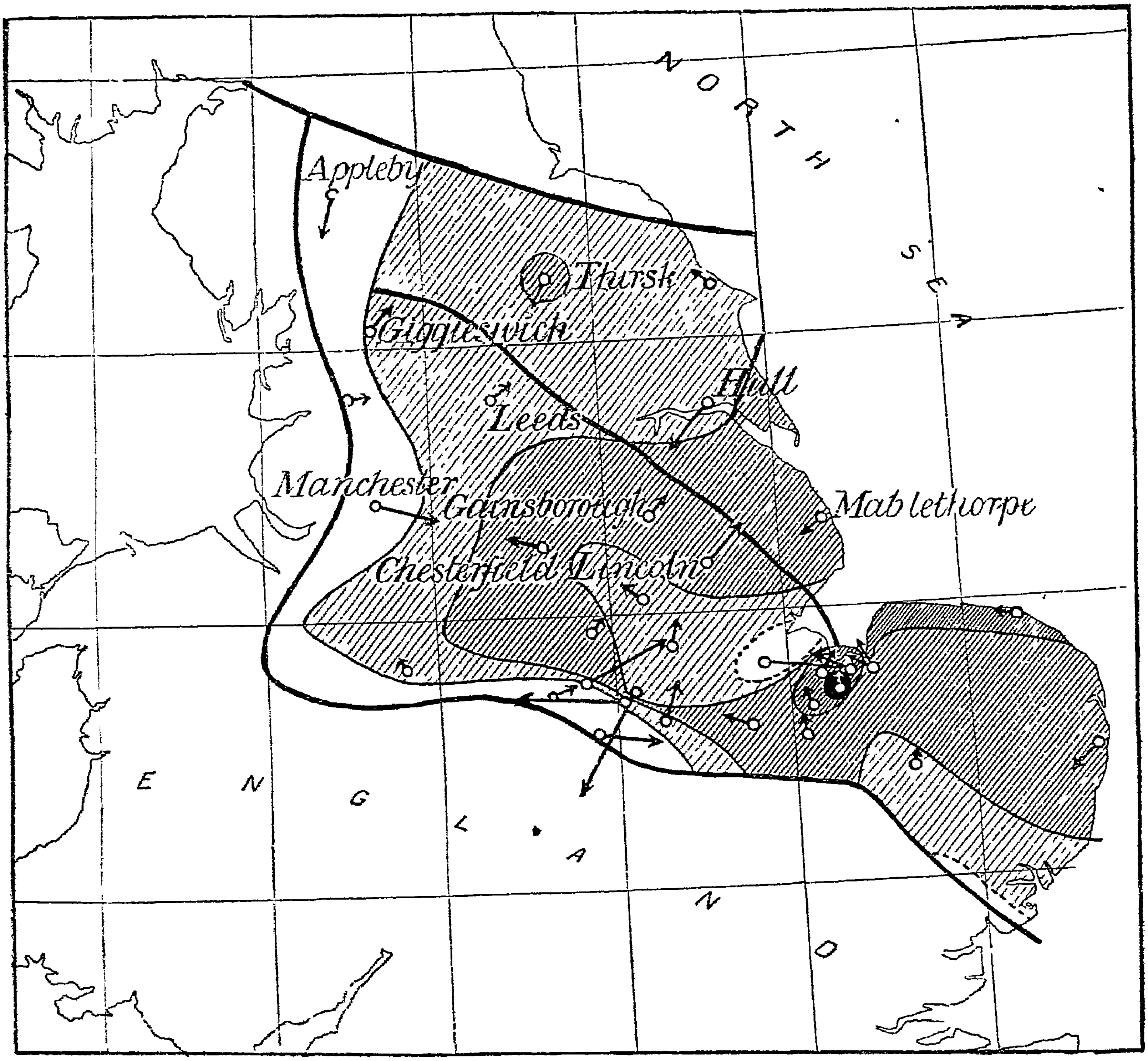
North-Eastern England

In the southern part of its course it traverses a region of high Vertical Force, and passes near the station at which the positive disturbance is a maximum, terminating in the Wash peak

The observations at Manchester and Chesterfield appear to indicate another centre of attraction in the limestone district of Derbyshire, but two stations are hardly sufficient to decide such a point. It is, however, very significant that the limestones of Derbyshire are intercalated with the basaltic rocks known locally as "toadstones," and although these do not cover a great area at the surface it is by no means

improbable, as Professor JUDD infoims us, that large masses of similar rocks occur at no great depth.

Fig 30



North-Eastern England

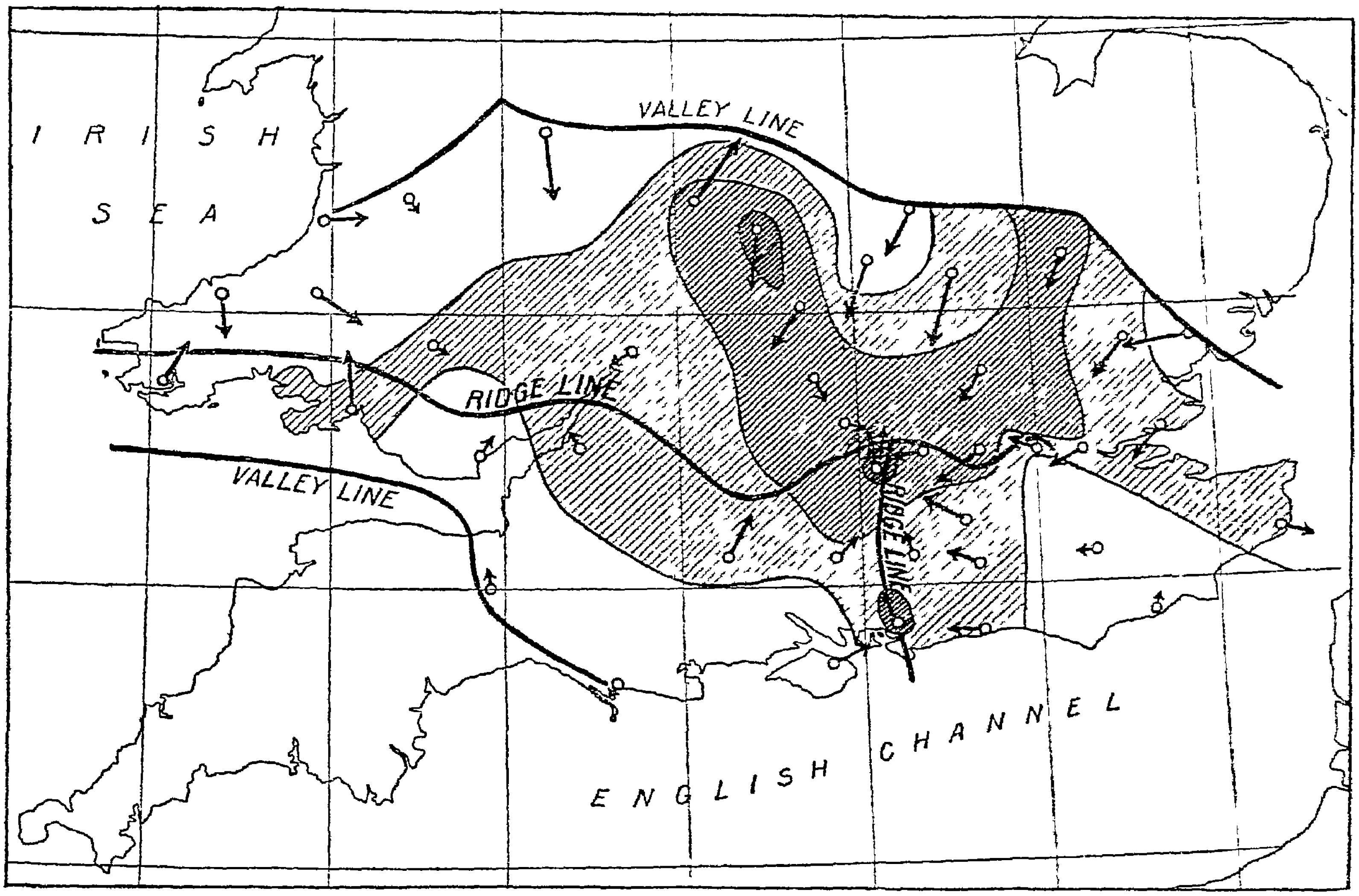
The Southern District.

The Southern District contains those parts of England and Wales to the south of the regions which have already been discussed with the exceptıon of Devonshire and Cornwall. In the latter outlying district the disturbing forces indicate on the whole an attraction to the south of the peninsula.

The Home Counties have been already discussed, and we only refer to the district again to point out that the ridge line which runs westward thıough the Reading peak

is continued into the Welsh coal-fields This fact is very important, and will be hereafter discussed when we consider the relations between the magnetic and geological peculiarities of this district.

Fig 31



Southern England and Wales

Ireland

In Ireland, as elsewhere, the Horizontal Forces tend towards the regions of greatest Vertical Force

At Coleraine and Waterfoot, stations to the north of the great mass of basaltic crystalline rocks in Antrim, the disturbing forces tend southward. At Cookstown Junction and Bangor, on its southern borders, they are directed to the north.

Kells is a point of maximum Vertical Force, and the forces at neighbouring stations are all directed towards it.

A clearly marked ridge line runs through a region of positive Vertical Force disturbance in Connemara, and extends to the north beyond it.

Another series of stations of high Vertical Force extend eastwards from Clare, and they apparently dominate all the region to the south of them. The direction of the disturbing force at Kilrush is anomalous. At Charleville there is a minimum of

Vertical Force, and therefore the direction of the Horizontal Force cannot be determined from the rule

The Wicklow and Arklow mountains, which are composed of granite, do not appear to exert any important effect on neighbouring stations, but there is a region of high Vertical Force in Wexford

Magnetic Map of the United Kingdom

In Plate XIII we have attempted to represent the magnetic state of the whole country by bringing together the results of our studies of the magnetic districts into which it may be divided

The valley and ridge lines are shown The unshaded parts are regions of negative Vertical Force disturbance, and the three shades employed indicate that the disturbance is greater than 0 but less than 01, greater than 01 but less than 02, and greater than 02 metric unit respectively

The lines representing the Horizontal Forces are drawn to a scale on which 1 mm. = 0 001 metric unit, except at Canna and Soa, where the disturbances are so large that the arrows would be inordinately long

Considerations which have been adduced in the foregoing discussion have led us to depart, in a few minor points, from the strict rules by which the valley and ridge lines have been drawn. Thus, the line which separates the Highland and Scotch Coal-field Districts has been drawn so as to include Crieff in the latter.

The rather uncertain valley line in Mid-Wales, which was taken from the Horizontal Force disturbances (Plate X), is replaced by the closely neighbouring ridge line, taken from the Declination disturbances (Plate X) This would necessitate valley lines on each side of it, the position of which is uncertain.

In Ireland too, the ridge line which runs from Antrim to the neighbourhood of Kells, is not very definite, and it is best to consider the clearly marked portion of it in Antrim as only doubtfully connected with the centre of attraction near Kells These points must be left for future investigation, but the unquestionable existence of widespread regional disturbance in the districts we have specially studied in England, together with the general agreement between the two maps of Scotland, deduced from the surveys of 1857 and 1886, leads us to hope that Plate XIII gives the first approximation to a map of the disturbing magnetic forces in play over the whole kingdom.

On the Relation between the Magnetic and Geological Constitution of the Magnetic Districts

Up to the present, we have discussed the various districts into which we have divided the country from the point of view of their magnetic peculiarities only.

It now remains to investigate the question whether any connection can be established between these and their geological characteristics

It is well known that certain varieties of crystalline rocks are often more or less magnetic, and that when they are permanently magnetised the poles are sometimes very irregularly distributed

We have, however, thought it worth while to investigate the magnetic state of some pieces of diorite and basalt, brought from Malvern and Canna respectively

The fragments were cut into small rectangular blocks delicately suspended and tested We have to thank Dr HOFFERT, Demonstrator in the Physical Laboratory of the Science Schools at South Kensington, for undertaking this part of the work. The observations were very tedious, and were, for the most part, made by Messrs GRAY, ANDERSON, and WILKINSON, students in the laboratory

Of several blocks from Malvern only one showed any polarity in its natural state, but when placed between the poles of an electromagnet it became magnetised by induction, so that the time of oscillation was reduced from 72^s to 56^s. The stone brought from Canna was part of a basaltic column and its upper, lower, east and west ends were marked It showed polarity, but the upper end was a north-seeking pole, so that it was magnetised in a direction opposed to that which would be induced by the magnetic field of the earth The moment due to the permanent magnetism was calculated by three different methods viz (1) by the difference of time of oscillation when the direction of the field (about twelve times as strong as that of the earth) was reversed, (2) by the deflection when the stone was placed E and W , and (3) by the difference of times when the position of the stone was reversed The results obtained were

0023 ·0027 and ·0019 C G S. unit

The periods of oscillation were 93^s and 86^s in the earth's field and the artificial field above described The general conclusion arrived at was that, as the volume was about 1 c c , the permanent intensity of magnetisation was about 0 002 C G S. unit, and that in a field of strength F the induced intensity was about 0 0015 F.

We have examined the relations between the magnetic disturbing forces and the geology of the area of the survey by means of a geological map which has been specially prepared for us under the kind superintendence of Professor JUDD (Plate XIV). Details are disregarded, but the principal masses of basaltic and non-basaltic crystalline rocks and the main groups of the sedimentary formations are clearly distinguished from each other. This may be compared with Plate XIII , in which the magnetic disturbing forces and the ridge and valley lines are shown

As the ridge lines are drawn according to an arbitrary rule, they are only intended to draw attention to the districts in which loci of attraction probably exist. We have no real knowledge of their distances from the stations between which they run In the geological map, therefore, the stations on each of the principal ridge lines have

been connected by lines, the spaces enclosed have been shaded, and thus the districts within which the main loci of attraction probably lie are clearly indicated

It must be remembered that the outlines of these districts depend largely on the accidental positions of stations which were selected without reference to them, and that we must rather expect such rough indications of relations between magnetic and geological facts as may serve to guide future investigations than complete and unmistakeable harmony

When regarded from this point of view, however, we think the results are very suggestive

In the Highlands one region of attraction encloses the Caledonian Canal, another is evidently in close relation with the basaltic masses in Skye, Glenmorven, and Mull, though, as has been pointed out, the main centre of attraction appears to be to the west of these islands

A third region in Scotland encloses the basaltic rocks of Arran and of the Scotch Coal-fields The fact that the Fifeshire basalt lies outside it is probably due only to the accidental circumstance that we have no stations between Stirling (47) and Dundee (21)

A fourth region evidently consists of the Antrim basalt. There is a fifth in Connemara, where the rocks are granite

With regard to North Wales, if we consider it as forming one district with Shropshire, we see that a line drawn through the centre of the basaltic rocks would first run from east to west, then nearly north, and finally turn to the west between Anglesea and Carnarvonshire The district is thus very irregular and the ridge lines do not give much information, but we must point out that the Horizontal Forces at the stations which border it all tend inwards towards the axis above suggested This is true of Holyhead (90) Llandudno (106), Llangollen (107), Shrewsbury (138), Aberystwith (55), and Pwllheli (128)

If this suggested relation is hereafter verified, every considerable mass of basaltic rock in the kingdom will be closely connected with a region of magnetic attraction Of smaller masses it is to be noted that at Falmouth (80) the disturbing force acts southward toward the serpentine of the Lizard, and that the relatively small masses in Pembrokeshire and Wexford are within another region of attraction Our measurements do not assign any particular importance to the outcrop of basaltic rock near Limerick (185) nor to the dykes in the north of England. It is, however, curious that a line drawn through Melton Mowbray and the Wash peak (indicated by circles) passes towards Wales through the only basalt in the Midlands

Taking the evidence as a whole, we think we are justified in saying that large masses of basaltic rock indicate regions of magnetic attraction.

The other crystalline rocks appear to be much less important magnetically Thus the Malverns, though a strong local centre, do not disturb a district of any considerable magnitude. If the effects of the two classes of rocks when they appear on the

surface are in such marked contrast, it may be open to question whether strong magnetic attraction in a district in which no crystalline rocks appear on the surface does not indicate not only crystalline but basic crystalline rocks beneath it

However this may be, there are two regions of attraction which are not connected with basaltic rocks on the surface, at all events as the main cause of their peculiarities

One of these runs westward from London to the South Wales Coal-field, and the directions of the disturbing forces in Wexford are such as would be caused if it crossed the Irish Channel This extension is doubtful, but the line is most clearly marked right across England, and its general direction is such as to make it almost impossible to avoid the conclusion that it is connected with the palæozoic ridge, the existence of which, long ago predicted by Mr GODWIN AUSTEN, has been proved by deep borings near London, and which is supposed to connect the Welsh and Belgian Coal-fields

If the palæozoic rocks are nearer the surface here than elsewhere, the crystalline rocks may approach it also within a moderate distance, and if they are susceptible to magnetisation the observed results would follow. The palæozoic rocks are supposed to form basins, and if those beneath them have a similar outline, it would be possible to explain a centre of attraction such as the Reading peak

The last region of attraction runs from the Wash through south-east Yorkshire toward the Cumberland Lakes, and, as Professor JUDD first pointed out to us, it includes the line to the north of the Humber, along which the oolitic and liassic strata thin out rapidly, and where, therefore, the crystalline rocks are probably suddenly brought much nearer to the surface It is continued northward toward the igneous masses in Cumberland

The centre of attraction which apparently exists at Kells, in Ireland, is not sufficiently accounted for by the surface geology of the district in which it is placed, and we have not felt ourselves justified in representing it as connected with the Antrim basalt, as the fact that this connection requires further confirmation has been already pointed out

On the whole, then, we think we may assert that every region of magnetic attraction, with the possible exception of that near Kells, is marked either by the presence of basalt, or by some geological peculiarity which makes it possible or even probable that within it crystalline rocks, capable of affecting the magnet, are nearer the surface than elsewhere. This is possible as regards the line of fault marked by the Caledonian Canal, and probable as regards south-east Yorkshire and the south of England.

The former district is remarkable from the fact that it was far more strongly affected by the earthquake of Lisbon than the rest of the British Isles, and this may, perhaps, indicate that it has special relations to the primary rocks, which would account for its magnetic importance.

On comparing the geological map with Plate XIII., on which the disturbing forces are shown for the whole country, a difficulty arises from the fact that the Vertical disturbing Forces appear, on the whole, to be greater over the districts in the east and south of England, where the later sedimentary rocks occur, than over Wales and Ireland, where, from their absence, it would *primâ facie* appear probable that the downward attraction would be greater

Of course many hypothetical explanations could be offered of the fact, such as that the primary rocks in England might possibly contain larger quantities of ferruginous matter, &c , but we must be content with observing that if a fairly uniform increase in the disturbing Vertical Force were to take place from east to west, it is very doubtful whether we should have detected it Probably it would cause a disturbance of the first order, the terrestrial lines would be deflected, and the disturbances at distant points would not be comparable.

If the true Vertical Force isomagnetics could be prolonged beyond Wales into another district in which the tertiary strata re-appeared, a southward trend in Wales would indicate an increase in the force On looking at Plate VIII we think it will be admitted that the slope of the lines of equal Vertical Force is greater in Wales than in England, which is in harmony with the existence of a southward bend, but the fact that they are extremely irregular and terminate on the west in the sea makes any certain deduction impossible We must, therefore, regard the distribution of the Vertical Force disturbances as presenting some difficulty, and must emphasise the necessity of using them only to compare neighbouring stations

The Causes of Local and Regional Magnetic Forces

It has long been known that distortions of the isomagnetics occur chiefly in the neighbourhood of crystalline rocks, and it has been generally assumed that this is due to so-called rock magnetism, the rocks being magnetised either permanently or by the inductive action of the earth's field.

Dr. NAUMANN, in the work already quoted ('Die Erscheinungen des Erdmagnetismus,' Stuttgart, 1887), has recently opposed this view

The arguments which may be brought against it are · (1) That rocks brought from considerable depths do not exhibit magnetic qualities until they have been for some time upon the surface , (2) That the effects which rocks or mountains produce on the magnet, even if very great when the distance is small, diminish so rapidly as the distance increases that they are quite insufficient to account for the widespread effects which are attributed to them; (3) That extensive local magnetic disturbance is associated rather with geological faults than with the presence of igneous rocks; (4) That the cause of the phenomena is to be looked for in the effects produced on

earth-currents by dislocations of the strata rather than in rock magnetism We will consider these arguments in the inverse order to that in which we have stated them

Mr PREECE, F R S, Chief Electrician to the General Post Office, has been good enough to have measurements of the earth-currents made in several of the districts in which we have found large disturbances of the Declination The currents flowing between various post-offices have been observed, and the directions and intensities of the currents noted

The following Table gives the data obtained on telegraph lines between Melton Mowbray and stations in its neighbourhood The letters P.D signify Potential Difference

Station	Bearing from Melton	Distance from Melton in miles	Earth currents in milliampères	Direction of current	Resistance of circuit in ohms	P D per mile in volts
Long Clawson	N 17° W	6.0	0.062	From	912	0.009
Oakham	S 50° E	9.0	0.045	Melton	477	0.002
Asfordby	W	2.5	0.070	in all	317	0.009
,, (Private Line)	W 15° N	1.9	0.067	cases	317	0.011

These potential differences per mile are much smaller than those which occur during magnetic storms, and which on the earth-current theory must, we suppose, be regarded as the causes of the deflections of the Declination needle which then take place During a very violent storm in 1881 Mr. PREECE found a P D. of 1.9 volt per mile We will take for comparison a less extreme case. Yearly records of the movements of the Declination needle, and of the registers of the earth-currents apparatus, are published by the Greenwich Observatory, and the Astronomer-Royal informs us that half an inch on the earth-current registers corresponds approximately to a P D of one volt in circuit

On September 10, 1886, an increase of 16′ took place in the Declination between 22^h and 22^h 5, and between 21^h 8 and 21^h 2 a change in the intensity of the current in one of the circuits occurred which corresponded to a change of P.D of about 0.9 volt. The direction of the current was such as to produce the observed movement of the needle, and if we regard the current as its cause, since the distance between the earth plates is 3 miles, we find that a deflection of 16′ was produced by a potential difference of 0.3 volt per mile

There are two earth-current circuits mutually at right angles, both of which are inclined at about 45° to the magnetic meridian. The second was but slightly affected, and, therefore, the effective difference of the potential was about $0{\cdot}3 \cos 45° = 0.2$ volt per mile nearly

At Melton Mowbray we find the disturbance of the Declination to be 33′ W. at one

of the two stations. It changes very rapidly, falling to 26′ W within a mile and a quarter, and to nearly 30′ E at Loughborough, which is thirteen miles distant. If currents produce it they must therefore be very local On the other hand we know that the disturbances produced during magnetic storms are simultaneous over areas such as that of the United Kingdom, and if the currents produce the deflections, they also must be widespread, so that if we assume that the same earth-current is required at Melton to produce a permanent deflection as is observed at Greenwich when an equal temporary deflection takes place, we are not overstating the case

Now, during the whole of the year 1886, there were only a few occasions on which the earth-currents were stronger than on that which we have selected as an example, and we may therefore say that on the earth-current theory there should be permanently at Melton a potential difference of 0 3 volt per mile (for the smaller deflection of 26′ is 1 6 times that observed at Greenwich on the selected occasion) an amount which is only registered at Greenwich during violent storms, and is nearly thirty times greater than that observed in a circuit only two miles in length in the neighbourhood of Melton itself.

So far we have dealt only with the magnitudes of the currents The case becomes much stronger when we consider their directions. On all the circuits out of Melton Mowbray referred to in the table on p 315, the earth-currents flowed from that station. Hence, on all the currents except that to Oakham, the direction of the current was such as to produce a deflection of the North Pole to the east, *i e*, in the opposite direction to that which was actually observed, while the P D. between Oakham and Melton was the smallest of those measured, being only 0 002 volt per mile. We do not lay stress on this, as the distance was perhaps rather too great, the main fact being that the earth-currents between Asfordby and Melton, on the same side of the latter town as that on which our second station was placed, were not only much too small (if we may judge from what is observed at Greenwich) to produce the observed deflection, but that they were actually in the wrong direction

To make certain that nothing in this argument depends upon the particular deflection obtained at Greenwich on the selected occasion, we have taken twenty other examples in which Declination disturbances of from 3′ to 22′ were accompanied by changes in the earth-currents. To avoid the necessity for correcting for diurnal variation they were chosen from the nocturnal hours when the magnet is normally steady, and occasions were selected on which one circuit only was appreciably affected Except in these particulars they were chosen haphazard

If we divide the E.M F per mile in volts (V) by the Declination disturbance (Δ) we get a number which expresses a relation between the two, and which, if they were cause and effect, ought to be constant The following table shows that the selected example was not particularly favourable to our views As the first station at Melton was on the south side of the town we assume that the potential difference was the same as that between Melton and Oakham. At the second station we take it to

be that between Melton and Asfordby. In both cases the whole is supposed to be effective, *i.e.*, it is not resolved along the magnetic meridian. A negative sign means that the current flowed in the direction opposite to that which would be required to produce the observed deflection.

Station		
Greenwich	Largest value of V/Δ	0·028
	Smallest ,, ,,	0·008
	Mean ,,	0·016
	Value on selected occasion	0·013
Melton Mowbray	Station 1	0·00006
	,, 2	0·00040

We are quite aware that an argument of this sort is open to criticism. The exact relation between earth-currents and magnetic storms is uncertain; in short circuits the earth-currents may be masked by those due to the earth-plate, and we have been compelled to assume that the conditions at Greenwich and Melton are the same. We should not, therefore, have put the argument forward had the conclusion depended on any nice balance of figures. As it is, we think it supports the view that the permanent earth-currents at Melton Mowbray are very much less, and less extended, than the temporary currents observed at Greenwich during ordinary but considerable storms, though the permanent Declination disturbance which they are supposed to produce is of the same order as the temporary deflections which are observed simultaneously with the currents at Greenwich.

Similar results were obtained near to the Reading and to the Wash disturbances. If these are produced by earth-currents they must circulate round the peaks, and thus the potentials at points on opposite sides of them should be different. Near the Wash the direction of the telegraph wires is not very favourable for a test, but we owe to the kindness of Mr. PREECE, to whom we must again express our obligation, a series of measurements made at Reading and Windsor. Of these it is only necessary to say that, though Reading and Windsor are on opposite sides of the focus of disturbance, and though the needle is deflected 11′ to the west at Windsor and 6′ to the east at Reading, so that the assumed current circulating round the peak ought to run in opposite directions through them, the observed earth-currents were so small that no measure of their magnitude could be taken.

It might be urged, in answer to these arguments, that the earth-currents by which local disturbances are produced are not mere surface currents, but that they flow through the mass of the earth, possibly where the strata are (as recently suggested by Professors LAMB and SCHUSTER) of higher conductivity.

To this it may be replied that the greater the depth at which the currents are

supposed to be situated, the more difficult is it to account for the fact that their effects are so local

A more conclusive answer is the fact that our observations show clearly that the needle is attracted to certain lines or points round which currents would have to circulate in the same direction (*i e*, with the hands of a watch) in order to account for the facts If the magnetic field of the earth is due to currents flowing from east to west, it is easy to imagine that they might be deflected by layers of more or less than average conductivity, but why should they form local eddies, in which the flow is always in the same direction? They must circulate similarly round the Reading and Wash peaks, round Kells, probably also round the Antrim basalt and the Toad-stones of Derbyshire In accordance with Commander CREAK's investigation, they must flow round magnetic islands in opposite directions in the two hemispheres They must flow in different directions on the two sides of the palæozoic ridge in the southern counties, of the Malverns, of the Yorkshire ridge, of the Scotch coal-field, and of the Caledonian Canal It may be possible to imagine physical causes which would account for such a state of things, but it does not appear easy to frame an hypothesis which shall be more probable than that involved in the theory of magnetic rocks

As regards Dr NAUMANN's view, that geological faults determine local magnetic action, it is disputed in its application to Japan by Dr. KNOTT, who has superintended the recent magnetic survey of that country. We find at Malvern a mass of crystalline rock bounded by a fault Either the rock or the fault may be supposed to be the cause of the attraction toward the ridge which is undoubtedly exerted on the needle, but whereas the rock is susceptible of magnetisation, and its effect on the magnet is precisely such as would be produced if it were magnetised by induction in the earth's field, there is no shred of direct evidence to prove that the fault is capable of causing or deflecting earth-currents, so as to make them flow in opposite directions on its opposite sides It is more probable that the action of faults is due to the displacement of crystalline rocks in their neighbourhood, and that a fault is, at all events, not a necessary cause of regional attractions is proved by our observations in Scotland Two great fault lines traverse that country, one coinciding with the Caledonian Canal and the other running from Stonehaven to the mouth of the Clyde. The first of these is, and the second is not, associated with a locus of attraction

Taking the next argument in order, we agree that mountains which exercise considerable influence on the magnet when close to them produce no effect at distances which are small relatively to the range of regional disturbances It must, however, be remembered that the depth from the surface to magnetic rocks concealed by overlying strata may not exceed the horizontal distances at which the Malverns and the Wash affect the Declination, and that their influence on the Vertical Forces might extend over vast areas If, as would often be the case, they were not horizontal, but approached the surface by a gentle slope, the magnet would certainly tend to turn

towards or away from the nearer and more elevated portions, according to the nature of their magnetisation

If the slope were less than the angle of Dip, the sides of the sub-terrestrial hill would be magnetised, so as to attract the north-seeking pole, and the empirical rule that the Horizontal Forces tend toward places of greatest Vertical Force would be amply and completely explained.

Dr NAUMANN's argument loses a great part of its force if we regard a widespread disturbance of the isogonals around a mountain range as due not to the direct action of the mountains but to a far reaching mass of rocks of which they are the culminating point

Lastly, as to the statement that rocks containing iron only become magnetic when brought to the surface—without impugning the actual observations, we can only say that so important a generalisation requires a much more extended basis of fact than any that is provided for it If the Malvern granites produce in their immediate neighbourhood a magnetic effect, it is evident that rocks, the susceptibility of which can only be detected by refined methods, may in large masses deflect the lines of force in the earth's magnetic field to an appreciable extent

On the whole then, while fully admitting that there is room for much further investigation on this head, and that any view is more or less hypothetical, we do not think that any theory has hitherto been proposed which offers fewer difficulties than that of rock magnetism

If the cause of the magnetic disturbance is induced magnetism, it would vary with the direction and intensity of the earth's magnetic field, but the change could hardly be great relatively to the secular variations of the elements It is therefore important to note that in the 'Phil Trans' for 1870 (vol. 160, p 274), Sir E SABINE gives the Declination at Greenwich and Kew as having been practically identical in 1842 5.

His values are —

Greenwich	23° 13′,
Kew	23° 11′

As according to his isogonals the Declination at Kew should, at that epoch have been 9′ greater than that at Greenwich, whereas, according to his table, it was 2′ less, the difference due to disturbance was − 11′ (*i e.*, the Kew value was too small), whereas in 1860 it was + 11′, in 1865 + 10′, and is now 10′ (p 270). If then we accept Sir E SABINE's table as correct, we must assume that a disturbance difference amounting in all to 22′ was established between 1842 5 and 1860, which has remained constant during the twenty-nine years which have elapsed since the latter epoch

We, therefore, wrote to Mr. WHIPPLE asking him for information as to the observations given by SABINE, as having been made at Kew in 1842·5. It should be noted that the authority quoted by SABINE is "Obs," which indicates an official origin for the value given

Mr. WHIPPLE writes, "I have now looked over all the MSS papers I can find, referring to this Observatory in 1842, and I am quite unable to find any reference to the observation quoted in the 'Phil Trans' for 1870, p 274 There was no Staff here at the date, nor any official publication, hence I do not see how "Observatory," or, as printed, "Obʸ," could be any authority Again, in SABINE's contribution, 'Phil Trans.,' 1849, p 208, Greenwich and Bushey are both given with the names of the observers, and I think there can be no doubt that if SABINE had observed here in 1842, he would have certainly quoted the observations in that table . . I fear we must consider it somewhat too hypothetical to trust to"

On the whole then, it appears to us most likely that the number given by SABINE was not deduced from an observation made in 1842, but was reduced to that epoch from the Kew Observatory records, which did not begin till some years afterwards, and that in the deduction some numerical blunder was made.

SUMMARY

We may now attempt to sum up the result of our studies of local and regional magnetic forces in the United Kingdom.

We have proved beyond possibility of doubt that two centres of attraction exist near Reading and the Wash respectively, and in the former case it is evident that the Horizontal disturbing Forces tend towards regions of maximum Vertical disturbing Force.

We have then gone through the United Kingdom district by district, and have shown that the same rule holds good everywhere. Putting aside stations of maximum or minimum Vertical Force and stations on valley lines (which do not furnish real exceptions) there are not half a dozen which are clearly anomalous in the whole 200 , nor has this result been obtained by splitting the whole area of the survey up into a large number of small districts which have been carved out so as to secure an appearance of uniformity of behaviour. Omitting the border stations on the south of Scotland, in Devonshire and Cornwall, the Channel Isles and Dover, the magnetic constitution of the rest of Great Britain is accounted for by five principal ridge lines only

The first is coincident for a great part of its length with the direction of the great fault of the Caledonian Canal. The second and third are connected with the masses of basalt in the Western Isles and the Scotch Coal-field respectively.

The fourth runs for 100 miles parallel to, if not coincident with, a line in Yorkshire and Cumberland along which geologists believe that crystalline rocks may occur near the surface, and sends out a branch toward the igneous rocks which occur in North Wales.

The fifth runs for nearly 200 miles from London to Milford Haven along another similar geological line, and sends out a branch to the south coast from near Reading.

It is probable that to these a sixth in North Wales and Shropshire should be added There are indications of other minor centres at Malvern, in Derbyshire and in the neighbourhood of Charnwood Forest, but the number of stations involved is small The vast majority are included in the few and simple systems above described

These results are not the outcome of the calculations only, for all the principal conclusions can be drawn from the observations alone Mr WELSH's survey though including fewer stations at which all the elements were determined than ours, though omitting almost altogether stations in the Western Isles, though worked up (as regards the Dip and Declination) by means of a different system of geographical coordinates, and in the case of all three elements by means of an assumption as to the linearity of the isomagnetics which we have abandoned, confirms our conclusions In four only of the 28 of his stations which fall within our districts, viz, Stornoway, Pitlochrie, Ayr, and Dumfries is there an important difference between us as to the direction of the disturbing forces.

In the most highly disturbed districts, at Loch Inver, Kyle Akin, Broadford, Glenmorven we agree as to the order of the magnitude and as to the direction of the forces. At Cumbrae and Fairlie we find closely neighbouring stations of positive surrounded by others of negative Vertical Force disturbance, and WELSH confirms us by a similar result at Ardrossan, not ten miles off

These coincidences between the results of the two surveys, and between the magnetic peculiarities and the geological constitution of districts cannot be accidental, and we venture to assert that, throughout the kingdom, the lines of Horizontal magnetic disturbing Force tend towards regions of maximum Vertical disturbing Force, which are themselves defined by the presence of crystalline rocks, and especially of basalt, either visible on the surface or concealed by superimposed masses of sedimentary strata

Beyond this general conclusion we do not wish at present to go. The detailed constitution of our principal magnetic districts (except in the case of the Reading and Wash disturbances) has yet to be investigated. We have not discriminated between various possible causes of a decrease in the Vertical Force, such as the removal of the attracting mass to a greater depth, or the formation of repulsive poles, either by induction or permanent magnetism We do not think the last word has been said on the cause or causes of local and regional forces All these require and will, in the future, no doubt, receive attention. We trust, however, that by following out to their logical conclusion the premises of a not improbable working hypothesis, we have succeeded in showing that local and regional forces obey certain simple laws, and that by means of these the kingdom can be divided into magnetic districts in which the relations between the direction of the disturbing forces and the main geological characteristics are so suggestive as to be worthy of careful statement and further investigation.

Tables for the Calculation of the Magnetic Elements in the Future

We conclude this paper with Tables, by means of which the values of the magnetic elements may be determined during the next few years for any place in the United Kingdom For this purpose we should know as accurately as possible—(1) the values of each element at the station at the epoch of the survey, (2) the secular change, (3) the local disturbance To determine the first of these, we give the next three Tables (VIII, IX, X), in which the values of the elements deduced from the general formulæ are given for all intersections within the kingdom of lines of latitude and longitude which correspond to whole degrees, together with the variation per degree of latitude and longitude The method of using them requires no explanation

The values in the Table of Horizontal Forces display some irregularities This is partly due to the fact, that a discontinuity is introduced along the line $H = 1\ 7$, on opposite sides of which we have used different formulæ The formula also, which is applied to the southern district, does not represent parallel lines, but lines of which the slope is a periodic function of the latitude We have slightly smoothed the irregularities, a process which will introduce some discrepancies between the numbers given by the Table and by the formulæ, but we have thought it better not to attempt to do away with all traces of the discontinuity.

Table VIII.—Declinations at Intersections of Degrees of Latitude and Longitude

Latitude N.	Longitude																								
	10° W		9° W		8° W		7° W		6° W		5° W		4° W		3° W		2° W		1° W.		0		1° E		2° E
60°											23° 15′ 2	42′ 3	22° 32′ 9	42′ 4	21° 50′ 5	42′ 3	21° 8′ 2	42′ 4	2[illegible] 15′ 5						
											23′ 2		21′ 8		20′ 2		18′ 8								
59											22° 52′ 0	40′ 9	22° 11′ 1	40′ 8	21° 30′ 3	40′ 9	20° 49′ 4								
											22′ 5		21′ 0		19′ 5		18′ 0								
58					24° 27′ 5	39′ 3	23° 48′ 2	39′ 4	23° 8′ 8	39′ 3	22° 29′ 5	39′ 4	21° 50′ 1	39′ 3	21° 10′ 8	39′ 4	20° 31′ 4								
					25′ 1		23′ 6		22′ 1		20′ 6		19′ 1		17′ 6		16′ 1								
57					24° 2′ 4	37′ 8	23° 24′ 6	37′ 9	22° 46′ 7	37′ 8	22° 8′ 9	37′ 9	21° 31′ 0	37′ 8	20° 53′ 2	37′ 9	20° 15′ 3								
					23′ 2		21′ 7		20′ 2		18′ 7		17′ 2		15′ 7		14′ 2								
56			24° 15′ 6	36′ 4	23° 39′ 2	36′ 3	23° 2′ 9	36′ 4	22° 26′ 5	36′ 3	21° 50′ 2	36′ 4	21° 13′ 8	36′ 3	20° 37′ 5	36′ 4	20° 1′ 1	36′ 3	19° 24′ 8						
			24′ 0		22′ 4		21′ 0		19′ 4		18′ 0		16′ 4		15′ 0		13′ 4		12′ 0						
55	24° 26′ 5	34′ 9	23° 51′ 6	34′ 8	23° 16′ 8	34′ 9	22° 41′ 9	34′ 8	22° 7′ 1	34′ 9	21° 32′ 2	34′ 8	20° 57′ 4	34′ 9	20° 22′ 5	34′ 8	19° 47′ 7	34′ 9	19° 12′ 8	34′ 8	18° 38′ 0				
	26′ 2		24′ 7		23′ 2		21′ 7		20′ 2		18′ 7		17′ 2		15′ 7		14′ 2		12′ 7		[illegible]				
54	24° 0′ 3	33′ 4	23° 26′ 9	33′ 3	22° 53′ 6	33′ 4	22° 20′ 2	33′ 3	21° 46′ 9	33′ 4	21° 13′ 5	33′ 3	20° 40′ 2	33′ 4	20° 6′ 8	33′ 3	19° 33′ 5	33′ 4	19° 0′ 1	33′ 3	18° 26′ 8				
	28′ 1		26′ 6		25′ 1		23′ 6		22′ 1		20′ 6		19′ 1		17′ 6		16′ 1		14′ 6		[illegible]				
53	23° 32′ 2	31′ 9	23° 0′ 3	31′ 8	22° 28′ 5	31′ 9	21° 56′ 6	31′ 8	21° 24′ 8	31′ 9	20° 52′ 9	31′ 8	20° 21′ 1	31′ 9	19° 49′ 2	31′ 8	19° 17′ 4	31′ 9	18° 45′ 5	31′ 8	18° 13′ 7	31′ 9	17° 41′ 8	31′ 8	17° [illegible]
	30′ 0		28′ 5		27′ 0		25′ 5		24′ 0		22′ 5		21′ 0		19′ 5		18′ 0		16′ 5		15′ 0		[illegible]		[illegible]
52	23° 2′ 2	30′ 4	22° 31′ 8	30′ 3	22° 1′ 5	30′ 4	21° 31′ 1	30′ 3	21° 0′ 8	30′ 4	20° 30′ 4	30′ 3	20° 0′ 1	30′ 4	19° 29′ 7	30′ 3	18° 59′ 4	30′ 4	18° 29′ 0	30′ 3	17° 58′ 7	30′ 4	17° 28′ 3	30′ [illegible]	16° 58′ 0
							26′ 2		24′ 8		23′ 2		21′ 8		20′ 2		18′ 8		17′ 2		15′ 8		14′ [illegible]		[illegible]
51							21° 4′ 9	28′ 9	20° 36′ 0	28′ 8	20° 7′ 2	28′ 9	19° 38′ 3	28′ 8	19° 9′ 5	28′ 9	18° 40′ 6	28′ 8	18° 11′ 8	28′ 9	17° 42′ 9	28′ 8	17° 14′ 1	[illegible]	16° 45′ 2
											22′ 5		21′ 0		19′ 5		18′ 0								
50											19° 44′ 7	27′ 4	19° 17′ 3	27′ 3	18° 50′ 0	27′ 4	18° 22′ 6								

Table IX — Horizontal Forces at Intersections of Degrees of Latitude and Longitude

Latitude N.	Longitude.																								
	10° W		9° W		8° W		7° W		6° W		5° W		4° W		3° W		2° W		1° W		0		1° E		2° E
60°										56	1 4610	55	1 4665	56	1 4721	55	1 4776	56	1 4832						
											368		369		368		369								
59									1 4923	55	1 4978	56	1 5034	55	1 5089	56	1 5145								
									368		369		368		369		368								
58					1 5180	56	1 5236	55	1 5291	56	1 5347	55	1 5402	56	1 5458	55	1 5513								
					369		368		369		368		369		368		369								
57					1 5549	55	1 5604	56	1 5660	55	1 5715	56	1 5771	55	1 5826	56	1 5882								
					368		369		368		369		368		369		368								
56			1 5862	55	1 5917	56	1 5973	55	1 6028	56	1 6084	55	1 6139	56	1 6195	55	1 6250	56	1 6306						
			368		369		368		369		368		369		368		369		368						
55	1 6175	55	1 6230	56	1 6286	55	1 6341	56	1 6397	55	1 6452	56	1 6508	55	1 6563	56	1 6619	55	1 6674	[illegible]	[illegible]				
	368		369		368		369		368		369		368		369		368		369		370				
54	1 6543	56	1 6599	55	1 6654	56	1 6710	55	1 6765	56	1 6821	55	1 6876	56	1 6932	55	1 6987	[illegible]	1 7043	57	1 7100				
	367		368		373		378		384		388		391		392		[illegible]		393		[illegible]				
53	1 6910	57	1 6967	60	1 7027	61	1 7088	61	1 7149	60	1 7209	58	1 7267	57	1 7324	56	1 7380	56	1 7436	55	1 7491				
	414		417		417		414		410		406		404		403		404		405		[illegible]				
52	1 7324	60	1 7384	60	1 7444	58	1 7502	57	1 7559	56	1 7615	56	1 7671	56	1 7727	57	1 7784	57	1 7841	[illegible]	[illegible]	[illegible]	1 7960	[illegible]	1 8021
							407		406		407		410		414		418		424		[illegible]		[illegible]		[illegible]
51							1 7909	56	1 7965	57	1 8022	59	1 8081	60	1 8141	[illegible]	1 8202	63	1 8265	[illegible]	1 8331	[illegible]	[illegible]	69	[illegible]
											409		414		420										
50											1 8431	64	1 8495	66	1 8561										

Table X — Dips at the Intersections of Degrees of Latitude and Longitude

Latitude N	10° W		9° W		8° W		7° W		6° W		5° W		4° W		3° W		2° W		1° W		0		1° E		2° E
60°															72°42'0	6'0	72°36'0	6'0	72°30'0						
															30'2		30'4		30'6						
59					72°42'0	6'0	72°36'0	6'0	72°30'0	6'0	72°24'0	6'1	72°17'9	6'1	72°11'8	6'2	72°5'6	6'2	71°59'4						
					30'2		30'4		30'6		30'8		31'1		31'3		31'5		31'7						
58					72°11'8	6'2	72°5'6	6'2	71°59'4	6'2	71°53'2	6'4	71°46'8	6'3	71°40'5	6'4	71°34'1	6'4	71°27'7						
					31'3		31'5		31'7		31'9		32'0		32'3		32'4		32'6						
57					71°40'5	6'4	71°34'1	6'4	71°27'7	6'4	71°21'3	6'5	71°14'8	6'6	71°8'2	6'5	71°1'7	6'6	70°55'1						
					32'3		32'4		32'6		32'9		33'1		33'2		33'5		33'7						
56					71°8'2	6'5	71°1'7	6'6	70°55'1	6'7	70°48'4	6'7	70°41'7	6'7	70°35'0	6'8	70°28'2	6'8	70°21'4						
					33'2		33'5		33'7		33'8		34'0		34'2		34'3		34'5						
55	70°48'4	6'7	70°41'7	6'7	70°35'0	6'8	70°28'2	6'8	70°21'4	6'8	70°14'6	6'9	70°7'7	6'9	70°0'8	6'9	69°53'9	7'0	69°46'9	7'0	69°39'9				
	33'8		34'0		34'2		34'3		34'5		34'7		34'8		35'0		35'2		35'4		35'6				
54	70°14'6	6'9	70°7'7	6'9	70°0'8	6'9	69°53'9	7'0	69°46'9	7'0	69°39'9	7'0	69°32'9	7'1	69°25'8	7'1	69°18'7	7'2	69°11'5	7'2	69°4'3	7'2	68°57'1		
	34'7		34'8		35'0		35'2		35'4		35'6		35'8		36'0		36'2		36'3		36'5		36'8		
53	69°39'9	7'0	69°32'9	7'1	69°25'8	7'1	69°18'7	7'2	69°11'5	7'2	69°4'3	7'2	68°57'1	7'3	68°49'8	7'3	68°42'5	7'3	68°35'2	7'4	68°27'8	7'5	68°20'3	7'5	68°12'8
	35'6		35'8		36'0		36'2		36'3		36'5		36'8		37'0		37'2		37'5		37'7		37'9		38'1
52	69°4'3	7'2	68°57'1	7'3	68°49'8	7'3	68°42'5	7'3	68°35'2	7'4	68°27'8	7'5	68°20'3	7'5	68°12'8	7'5	68°5'3	7'6	67°57'7	7'6	67°50'1	7'7	67°42'4	7'7	67°34'7
									37'5		37'7		37'9		38'1		38'4		38'6		38'9		39'2		39'5
51									67°57'7	7'6	67°50'1	7'7	67°42'4	7'7	67°34'7	7'8	67°26'9	7'8	67°19'1	7'9	67°11'2	8'0	67°3'2	8'0	66°55'2
									38'6		38'9		39'2		39'5		39'8		40'2		40'5		40'8		41'2
50									67°19'1	7'9	67°11'2	8'0	67°3'2	8'0	66°55'2	8'1	66°47'1	8'2	66°38'9	8'2	66°30'7	8'3	66°22'4	8'4	66°14'0

The determination of the rate of secular change is a more difficult matter, but at the completion of our work we have more data at our disposal than in the earlier stages, when coefficients had to be chosen for the reduction of observations The plans adopted in the case of each element are as follows —

1 *Declination*

We have deduced from Sir F EVANS's Map for 1872 (*loc. cit*) the Declinations at 24 points distributed uniformly all over the kingdom, and have compared them with the values given by our Table VIII, on page 238 The secular corrections thus calculated show an increase from east to west amounting to about 0′ 11 and 0′ 14 per degree of longitude in latitudes 58° and 52° respectively.

There is also an increase with latitude above the latitude 52°

The results are exhibited in the following Table —

TABLE XI —Mean Secular Change of Declination per annum between 1872 and 1886

Latitude	Longitude.					
	10° W	8° W	6° W	4° W	2° W	0°
°	′	′	′	′	′	′
60				9 1	9 0	
58		9 6	9 2	8 8	8 9	
56		9 1	8 6	8 3	8 2	
54	9 3	8 5	8 1	7 8	7 8	7 9
52	9 4	8 9	8 1	7 8	7 8	7 7
50				8 3	8 0	

This is in fair accord with the observations at individual stations given on p 88, but as Sir FREDERICK's isomagnetics were not deduced by any definite system of calculation, but little importance must be attached to minor variations and discrepancies

We do not, for instance, feel justified in assuming that there is a real increase of the secular coefficient to the south of lat 52°, especially as M MOUREAUX's values in France for corresponding intervals are less than ours

It is also in accord with the results of the comparison of our observations with those of WELSH, which show that between 1857–1886 the secular change in Scotland increases with the latitude and with west longitude We think, therefore, that the most that can be said at present is that the secular change at the time of our survey, as given by the mean of the Greenwich and Kew values (p. 91), was in the neighbourhood of London about 6′ 5 per annum, and that a comparison with Sir F EVANS's results shows that it is about 1′ 5 greater in the south-west of Ireland, and about 2′ larger in the north-east of Scotland

Hence by smoothing the irregularities in Table XI above, and reducing all the numbers so as to give the present rate at London we get the figures in Table XII., p 325.

These may be checked in future by means of the observations at Kew, Greenwich, Falmouth, Stonyhurst, and Valentia, but we do not think a comparison of two observatories only leads to trustworthy laws of rate of change of secular variation with geographical position There can be no question that Declination change on the whole increases with the latitude, yet the secular coefficient is less at present at Stonyhurst than at Greenwich and Kew It is therefore very much to be regretted that the observations so sedulously carried on by the late Provost LLOYD, at Trinity College, Dublin, have of late years been interrupted

We may point out that the best position for a magnetic observatory would be in the centre of a widespread region of low Vertical Force The disturbing rocks would probably produce less effect in such a situation, and all the phenomena might be expected to be more normal Unfortunately Kew and Greenwich are within the range of the Reading disturbance, and Stonyhurst is in a region where the Vertical Force changes very rapidly both to the east and west of that place

2 *Inclination*

We have also in the case of the Dip found the secular change between 1842 5 and 1886 by comparing Table X with Sir E SABINE's map for the earlier epoch The result confirms the conclusions previously arrived at, viz , that the secular change diminishes with latitude and increases with west longitude As, however, the north-westerly stations are on the line of minimum change they probably do not form real exceptions to the general rule Partly by this method and partly by the map of assumed values (p 85) we have selected those given in Table XII

3. *Horizontal Force*

This element when treated in the same way gives since 1842 5 a greater rate of secular change in the south than in the north, and in the west than in the east The latter conclusion is in accord with that of M MOUREAUX

We are now inclined to think that the high value of the secular change between the years 1883–85 at Greenwich (·0028) led us to assume rather too high a value (·0022) for the south of England

The mean annual change for Greenwich between the years 1883–87 is 0020 and for Kew 0017 The values for Greenwich in 1857 and 1887 are 1 769 and 1 818, but in 1861 a new instrument was introduced, and to make the two values comparable we must subtract 0 016 from the first This leads to a difference of 0 065 or a secular change of 0·0022

Though this is a little greater than the present rate, we do not think the evidence is sufficient to justify us in altering the rates obtained for Scotland by a comparison for the interval 1857–87, and we assume that they are valid at the present time For the neighbourhood of London we take 0 0019, and by the aid of SABINE's and WELSH's papers we have arrived at the values given in the following Table .—

TABLE XII —Rates of Secular Change per Annum in the Declination, Inclination, and Horizontal Force, in terms of Minutes of Arc for the Declination and Dip, and of Metric Units for the Horizontal Force
(N B —The two former elements are diminishing, the Horizontal Force is increasing)

Latitude	Longitude												
	10° W	9° W	8° W	7° W	6° W	5° W	4° W	3° W	2° W	1° W	0°	1° E	2° E
60°							8 6 1 0 0015	8 4 0 9 0015	8 3 0 9 0015	8 1 0 9 0014			
59°			9 0 1 2 0018	8 9 1 1 0017	8 7 1 1 0017	8 6 1 1 0017	8 4 1 0 0016	8 3 1 0 0016	8 1 1 0 0016				
58°			8 8 1 3 0019	8 6 1 2 0018	8 5 1 2 0018	8 3 1 2 0018	8 2 1 1 0017	8 0 1 1 0017	7 9 1 1 0017				
57°			8 6 1 4 0020	8 5 1 3 0019	8 3 1 3 0019	8 2 1 3 0019	8 0 1 2 0018	7 9 1 2 0018	7 7 1 2 0018				
56°			8 4 1 4 ·0020	8 2 1 4 0019	8 1 1 4 0019	7 9 1 3 0019	7 8 1 3 0018	7 6 1 3 0018	7 5 1 2 0018				
55°		8 4 1 6 0021	8 2 1 5 0021	8 1 1 5 0020	7 9 1 5 0020	7 7 1 4 0020	7 6 1 4 0019	7 5 1 4 0019	7 3 1 3 0019				
54°	8 3 1 7 0021	8 1 1 6 0021	8 0 1 6 0021	7 8 1 6 0020	7 7 1 5 0020	7 5 1 5 0020	7 4 1 5 0019	7 2 1 1 0019	7 1 1 4 0019	6 9 1 4 0018	6 8 1 3 0018		
53°	8 1 1 8 0022	8 0 1 7 0022	7 8 1 7 0022	7 7 1 7 0021	7 5 1 6 0021	7 4 1 6 0021	7 2 1 6 0020	7 1 1 5 0020	6 9 1 5 0020	6 8 1 5 0020	6 6 1 4 0019	6 5 1 1 0019	6 4 1 4 0018
52°	7 9 1 9 0022	7 7 1 8 0022	7 6 1 8 0022	7 4 1 8 0021	7 3 1 7 0021	7 1 1 7 0021	7 0 1 7 0020	6 8 1 6 0020	6 7 1 6 0020	6 [illegible] 1 6 0019	6 4 1 5 0019	6 2 1 5 0019	6 1 1 5 0018
51°					7 1 1 8	7 0 1 8 0021	6 8 1 8 0020	6 7 1 7 0020	6 5 1 7 0020	6 4 1 7 0019	6 2 1 6 0019	6 1 1 6 0019	5 9 1 6 0018
50°						6 7 1 9 0022	6 6 1 9 0021	6 4 1 8 0021	6 3 1 8 0021				

The local disturbance at any place may be estimated from the disturbances at neighbouring stations as given in Plates IX, X, and XII It is evident that this correction must be somewhat uncertain, but the maps will at all events give information as to whether the disturbance is likely to be large or small As an example of the use of the tables and maps, we calculate the Declination for lat 52° 30′ N and long 1° 30′ W on July 1 1889

From Table VIII it was

$$18^\circ\ 29' \ 0 + 8' \ 2 + 15' \ 2 = 18^\circ\ 53' \ 4 \text{ on January 1, 1886}$$

From Table XII the secular change is

$$-(6\ 5 + 1 + 15) \times 3\ 5 = -23' \ 6$$

which reduces the value to 18° 29′ 8

From Plate IX we see that the station is in a region of negative disturbance, and that the true Declination is probably less than the calculated amount by 15′ or 20′

It is obvious that for this purpose much rougher methods of calculation would suffice, but the main reason for making the process as accurate as possible is that the values of disturbing forces can only be determined if the rates of secular change are carefully discussed and accurately known As we hope that these will be further investigated, we give the fullest data at our disposal for the calculation of the undisturbed values of the elements.

INDEX

IV. On the Effects of Pressure on the Magnetisation of Cobalt.

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Communicated by Professor J. J. THOMSON, F.R.S.

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[PLATES 15, 16.]

§ 1. IN August, 1888, I commenced an investigation into the effects of pressure on the magnetic properties of cobalt, at the suggestion of Professor J. J. THOMSON. In his "Applications of Dynamics to Physics and Chemistry," he has arrived at certain conclusions as to the relations of magnetisations and mechanical strains, and the primary object of the following investigations was to obtain experimental data, whereby these conclusions might be tested.

During the process of the experiments, numerous points presented themselves whose elucidation seemed of importance for a satisfactory comparison of theory and experiment. The experiments have thus included a wider range than was originally intended, and have occupied a considerable amount of time.

Phenomena resembling very closely in some points those to be presently discussed have been noticed by previous observers in iron and nickel. As frequent references to these will be found essential for an adequate discussion of the phenomena presented by cobalt, a brief outline of the more prominent characteristics is here interposed.

§ 2. If a bar of some magnetic metal, subjected to a constant longitudinal stress, be exposed to the action of a magnetising current in a surrounding spiral it becomes magnetised, and in general simultaneously alters its length. If the current cease to flow the bar retains in general part of its magnetisation, and it may not return to its original length. On the other hand, if the bar while under the action of a constant current in the magnetising spiral, be subjected to a longitudinal stress, it in general simultaneously alters its magnetisation. Also, if the stress cease to act the magnetisation may not return to its original value. Phenomena similar to the last are also shown by the magnetisation which is residual after the break of the current.

These phenomena all depend on the *intensity* of magnetisation in the bar. The mode of this dependence will be discussed in considerable detail presently, but erroneous ideas are not unlikely to arise unless the following practical considerations be kept clearly in view.

When a bar whose diameter of cross section is comparable with its length is magnetised in a uniform field, there seems good reason to believe that the magnetisation it possesses is very decidedly less in the interior near the axis than it is near the surface, and when the bar is possessed merely of residual magnetism, the difference between the interior and surface distributions seems completely proved When, as usual, the magnetising field is not uniform, there is in general still less reason to expect a uniform intensity of magnetisation Thus, in general, a bar may be regarded as composed of a large number of elements, throughout each of which the magnetisation may be treated as sensibly uniform, but which may differ widely amongst themselves The term *intensity* applied to the magnetisation of such a bar merely means some sort of average of the intensities throughout the several elements

Now the application of a given stress to such a bar may affect very differently the magnetism of different elements It may, for instance, increase the magnetisation of some of them and diminish that of others Experiments as a rule give merely a sort of integral of the effects on the different elements, and this ought to be clearly recognised

In particular I would point out that the magnetic moment of a bar, as determined by some definite method, may have the same value when the bar is exposed to the action of a certain current in a spiral, as it has after the break of a larger current in the spiral, and that notwithstanding there may be an important difference between the distributions of magnetism in the two cases There is thus no *à priori* reason to expect the same phenomena to follow the application of a given longitudinal stress in the two cases

Effects of Magnetism on the Length of Bars

§ 3 Such effects have been observed by JOULE,* MAYER,† Professor BARRETT,‡ and Mr SHELFORD BIDWELL §

JOULE's experiments dealt with iron and steel His method of experiment consisted in running a current through a magnetising spiral and observing the instantaneous changes in the length of the bar accompanying the make and break of the current Observations were taken with several currents gradually rising in strength. Apparently JOULE did not demagnetise his rods in the course of his experiments, nor does he seem to have examined the effect of making and breaking the same current more than once. He concluded that in iron or soft steel bars free from stress the institution of a current in a surrounding spiral is followed by an increase in the length of the bar proportional to the square of the intensity of the magnetisation

* 'Phil Mag,' vol. 30 (3rd series), 1847, pp. 76–87 and 225–241

† *Ibid*, vol. 46 (4th series), 1873, pp. 177–201.

‡ *Ibid*, vol 47 (4th series), 1874, p. 51, also 'Nature,' vol 26, 1882, p. 585

§ 'Phil. Trans,' A, vol. 179, 1888, pp. 205–230. Also 'Roy. Soc. Proc,' vol. 40, 1886, pp. 109–133 and 257–266.

induced in the bar, and that on the break of the current there remains an increase in length proportional to the square of the intensity of the residual magnetisation JOULE also found that iron and soft steel under tension lengthened or shortened when magnetised according as the tension was small or great He believed the shortening proportional to the product of the bar's magnetisation into the strength of the magnetising current This last conclusion, however, is certainly not warranted by the results of his experiments

On JOULE's p 237, are given the results of an experiment, numbered 23, in which a soft steel wire exposed to a certain tension showed an increased length while exposed to a certain current, lengthened further when the current was broken, and then shortened on the make of a stronger current This suggests that for the wire in question under a given load there existed a critical intensity of magnetisation, and that the bar lengthened or shortened according as the magnetisation was less or greater than the critical value The true interpretation of this experiment seems to have been entirely missed by JOULE, and it would not appear to have attracted the notice of subsequent observers. It is thus only within the last few years that the existence of a critical field, or state of magnetisation, has been brought to light by the decisive experiments of Mr SHELFORD BIDWELL

§ 4 MAYER experimented with rods free from longitudinal stress, and he gives the results of the make and break of only one strength of current The current was, however, twice made and broken in some at least of the experiments, and the results of both the magnetic cycles are recorded Some of the results, given in a table on his p 200, are so suggestive in the light of our present knowledge that I reproduce them here in a form slightly altered to suit the present inquiry The numbers in the first column distinguish the rods used, all the other numbers give the lengthening, or when with a minus sign the shortening, of the rods in terms of an arbitrary unit.

Rod No	Increase in length of rods.		
	First make of circuit.	After first break	Second make of circuit
Iron 1	1 15*	4	1 1
Iron 2	1 6	4	1 6
Iron 3	2 0	1 1	2 5
Iron 4	2 5	1 35	2 5
Iron 5	1 65	1 05	2 05
Iron 6	1 4	55	1 45
Steel 7	8	1 4	1 15
Steel 8	— 25	25	— 25
Steel 9	— 4	— 15	— 35

* In original 1 25, but this seems a misprint.

On breaking the second currents the rods all returned to the length they possessed on the break of the first currents, so that the changes in the lengths of the rods became cyclic after a single make and break of the current The last three experiments are, it will be observed, very far from agreeing with JOULE's laws

The experiments suggest that it is mainly on the intensity of the existing magnetisation that the phenomena depend, but the divergence between the figures in the second and fourth columns indicates that the previous history of the material may, in some cases at least, be of considerable importance

In all the iron rods the induced magnetisation is clearly much below the critical In 7, the softest of the steel rods, the induced magnetisation though below is clearly approaching the critical In rod 8, the critical magnetisation lies between the induced and the residual, and if a critical magnetisation existed in the case of rod 9 it must have been less than the residual

§ 5 Of Mr SHELFORD BIDWELL's experiments those described in the 'Philosophical Transactions,' appear the most complete In these, he first demagnetised the specimen, and then sent through a surrounding spiral a series of currents gradually rising in strength For each strength of current the alterations in the length of the rod accompanying two successive makes were observed and recorded. The Tables III and IV, pp 220 and 221, for an iron rod show a small but unmistakeable difference between the effects of the first and second makes of a current of given strength In Table V. p 222, for a cobalt rod the differences might well be attributed to experimental errors

Mr. SHELFORD BIDWELL found critical fields both for iron and cobalt Iron he found to lengthen or shorten when magnetised according as the magnetising field was below or above the critical, while cobalt presented exactly the opposite phenomena A nickel rod shortened in all fields up to 1400 C G.S units, and the experiments left the existence of a critical field an open question The critical fields for several specimens of iron varied from about 270 to 380 C G S units, and the maximum elongations appeared in fields of about 80 C G S units In one cobalt rod the maximum shortening occurred in a field of about 400 C G S units, and the critical field considerably exceeded 800 C G.S units A second softer specimen showed a maximum shortening in a field of about 300 C G.S units, and had a critical field of 750 C G.S. units In these experiments, the rods were free from stress Mr SHELFORD BIDWELL's experiments, however, described in the 'Proceedings of the Royal Society,' show that the critical fields for iron are lowered by tension, a result in exact accordance with JOULE's experiments

§ 6. Professor BARRETT found a cobalt rod to lengthen when exposed to a comparatively weak field. The data he supplies are too limited to enable one to judge whether his later experiments on the subject were wholly free from the objections which he himself believed to have invalidated the conclusions based on his earlier experiments. I scarcely think his results afford any reasonable ground for doubting

the accuracy of Mr SHELFORD BIDWELL's conclusions at least for the actual cobalt rods of his experiments

Effects of Longitudinal Stress on the Magnetisation of Rods and Wires

§ 7 A preliminary general idea of these effects may be obtained from a consideration of the phenomena presented by a soft iron wire stretched beyond its original limit of perfect elasticity, and then subjected, in a weak magnetic field, to a stress cycle consisting of the application and removal of a definite tension for which it is perfectly elastic

The first application of the tension causes a large increase in the magnetisation of the wire, and on the removal of the tension this increase nearly all remains, or may even be added to The next one or two tension cycles cause a distinct progressive increase in the magnetisation. Presently, answering to the tension cycles, there appears a nearly cyclic change in the magnetisation, in which the maximum magnetisation appears when the tension is "on" Also the magnitude of the cyclic effect remains, at least approximately, constant for, at all events, a large number of tension cycles

Subtracting the effect of the final cyclic change of magnetisation from the change accompanying the first application of tension, we get what may be fairly regarded as that portion of the effect of the first tension which is of a permanent and non-cyclic character This quantity I have here termed the "shock-effect" If the magnetising current be broken, the first application of tension or of pressure has a large shock-effect, which invariably consists in a diminution of the residual magnetisation The effects of beating,* or other mechanical agitation of a rod, are in many respects very similar.

The experiments on the effects of stress on magnetism which are of most importance for our present purpose are those of VILLARI,† Sir W THOMSON,‡ Mr SHIDA,§ and Professor EWING ‖ All these observers have, at least in practice, recognised the distinction between the effect on a bar's magnetisation of the first few applications of a given longitudinal stress and its subsequent applications

§ 8 VILLARI unfortunately gives no clue to the strength of his magnetising fields, save the number and relative size of the DANIELL or BUNSEN elements he employed He discovered that the non-cyclic effects of the first tension and the cyclic effects of the later tension cycles in bars—or rather stout wires—of iron in weak magnetic fields are of the character indicated in the last paragraph. In stronger fields he found

* See WIEDEMANN's 'Elektricität,' vol 3, p 666, &c

† POGGENDORFF's 'Annalen,' vol 126, 1865, pp 87–122

‡ 'Phil Trans,' 1879, pp 55–85, or 'Mathematical and Physical Papers,' vol. 2

§ 'Roy Soc Proc,' vol 35, 1883, pp 404–454

‖ 'Phil Trans,' 1885, pp. 523–640, 1888, A, pp 325–337.

that while the first application of the tension still increased the bar's magnetisation, the cyclic effect had changed sign On his p 91, in experiment 3, is given an instance where a soft iron bar, exposed to one of his strongest fields, lost magnetisation on the very first application of the tension From the data supplied, however, it will be found that the shock-effect of this first tension was still an increase of magnetisation. Similar cases are also referred to on his p 97, but no data are given from which the sign of the shock-effect can be deduced

From VILLARI's experiments it obviously follows that a critical field, or intensity of magnetisation, must exist where the cyclic effects of tension vanish The field in which the effect vanishes is called by Sir W THOMSON the "VILLARI critical point" He and Professor EWING have made it clear that the field in which this phenomenon occurs depends on the amount of the tension which is applied and removed, and also on the existence of any permanent load Professor EWING also prefers to define the VILLARI point by reference to the magnetisation existing in the rod, and not to the field producing it. Some of VILLARI's steel bars behaved like iron, but in others a critical field, if existent, was lower than the lowest field he employed

VILLARI also observed the fall in the residual magnetisation caused by the first tension after the break of the current He further observed that after several tension cycles there appeared a cyclic change in the residual magnetisation Without making any restriction as to the strength of the pre-existing fields, he lays down the law, that in soft iron the residual magnetisation in the cyclic state is greatest when the tension is "on," while in hard iron and steel, it is greatest when the tension is "off" I can find no clue as to the actual strength of the pre-existing field or fields he employed.

Fitting an iron core inside a coaxial iron tube of the same length, VILLARI found that in fields between certain limits, the core and the tube showed, in response to cyclic changes of tension, cyclic changes of magnetisation of opposite signs The phenomena, as VILLARI himself pointed out, are exactly in accordance with the view, stated in § 5, that the intensity of magnetisation is greater on the surface of a bar than in its interior.

§ 9. Sir W THOMSON experimenting on a soft iron wire permanently stretched, and then exposed to cycles "on" and "off" of some weight causing no further permanent extension, found a critical field which was in general higher the smaller the weight employed in the tension cycles The critical fields obtained by the magnetometric method—calculated presumably for the centre of the magnetising coil—averaged about 20 C.G.S. units, and the fields at which the cyclic effect was a maximum, were roughly about a quarter of the critical

By the ballistic method with an induction coil at the centre of a much longer magnetising coil, critical fields such as 6·5 C.G.S. units were obtained. This Sir W THOMSON apparently assigns to the experimental wire having its magnetisation greatest at the centre of the magnetising spiral, so that this portion of the wire would

be brought into the critical state by much weaker currents than the portions under the ends of the spiral

In his § 210, Sir W THOMSON states that a VILLARI critical point was found in the central portions of a nickel bar, but from a note to § 239, it would appear that with the magnetometer opposite an end of the bar, no trace of a neutral field was obtained With a magnetometer as usual opposite one end, he found in nickel and cobalt a cyclic effect the reverse of that occurring in iron in fields below the VILLARI point, *i e*, the magnetisation was least when tension was "on"

§ 10 Mr SHIDA obtained VILLARI critical fields of 15 and 10 C G S units respectively for two different specimens of soft iron wire, which had been permanently stretched, and on which the tension cycles were performed with weights not much less than those causing the permanent extension With a pianoforte steel wire, he found that in the cyclic state tension "on" co-existed with a minimum of magnetisation in all his fields.

§ 11. The first thing in Professor EWING's researches that concerns us is his method of demagnetising wires. This consists in subjecting them to the action of a series of rapidly reversed currents diminishing in intensity In § 19 of the first of his papers referred to above, he states that his wires were thereby reduced to a standard condition—different possibly, he admits, from their condition previous to their first magnetisation

In his § 107 Professor EWING defines the VILLARI critical point as 'that value of the magnetisation $\mathfrak{I}$ at which reversal occurs in the sign of the magnetic difference produced by two (assigned) states of longitudinal stress." He uses the term as applicable (1) to that value of $\mathfrak{I}$ at which no magnetic change accompanies the *repeated* alternation from one to the other of two assigned states of stress, (2) to the value of $\mathfrak{I}$ answering to the intersection of the two curves giving the relations of $\mathfrak{I}$ to $\mathfrak{H}$—the field,—when $\mathfrak{H}$ is gradually raised from zero, and separate experiments are completed with the wire in each of the assigned states of stress. He also proposes that one of the assigned states of stress should be that answering to no load. It will be noticed that Professor EWING's first usage of the "VILLARI critical field" is that which accords with Sir W THOMSON's

In originally soft annealed iron wires stretched beyond the limit of perfect elasticity, Professor EWING found in every case distinct VILLARI points of the first kind These existed in lower fields the greater the weight producing the second assigned state of stress. If in a certain field the cyclic state has been reached for the loads 0 and Q, for which the magnetisation has the critical value, and the load be gradually raised from 0, then there ensues a rise in the magnetisation until the load reaches some value P less than Q, and then an equal fall as the load is increased to Q. Even in the weakest fields Professor EWING seldom found the magnetisation increase continually when the load was raised nearly to the limit of perfect elasticity.

From the figures on Professor EWING's Plate 63, it will be seen that the first

application of load has effects very similar to those just described. It will appear, however, from fig 42, § 88, that the critical fields at which the first application of a given load has no effect on the magnetisation are distinctly higher than the VILLARI points for the cyclic applications of these loads This figure also shows that the first application of a moderate load produces a very large increase of the magnetisation in weak fields, whereas in the strongest field of 34 C G S. units the effect is trifling

From fig 40 it will be seen that in the cyclic state the effects of tension cycles on the magnetisation residual after a field of 3·33 C G S units were of the same general character as appeared with an equal induced magnetisation

Soft annealed wires, when none but very small loads were used, were found by Professor EWING to possess in weak fields a maximum of magnetisation with tension "on," agreeing so far with stretched wires With greater loads the effects become reversed after the first application of the load. Professor EWING attributes this to *hysteresis*, and shows that when tapped these wires behave as stretched wires with low VILLARI points

From fig 43, Plate 64, it will be seen that soft wires respond to the first application of tension in a precisely similar way to stretched wires, but have much lower critical fields. The phenomena accompanying the application of load cycles to a soft annealed wire possessed of the magnetism residual after the break of a field of 34 C G S unit, are discussed in §§ 83–85 and shown in fig 39 They are unquestionably of the same general character as those occurring with an equal induced magnetisation

Iron wires, both when stretched and annealed, and a pianoforte steel wire stretched after annealing, were exposed by Professor EWING to gradually increasing magnetising currents, and the curves connecting $\mathfrak{I}$ and $\mathfrak{H}$ are given in his Plates 64 and 67

The specimens all showed VILLARI points of the second kind, which agreed with those of the first kind in being as a rule lower, the heavier the load These critical fields, or magnetisations, of the second kind were distinctly higher than the corresponding ones of the first kind.

Simultaneously observations were taken of the residual magnetisation whose results are shown in figures in Plates 67 and 68 In all the specimens the magnetisation residual after weak fields, and also its ratio to the induced magnetisation, were found to be largely increased by the presence of a load, the increase being greatest with the heaviest loads used As the fields were raised the influence of the loads diminished, and in the annealed iron and the steel wires the residual magnetisation and its ratio to the induced became eventually less under the heavier loads than in the absence of a load

§ 12 In his papers in 'Phil. Trans.,' A., 1888, Professor EWING found that in fields from 0 to 116 C G S. units, the presence of a tension on a thin nickel wire diminished its induced and residual magnetisations, and also the ratio of the residual to the induced. Also cyclic changes of tension "on" and "off" were accompanied by

cyclic changes of magnetisation, in which the magnetisation, whether induced or residual, was least when the tension was "on"

The effects of pressure on a nickel bar were exactly the opposite of those of tension, and the increase in the susceptibility was extremely large, especially near the "Wendepunkt"—or point where $\mathfrak{I}/\mathfrak{H}$ is a maximum Professor EWING looked for a VILLARI point in low fields but found no trace of its existence

Theoretical Considerations

§ 13. The experimental results already stated will enable Professor J J THOMSON's theoretical conclusions to be understood

Let e denote the strain in an isotropic elastic cylinder parallel to its axis, f and g the strains in the cross section perpendicular to and along the radius vector Also let $\mathfrak{H}$, $\mathfrak{I}$ and κ denote respectively the strength of the magnetic field, the intensity of magnetisation, and the coefficient of magnetic induction, assumed everywhere the same throughout the cylinder

In the case of a uniform longitudinal stress $g = f$. Thus, assuming

$$\frac{d\mathfrak{I}}{dg} = \frac{d\mathfrak{I}}{df} \qquad \text{(1)},$$

Professor THOMSON obtains for this case, on his p 51, two equations (48), of which the first may be put in the slightly altered form

$$\kappa \frac{de}{d\mathfrak{I}} = \mathrm{C}\left(1 - \mathfrak{H}\frac{d\kappa}{d\mathfrak{I}}\right)\left(\frac{d\mathfrak{I}}{de} - 2\sigma\frac{d\mathfrak{I}}{df}\right) \qquad \text{(2)}.$$

Here C denotes a positive constant depending only on the elastic properties of the cylinder, and σ is POISSON's ratio

All the differential coefficients must of course apply to some one definite state of the cylinder, and in determining $d\mathfrak{I}/de$ and $d\mathfrak{I}/df$ the strength of the field must be kept constant

It is pointed out by Professor THOMSON that according to EWING's experiments $1 - \mathfrak{H}\, d\kappa/d\mathfrak{I}$ is positive for iron. It is obviously positive for any magnetic metal in any field exceeding that which answers to its "Wendepunkt." It also appears to be positive for nickel and cobalt throughout the experiments of Professor ROWLAND * Thus it is probably safe to regard it as essentially positive

Under a uniform longitudinal stress

$$g = f = -\sigma e.$$

* 'Phil. Mag.,' vol 48, 1874, p. 321

If then, through a small increase in the stress, there be increases δe, δf, δg in the strains, we must have

$$\delta g = \delta f = -\sigma\delta e \quad . \quad (3)$$

If $\delta\mathfrak{I}$ be the consequent increase in the magnetisation, the field remaining constant,

$$\delta\mathfrak{I} = \frac{d\mathfrak{I}}{de}\delta e + \frac{d\mathfrak{I}}{df}\delta f + \frac{d\mathfrak{I}}{dg}\delta g,$$

and so by (1) and (3)

$$\delta\mathfrak{I} = \delta e\left(\frac{d\mathfrak{I}}{de} - 2\sigma\frac{d\mathfrak{I}}{df}\right).$$

Thus from (2)

$$\kappa\frac{de}{d\mathfrak{I}} = \mathrm{C}\left(1 - \mathfrak{H}\frac{d\kappa}{d\mathfrak{I}}\right)\frac{\delta\mathfrak{I}}{\delta e} \quad (4),$$

where $\delta\mathfrak{I}$ and δe denote the small increments in the magnetisation and longitudinal strain which follow a small change in the longitudinal stress

If (4) were a complete representation of the state of matters, then $de/d\mathfrak{I}$ and $\delta\mathfrak{I}/\delta e$ should always have the same sign, and should vanish for the same value of the magnetisation

It should be carefully noticed that in employing (4) we tacitly assume the changes in the strains to be very small, and thus the value of $\delta\mathfrak{I}/\delta e$ should be obtained by experiments in which the change in the longitudinal stress is very small

Thus, in particular, the magnetisation at which $\delta\mathfrak{I}/\delta e$ vanishes is the VILLARI point, as determined by experiments in which the difference between the two assigned states of stress is very small. And thus, in accordance with the experiments of Sir W THOMSON and Professor EWING, this theoretical VILLARI point may well be very much higher than those hitherto found by experiment

It may also be as well to point out that the magnetisation answering to the vanishing of $de/d\mathfrak{I}$ is not the critical magnetisation observed by Mr. SHELFORD BIDWELL where the rod resumes its original length, but the much lower magnetisation where the rod, if iron, has its greatest extension, if cobalt, its greatest shortening

Comparison of Theory and Experiment.

§ 14. The experiments already recorded sufficiently show that in iron, there is in weak fields, and again, in very strong fields, an agreement in sign between $de/d\mathfrak{I}$ and $\delta\mathfrak{I}/\delta e$, both expressions changing from positive to negative as the strength of the field is raised. It will be noticed, however, that the magnetisations obtained experimentally for the VILLARI point are much lower than those obtained for the point of maximum lengthening of the rod. This difference is certainly in some measure accounted for

by the lowering of the VILLARI point by the comparatively large stresses practically employed. Professor THOMSON also has pointed out that MAXWELL's distribution of stress should produce an independent system of strains in the bar, the strain in the direction of the lines of force being an extension. Though apparently a very small effect this would tend in iron to raise the magnetisation where $de/d\mathfrak{I}$ vanishes somewhat over the VILLARI point.

In nickel we see from the experiments of Sir W. THOMSON, Mr. SHELFORD BIDWELL, and Professor EWING, that $de/d\mathfrak{I}$ and $\delta\mathfrak{I}/\delta e$ are both negative in all ordinary fields. If the results obtained by Sir W. THOMSON in the central portions of a nickel bar, in a strong field, be accepted, then there does actually exist a very high critical magnetisation at which $\delta\mathfrak{I}/\delta e$ vanishes. There should accordingly be, not in weak fields where Professor EWING looked for it, but in very high fields, a critical magnetisation where a nickel bar would cease to contract. Such a phenomenon cannot be said to have its existence demonstrated by Mr. SHELFORD BIDWELL, but at the same time it is certainly not disproved by his experiments.

In cobalt according to Sir W. THOMSON, $\delta\mathfrak{I}/\delta e$ is negative in weak fields, and so also is $de/d\mathfrak{I}$ if we accept the results of Mr. SHELFORD BIDWELL. According to the latter observer, however, $de/d\mathfrak{I}$ changes sign in fields much higher, it is true, than the corresponding fields for iron, but still easily obtainable. Further, the cobalt not only recovers from the very considerable shortening it has experienced, but lengthens considerably as the field is raised. It would thus appear impossible to assign the phenomenon in any essential degree to the MAXWELL effect. If then Professor THOMSON's theoretical conclusions are sound, $\delta\mathfrak{I}/\delta e$ should change sign from negative to positive in a cobalt rod for a magnetisation quite within the reach of experiment.

Experimental Verification of Theory in Cobalt.

§ 15. The question whether or not a VILLARI point of the above kind exists in cobalt was the primary object of the following experiments, and a decided answer in the affirmative was obtained. Under the moderate stress employed, $\delta\mathfrak{I}/\delta e$ was negative in weak fields and positive in strong, and an unmistakeable VILLARI point appeared in a field of about 120 C.G.S. units.

This, it will be noticed, is a much lower *field* than those which Mr. SHELFORD BIDWELL obtained for the vanishing point of $de/d\mathfrak{I}$ in either of the specimens he employed, and the MAXWELL effect would here tend to lower these latter fields. In my specimen, however, as the field was raised from 120 to 300 C.G.S. units the *magnetisation* increased by less than 20 per cent. The accordance of theory and experiment would thus, in reality, appear to be closer in cobalt than in any recorded experiments on iron.

Preliminary Sketch of Phenomena Observed

§ 16. As the phenomena observed are somewhat complex, their mutual relationships might be hidden under the multiplicity of facts, which the full discussion of each separate phenomenon introduces A brief outline of some of the more important and certain results will, it is hoped, serve as a key to the subsequent more complete discussion

It is important to notice that all the phenomena were observed in a given specimen of cobalt, magnetised by a current in a given coil, and that it was found by repeating several of the sets of observations at various periods of the research, that the character of the specimen had not to all appearance been altered during the process of the experiments. The importance of this, when an attempt shall be made to connect the phenomena together, and explain them as consequences of one or more fundamental principles, will be at once obvious to any one who has noticed the variety in the phenomena observed in different specimens of iron

At the same time, when various specimens of the same magnetic metal—or even of different magnetic metals—are exposed to gradually increasing magnetic fields, and curves drawn in which the abscissæ give the strength of the field and the ordinates the intensity of magnetisation, experiment shows that the fields in which corresponding points of the several curves occur may widely differ, but that there exists a general resemblance in the form of the curves, which may be made much closer by properly altering the scale of abscissæ

Thus, while the particular field at which a certain phenomenon presents itself in a particular specimen may so largely depend on the individual peculiarities of the specimen that it may be of little value as an isolated fact, yet the determination of the point on the curve where the phenomenon occurs, and the position it occupies relative to the points where other phenomena occur, may lead to the recognition of general laws.

Phenomena Observed in the Induced Magnetisation

§ 17. The "Wendepunkt," or point where the coefficient of induced magnetisation is a maximum, is a point in the curves that can easily be recognised Very probably the most satisfactory comparison of the phenomena observed in different specimens would be obtained by expressing the magnetisation of each specimen in terms of the magnetisation it possesses at its Wendepunkt as a unit. The Wendepunkt of the specimen employed occurred in a field of about 35 C G S. units.

For the reasons just stated, the relations of both the induced and residual magnetisations to the strength of the field, though not a primary object of investigation, were observed under various conditions as to pressure. These relations, in the case of the induced magnetisation, are shown in figs. 1–4 (Plate 15).

The difference between the effects on the induced magnetisation of the first few and

subsequent applications of longitudinal stress, referred to in § 7 as occurring in iron, is equally characteristic of cobalt, so long as the magnetisation is not much in excess of that of the critical or VILLARI point In the present experiments the stress applied was a pressure, and so δe being always negative, and the magnetisation in the weaker fields greatest when the rod was under pressure. $\delta \mathfrak{J} / \delta e$ was, as already stated, negative in fields below the critical

The effect of the first application of pressure may, as explained in § 7, be regarded as the sum of the cyclic effect and of a non-cyclic or shock-effect In all fields below 200 C G S units the shock-effect gave an unmistakeable increase of magnetisation In all stronger fields, up to at least 725 C G S units, the magnitude of the shock-effect was so small that its sign even was a doubtful matter Thus, in cobalt, the shock-effect, so long at least as it is an appreciable quantity, gives an increase of magnetisation This consequence of the shock-effect, even in fields much above the VILLARI point, is known from VILLARI's experiments, recorded in § 8, to be also characteristic of iron

The absolute measures, both of the shock-effect of the first pressure and of the cyclic effect, possessed maxima in the neighbourhood of the Wendepunkt. Relative, however, to the magnetisation existing prior to pressure, both these effects increased continually in importance as the strength of the field was reduced within the actual limits of the experiments

The total effect of the first application of pressure had a critical field of about 160 C G S units, which, it will be noticed, is decidedly higher than the critical field for the cyclic effect As stated in § 11, in discussing Professor EWING's experiments, the existence of a higher critical field for the first application of stress than for the cyclic application, is equally characteristic of iron.

As it is important to know how far, if at all, the nature and magnitude of the cyclic effect of pressure depend on the circumstances attending the process of magnetisation, the rod was sometimes subjected to pressure before and during its introduction into the magnetising spiral It was found that there was no material alteration in the cyclic effect. The mode of variation of the effects of the first and of the cyclic application of pressure with the strength of the field is exhibited in figs 5–8 (Plate 15)

Phenomena Observed in the Residual Magnetisation

§ 18 Comparing figs 1 and 9, it will be seen that the residual magnetisation approaches "saturation" in much lower fields than does the induced.

The intensity of the residual, as of the induced, magnetisation was found to depend on the treatment of the rod during the flow and the break of the current. When the rod was free from pressure at the instant the current was broken, the residual magnetisation in all fields below 120 or 130 C G.S. units was increased by the appli-

cation of pressure cycles during the flow of the current In stronger fields no certain effect could be attributed to the pressure cycles. The effect, as may be seen from fig. 10, was greatest in very weak fields

The existence of pressure during the break of the current proved to have a considerable influence on the intensity of the residual magnetisation As might have been anticipated it increased the residual magnetisation in weak fields, but the effect changed its sign in fields far below the VILLARI point, and showed no signs of vanishing even in a field of 400 C G S units

The ratios of the residual magnetisation to the strength of the pre-existing field and to the intensity of the induced magnetisation under various conditions as to pressure are shown in figs 11 and 12 (Plate 16) From these it appears that in the absence of all pressure the intensity of the residual magnetisation is extremely small in weak fields In this case the phenomena closely resemble those noticed by many observers in iron The only essential difference apparently is that the fields at which certain phenomena appear in cobalt are much higher than the fields at which the corresponding phenomena appear in ordinary iron This is in exact agreement with the views already expressed as to the relative positions of the Wendepunkt

The application of pressure cycles during the flow of the current increased immensely the intensity of the residual magnetisation in weak fields The increase, in fact, due to this cause in the residual magnetisation in the weakest experimental fields, was even greater proportionally than in the induced, so that the ordinates of curves *c* and *d* of fig. 12 continually increase as the strength of the field is reduced.

The effects of the application or removal of pressure in shaking out the residual magnetisation were also observed, and are shown in Curve II, fig 13, and in figs 14 and 15 From the first of these curves it appears that the percentage of residual magnetisation shaken out by a series of pressure cycles continually diminished as the strength of the field was raised

When the rod was free from pressure during the break of the current, the percentage of the residual magnetisation, shaken out by the first application of pressure, was in weak fields decidedly diminished by the previous application of pressure cycles during the flow of the current In fields over 70 or 80 C.G S units, however, the effect of previous pressure cycles was extremely small In weak fields the removal of a pressure that had existed during the break of the current was fully as effective in shaking out residual magnetisation as was the application of a pressure when the rod, during the break of the current, remained free from pressure In fields over 30 or 40 C G.S. units, however, the removal of pressure became decidedly less effective than the application

As has been stated in §§ 8 and 11, VILLARI and Professor EWING found that cyclic applications of stress were followed by cyclic changes in the residual magnetisation of the same character as those produced in the induced magnetisation. As it seemed important to test the generality of these conclusions, the cyclic effects of pressure on

the residual magnetisation were here observed when the treatment of the rod during the flow and break of the current was varied

When pressure cycles were applied during the flow of the current, and pressure was "on" during its break, the cyclic effect had invariably the same sign while the strength of the field was raised from 11 to 400 C G S units. The effect in this case was in the same direction as in the induced magnetisation below the VILLARI point, *i e*, the magnetisation was greatest when pressure was "on"

When, however, the rod was free from pressure during the break of the current, a critical field was found to exist In the magnetisation residual after fields above the critical, the magnetisation was least when pressure was "on" This critical field was raised by the application of pressure cycles during the flow of the current, but it was in any case much lower than the critical field for the induced magnetisation It follows, of course, that the intensity of the critical residual magnetisation was very much less than that of the critical induced

The amounts of the cyclic effects under the various conditions are shown in the fifth columns of Tables II and VIII, and in the sixth columns of Tables VI and VII. These results show conclusively that for cobalt, and so in all probability for iron, the relations between stress and residual magnetisation must be largely dependent on circumstances other than the so-called intensity of magnetisation This, as pointed out in § 2, was *à priori* far from improbable, and should act as a warning against extending to residual magnetisation laws proved only for the induced

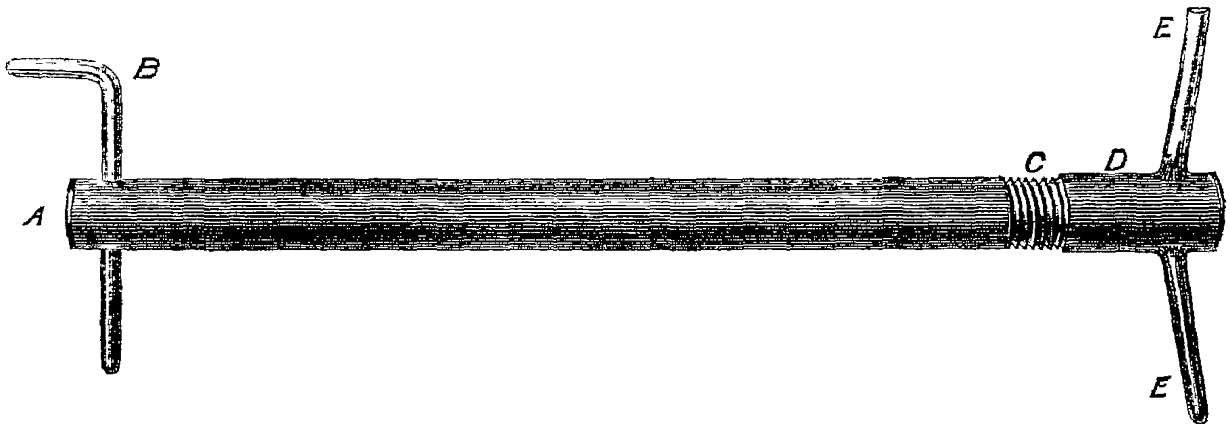
The fact that in fields over 60 or 70 C G S units both the intensity of the residual magnetisation and the character of the cyclic effect of pressure are very little affected by the application of pressure cycles during the flow of the current, while markedly influenced by the existence of pressure during the break of the current, seems worthy of special notice.

All the phenomena observed in the residual magnetisation apparently altered but little in magnitude, as the strength of the field was raised from 100 to 400 C G S. units This further emphasises the difference between them and the phenomena observed in the induced magnetisation

The Apparatus.

§ 19 The rod experimented on was supplied by Messrs JOHNSON and MATTHEY. The mean of a series of measurements gave 16 98 cms for its length and ·546 sq. cm. for the area of its cross section. During the experiments the rod was contained in a brass tube, the closed end of which was sufficiently thick to afford an unyielding resistance to the rod when under pressure. The rod could be slid in or out of the tube, but the fit was tight enough to prevent any lateral movement. On the outside of the brass tube, at its open end, a screw of small pitch was cut, answering to the screw cut on the inside of a brass cap The cap carried a projecting arm, forming a

diameter of a cross section, by means of which a considerable couple could be exerted The pressure was applied to the cobalt by means of a sort of ram-rod rigidly attached to the inside of the brass cap When the cap was screwed on this ram-rod advanced into the brass tube, and came to bear on the end of the cobalt rod



Drawing of the brass tube containing the cobalt rod, removed from the coil

A Closed end of brass tube
B Pin which can be slipped through the holes in the cheeks attached to one face of the coil and, as in the figure, through the diametral hole bored in the solid end of the brass tube
C Screw cut on the outside of the brass tube at its open end
D Brass cap with screw cut on its inside answering to C, and carrying ram-rod arrangement for applying pressure to the cobalt rod
EE Projecting arm, by means of which the cap is screwed on

In the original experiments, conducted in August, 1888, the brass tube was firmly imbedded in a block of wood, fixed inside a magnetising coil This coil had a length of 19 8 cms, and consisted of five layers of very thick copper wire A current of one ampère produced a field varying from 9 26 C G S units at the centre of the rod to 6 85 units at its ends This, it must be admitted, is not a very uniform field, but the fall occurred mainly within a centimetre or two of the ends of the rod.

In this preliminary investigation the observations were taken somewhat roughly. The reason they are referred to here is that they clearly showed the existence of a cyclic and a non-cyclic effect of pressure, and also the existence of a VILLARI point for the former effect. Further, the result of experiments on several different occasions agreed in giving 120 C.G S. units as a close approximation to the strength of the field due to the current at the centre of the rod when the VILLARI point appeared. The mode of variation of the several effects with the strength of the field showed also a general agreement with the subsequent more accurate observations taken with a different coil. This seems to warrant the belief that the phenomena observed do not owe any of their essential features to peculiarities of the apparatus

When it was attempted to obtain accurate results with the original apparatus various difficulties were encountered. In overcoming these I am much indebted to the ingenuity of Mr. BARTLETT, the assistant at the Cavendish Laboratory, who constructed the new apparatus.

The new coil had a length of 18.15 cm., an internal diameter of 1.4 cm., and an external diameter of 7 cm. It consisted of thirteen layers of copper wire of .24 cm. diameter when covered. In the solid end of the brass tube holding the cobalt a diametral hole was bored at right angles to the length of the tube, and a corresponding hole was made in each of two cheeks projecting from one of the faces of the coil. By slipping a pin through these holes the tube was firmly secured in a fixed position relative to the coil, and when the pin was withdrawn the tube and its contained cobalt could be removed and replaced without the least risk of disturbing the remainder of the apparatus.

A current of 1 ampère in the new coil produces a field varying from 65.7 C.G.S. units at the centre to 43.9 units at the ends of the cobalt rod. Throughout the greater portion of the rod, however, the field is very nearly the same as at the centre. These figures refer entirely to the action of the current, no attempt having been made to allow for the action of the rod itself.

In all the following calculations the values assigned to the field $\mathfrak{H}$ are calculated from the action of the current alone at the centre of the rod, and so are somewhat higher than the mean values of the actual fields. The difficulty of determining the actual fields at the different elements of the rod would be very great even treating the permeability as constant, a most erroneous supposition, and, considering the differences to be expected between different specimens of the same metal, an attempt at great accuracy in this department seems entirely supererogatory.

§ 20. Throughout the whole investigation the ordinary magnetometric method was employed. The axis of the coil was perpendicular to the magnetic meridian, and the magnetometer was situated in the direction of this axis produced. The magnetometer needle carried a mirror which formed on a millimetre scale, set parallel to the magnetic meridian, an image of a vertical wire placed across a slit in the centre of the scale behind which a lamp stood. The direct action of the coil current was neutralised by a compensating coil traversed by the same current. The strength of the current was recorded by a Thomson graded ammeter in circuit with the coil. The current was derived from the storage cells of the laboratory, and its strength was varied by changing the number of cells and the resistance in a wire bridge in circuit with the coil.

§ 21. In order to render apparent the relative magnitudes of the several phenomena some common system of magnetic measurements was essential. Thus all the scale readings have been reduced, and in the following tables and diagrams only C.G.S. units appear. From the previous remarks as to the method of calculating $\mathfrak{H}$ it will be clearly understood that the values given for the strengths of the fields, and consequently for the coefficients of magnetic induction κ, cannot rigidly be held to apply to the actual condition of the rod, in which $\mathfrak{H}$ and κ varied from point to point, but must be regarded as giving an approximation to the mean state of the rod, and more

particularly as indicating by their variation the amounts of the changes actually occurring

It should also be understood that in calculating the intensity of magnetisation $\mathfrak{J}$ the rod was treated as magnetised solenoidally In some of the experiments, *e g*, those on residual magnetism, a direct calculation on this basis might have introduced a considerable error, owing to the small distance of the rod from the magnetometer The actually observed readings, however, were first reduced by direct comparison of the readings obtained in an independent set of experiments, in which the rod in a given magnetised condition was placed first of all in the positions it was about to occupy in the course of the main observations, and then in a certain standard position, where its distance from the magnetometer was so great that the precise position of the "poles" was of comparatively little moment As stated in § 2, $\mathfrak{J}$ in reality doubtless varies from point to point of the rod, and the values here recorded indicate merely a sort of average.

Disturbing Agencies

§ 22 Before discussing the experiments in detail it will be as well to refer to two disturbing agencies, the one of importance mainly in weak fields, the other in strong

The former, the residual effects of previous magnetisations, affected very considerably some of the earlier experiments here recorded. In weak fields so important was this that the rod was sometimes found, after showing a considerable induced magnetisation, to possess, on breaking the circuit, residual magnetism of opposite sign. The application of pressure in general increased this residual magnetisation of opposite sign, sometimes to a very considerable extent. Or, supposing the residual magnetisation to be at first of the same sign as the preceding induced, it might change sign on the application of pressure These effects were produced at pleasure with the greatest ease when the rod was treated in accordance with the following hypothesis —

Calling the ends of the rod A and B, a weak current makes A, say, a north pole, and on breaking the current A is left with a quantity, N_1', of northern polarity A smaller reverse current shakes out only so much of this residual magnetism, leaving a quantity, N_2, in A, which exists alongside of a larger quantity, S_1, of southern polarity. The end A thus appears a south pole of strength $S_1 - N_2$ On breaking this second current, A is left with a quantity S_1', less than S_1 of residual southern magnetism, and a quantity N_2', probably less than N_2, of residual northern magnetism Whether A appears a south or a north pole depends on whether S_1' or N_2' is the greater On applying pressure, a considerably greater proportion of S_1' is shaken out than of N_2', and the polarity of the end A may thus change sign.

Similar phenomena, proceeding doubtless from the same cause, are described by WIEDEMANN as occurring in iron. They fit in well enough either with WEBER'S theory or with the view that the deep seated magnetic molecules are less affected by weak fields than are the surface molecules.

The total removal of this disturbing agency would require the complete demagnetisation of the rod. The method of demagnetisation employed at first consisted of applying a few reverse currents gradually diminishing in intensity. The magnetisation left in the rod seemed infinitesimal, but the unsatisfactory character of the result will be easily seen on reference to Tables I. and II. The method finally adopted was to expose the rod to a succession of diminishing reverse currents—the first current in the case of a weak field considerably exceeding that in existence during the pressure cycles—and then to tap it vigorously. The first attempt was by no means always successful, and the loss of time was a distinct objection to the method.

It is, of course, impossible to be sure that the rod was ever demagnetised, in the sense of being restored to its condition prior to its first magnetisation, but it was, at all events, reduced to such a condition that successive experiments made with the same field showed an excellent agreement. In strong fields demagnetisation appeared to be, at least for many purposes, of very little importance.

§ 23. The second disturbing agency, the heating of the coil and thence of the rod, by means of the current, was troublesome only in fields over 400 C.G.S. units. The heating of the coil wires increases their resistance, and so tends to diminish the strength of the field; while the heating of the rod increases its permeability, as has been shown by Professor ROWLAND.* For a rise of temperature of 225° C. he found the maximum value of the coefficient of induced magnetisation to increase by about 70 per cent., and to appear in a distinctly lower field.

This explains why, in some of my stronger fields, the scale reading kept altering in a direction indicating a progressive increase in the rod's magnetisation. This did not, of course, annul the cyclic changes accompanying cyclic changes of pressure, but it rendered impossible any very great accuracy in the determination of their magnitudes. The determination of the effect of the first application of pressure was found particularly difficult.

Lest it should be supposed that the cyclic changes in the magnetisation may in reality be due to the heating and cooling of the rod produced by the application and removal of pressure, I would here point out that the cyclic changes of magnetisation alter in sign in a comparatively low field, while the increase in the permeability accompanying heating was observed by ROWLAND in fields of all strengths, from 50 to 1470 C.G.S. units.

First Series of Experiments.

§ 24. With the new apparatus four principal series of experiments were performed. The first series took place in December, 1888, and the results are given in Tables I. and II. The general order of conducting these experiments was as follows:—

The rod, after being demagnetised by reverse currents alone without tapping, was removed to a distance. The number of storage cells and the bridge resistance were

* 'Phil. Mag.,' 4th series, vol. 48, 1874, p. 321.

then adjusted to give the field desired. When the ammeter and the scale reading became steady the rod was brought up and pushed rapidly into the coil, and the pin secured The difference between the scale readings when stationary, before and after the introduction of the rod, supplied the data for calculating the initial value of the induced magnetisation

A considerable number of cycles of pressure "on" and "off" were then performed, the scale reading, when become stationary, being usually taken after each "on" and "off" Sometimes when readings had been taken of the effects of the first four or five pressure cycles, some five or six cycles were applied in rapid succession and then the taking of readings once more resumed When a sufficient number of readings had been taken for the calculation of the average effect of a pressure cycle, the circuit was suddenly broken, the rod being always free from pressure during the break.

When the scale reading had become stationary after the break and had been observed, some six pressure cycles were applied, readings being taken after each "on" and "off" A series of six pressure cycles were then applied whose effects were not observed These were followed by some six or seven more pressure cycles whose effects were observed Finally the rod was removed to a distance and the magnetometer zero taken. The difference between the scale reading taken immediately after the break of the current and that taken after the removal of the rod supplied the data for calculating the initial value of the residual magnetisation

TABLE I.

$\mathfrak{H}$	$\mathfrak{I}_1$	κ_1	$\mathfrak{I}_2-\mathfrak{I}_1$	κ_2	First "on"	Cyclic "on"—"off"	$\frac{\text{Shock-effect}}{\mathfrak{I}_1}$	$\frac{\text{Cyclic "on"—"off"}}{\mathfrak{I}_2}$
·80 S (13)	1·54	1 9	1 08	3 3	77	..	.	.
1 15 N (12)	4 01	3 5	3 09	6 2	3 09	80	57	11
1 44 N (15)	4 63	3 2	2 47	4 9	2 93	65	49	09
1 61 S (14)	3 55	2 2	2 32	3 6	1 39	·11	36	02
1 9 N (11)	8 9	4 7	2 8	6 2	3 7	1 2	27	10
4 3 N (10)	19 6	4 5	6 9	6 1	7 4	1 6	30	059
7 8 N (9)	38 0	4 8	9 7	6 1	10 0	2 8	19	059
9 9 N (8)	50 0	5 0	13·1	6 4	13 9	3 0	21	048
18 4 N (2)	107 5	5 8	15 7	6 7	19 1	5 0	·13	041
19 2 S (1)	113 7	5 9	13 9	6 6	16 7	4 8	11	037
25 6 S (3)	148 5	5 8	16 8	6 5	24 4	5 8	12	035
36 8 S (4)	218 1	5 9	15 2	6 3	20 6	7 2	·06	031
66 9 S (5)	368 8	5 5	10 6	5 7	15 0	4 4	029	012
100 7 N (6)	..	.		.	8 4	1·4	015*	003*
126 6 N (7)	..	..	.		4 0	— 0 7	009*	— 001*

* The values employed here for $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are derived from subsequent tables

TABLE II

$\mathfrak{H}$	$\mathfrak{I}_1'$	$\mathfrak{I}_2'$	First "on"	Cyclic "on"—"off"	$\mathfrak{I}_1/\mathfrak{I}_2$	$\frac{\text{Shock-effect}}{\mathfrak{I}_1'}$	$\frac{\text{Cyclic "on"—"off"}}{\mathfrak{I}_2}$
1·15 N (12)	3·40	2·17	− ·62	+ ·46	·48	− ·32	·21
1·44 N (15)	2·62	1·23	·62	+ ·51	·37	·43	·41
1·61 S (14)	4·32	N	4·32	..	·74	1·0	
1·9 N (11)	5·9	3·4	1·1	+ ·42	·50	·26	·124
4·3 N (10)	8·6	4·9	2·0	+ ·49	·33	·29	·100
7·8 N (9)	13·1	6·9	3·7	+ ·54	·28	·32	·078
9·9 N (8)	16·4	9·0	4·6	+ ·62	·26	·32	·069
18·4 N (2)	27·9	15·7	8·6	+ ·26	·23	·32	·017
19·2 S (1)	29·9	19·0	5·4	+ ·15	·23	·19	·008
25·6 S (3)	39·4	25·3	9·5	+ ·15	·24	·25	·006
36·8 S (4)	51·0	34·6	12·9	− ·22	·22	·25	− ·006
66·9 S (5)	66·1	40·4	17·2	− 1·07	·18	·24	− ·027
100·7 N (6)	68·5	45·6	16·9	− 1·20	·14	·23	− ·026
126·6 N (7)	70·9	48·6	17·8	− 1·38	·13	·23	− ·028

§ 25 In Table I. $\mathfrak{H}$ is the strength of the field calculated as is explained in § 19, $\mathfrak{I}_1$ is the intensity and κ_1 the coefficient of induced magnetisation in the rod previous to the application of any pressures, while $\mathfrak{I}_2$ is the intensity and κ_2 the coefficient of induced magnetisation in the rod free from pressure after all the pressures cycles have been applied Thus $\mathfrak{I}_2 - \mathfrak{I}_1$ is the increase in the magnetisation taking place during the application of the pressure cycles Their application occupied a considerable time, during which absolute steadiness in the current could hardly be expected, and so $\mathfrak{I}_2 - \mathfrak{I}_1$ is probably in no case an absolutely exact measure of the increase in the magnetisation due to the pressure cycles By *first "on"* is meant the increase in the magnetisation caused by the first application of pressure *Cyclic "on" — "off"* gives the average algebraic excess of the magnetisation existing when pressure is "on" over that existing when pressure is "off." As already explained the *shock-effect* is the algebraic excess of the *first "on"* over the *cyclic "on" — "off"*

The second last column gives the ratio of the shock-effect to the intensity of the magnetisation prior to pressure, while the last column gives the ratio of the cyclic "on" — "off" to the intensity of magnetisation in the cyclic state. Thus the figures in these two columns may fairly be regarded as measuring the importance in the different fields of the shock-effect and the cyclic effect respectively.

In the weakest field the last three columns have no entries, because the experiments determining the cyclic effect of pressure failed to indicate a clear result.

The entries in the two last fields indicate, as they do elsewhere, that no observations of the corresponding quantities were obtained. In these fields when the compensating coil exactly balanced the coil current so that in the absence of the rod the spot was at the centre of the scale, the introduction of the rod drove the spot off the scale The compensating coil was thus moved so as to neutralise part of the effect

of the rod, and so bring the spot near the centre of the scale In the absence of the rod the spot was now off or nearly off the scale, so that accurate measures of the induced magnetisation were impossible

The effect of the changed position of the compensating coil was allowed for in reducing the observations A similar use of the compensating coil was subsequently made in some of the higher fields It had the disadvantage of making the scale reading more disturbed by slight irregularities in the current In these strong fields, however, the cyclic changes were so small compared to the total induced magnetisation that the alternative of increasing the distances of the coils from the magnetometer possessed greater disadvantages

§ 26 In Table II, $\mathfrak{H}$ is the field existing prior to the break of the current. $\mathfrak{I}_1'$ is the residual magnetisation prior to the application of any pressure, $\mathfrak{I}_2'$ that finally existing after what was approximately a constant number of pressure cycles A minus sign represents, as elsewhere, a diminution in the magnetisation, and when placed at the head of a column it applies to all the entries in that column The *shock-effect* is obtained by subtracting from the algebraic value of the total effect of the first pressure the algebraic value of the *cyclic* "*on*" — "*off*"

The second last column gives the ratio of the *shock-effect* to the intensity of the residual magnetisation prior to pressure, while the last column gives the ratio of the *cyclic* "*on*" — "*off*" to the intensity of the residual magnetisation in the cyclic state

§ 27. In both the Tables I and II the letter N signifies that a certain end of the rod, which we shall call A, was a north pole ; S, that it was a south pole The numbers in brackets in the first columns give the order in which the experiments took place

It was intended to apply as nearly as possible the same degree of pressure in every single case Still, as the pressure was applied by hand, a certain amount of irregularity was bound to exist The smallness in the variations shown by the individual observations of the cyclic effect in each separate field, and the smoothness of the several curves obtained prove that throughout any single series of experiments a pretty uniform standard must have been maintained As a considerable interval of time elapsed between some of the successive series of experiments, notably those made in December and January, the standard probably varied somewhat from one series to another ; fig. 7, at all events suggests that in February the standard pressure was somewhat less than in December.

§ 28. The results of Table I and II will be presently discussed along with the corresponding results from subsequent tables, but certain peculiarities illustrative of the first difficulty stated in § 22, claim a special attention

A glance at the values of κ_1 and κ_2 in Table I., shows that the supposed demagnetisation following the strong field (7) had not sufficed to remove all its effects A comparison of the fields (14) and (15), or of (12) and (13), shows a much smaller susceptibility for currents making A a south pole, than for those making it as in (7) a north pole. From the results subsequently obtained with a more perfect system of

demagnetisation, and contained in Tables IX and X., it appears that in those of the eight weakest fields of Table I, in which A was a north pole, the values obtained for κ_1, κ_2, and the cyclic effect are all undoubtedly somewhat too great

The effects of the residual polarity appear still more decidedly in Table II in the case of field (14) Here, what must be regarded as an abnormally large amount of residual magnetism, was originally present Every trace of this was, however, removed by a single pressure, and with subsequent pressures magnetism of opposite sign became apparent An even more striking case, that of field (13), does not appear in Table II because no absolute measures were taken. In this case, the residual magnetism on the break even of the current was of opposite sign to the induced, and it increased on the application of pressure

Second Series of Experiments.

§ 29 In the next principal series of experiments—taken in January, 1889—the method of taking the observations was much the same as in the first series, but the distances between the pieces of the apparatus were varied Thus the distance of the centre of the rod from the magnetometer needle was 73 6 cm, 50 cm, or only 39·2 cm, according as the quantity under examination was the intensity of the induced magnetisation, the effect of pressure on the induced magnetisation, or the residual magnetisation The compensating coil and the scale remained screwed to the beam carrying the apparatus, while the three positions of the magnetising coil and the corresponding positions of the magnetometer were indicated by pencil marks on the beam There was no shaking of the rod, or variation in the current caused by the movements of the coil and magnetometer

The fields in this series of experiments varied from 84·8 to 725 C G S. units. Preliminary experiments in the weakest of these fields, showed that demagnetising the rod had no appreciable influence on the magnitude of the cyclic effect of pressure There was thus in none of the recorded experiments, as given in Tables III. and IV, any attempt at demagnetisation, but the weaker fields were taken first so as to avoid the disturbing influence of any residual effects that might originate with very strong currents

TABLE III

$\mathfrak{H}$	$\mathfrak{I}_1$	κ_1	$\mathfrak{I}_2 - \mathfrak{I}_1$	First "on"	Cyclic "on"—"off"	$\frac{\text{Shock-effect}}{\mathfrak{I}_1}$	$\frac{\text{Cyclic "on"—"off"}}{\mathfrak{I}_2}$
84 8	409	4 8	7 4	+ 9 2	+ 2 2	+ 017	+ 005
106 4	472	4 4	4 6	+ 5 0	+ 0 7	+ 009	+ 001
126 6	526	4 1	..				
155 *m*	554	3 6	2 3	− 1 2	− 2 1	+ 002	− 004
181	582	3 2		− 1 8	− 2 4	+ 001	− 004
210	611	2 9	3 1		− 3 2	.	− 005
270 *m*	629	2 3	1 3	− 2 4	− 3 6	+ 002	− 006
282	624	2 2				.	
293	613	2 1	1 1	− 2 4	− 2 9	+ 001	− 005
339	654	1 9	.	.			
357	669	1 9		− 4 5	− 3 2	− 002	− 005
406 *m*	690	1 7	7 9	− 2 8	− 3 8	+ 001	− 005
541	707	1 3	17 0	− 1 5	− 2 3	+ 001	− 003
633	777	1 2	34	− 5 2	− 3 4	− 002	− 004
725	791	1 1	$\begin{Bmatrix} 22\ 9 \\ 1\ 3 \end{Bmatrix}$		− 2 2	..	− 003

TABLE IV.

Number of fields	Limiting fields	Average first "on"	Average cyclic "on"–"off"	Average shock-effect — $\mathfrak{I}_1'$
7	85–210	— 16 4	— 1 14	— 21
8	250–355	17 4	1 09	21
4	400–725	14 6	80	19

§ 30. In Table III., which gives the observations on the induced magnetism, the headings have the same meanings as in Table I. When *m* is attached to the strength of the field, two or three separate experiments were conducted with fields of approximately this strength, and the results given in the table are the averages of those obtained. The blanks are mainly due to unsteadiness in the scale-readings brought about by fluctuations in the strength of the magnetising current.

In no case probably, as already explained, does the column headed $\mathfrak{I}_2 - \mathfrak{I}_1$ give with perfect accuracy the change in the magnetisation produced by the pressure cycles alone, and in fields over 400 units the changes under this head must mainly be due to a totally different cause, viz., the heating of the rod In these high fields the scale-reading did not remain stationary in the absence of pressure, but showed a progressive increase of magnetisation. To obtain the cyclic effect, the pressures were applied and removed at as nearly uniform intervals as possible, and the changes in the magnetisation occurring in successive intervals were compared.

In the strongest field, after the application of a large number of pressure cycles,

during which an increase of 22·9 units of magnetisation appeared, the current was broken, and after a short interval re-made, before the rod had cooled much. During a second almost equally numerous series of pressure cycles an increase of only 1·3 unit of magnetisation appeared. At the commencement of the second part of the experiment most of the coil was doubtless colder than the contained rod, so that the temperature of the rod would not alter much for some time after the re-make of the current. This last experiment indicates, I think conclusively, that in fields of this strength the permanent change in the magnetisation brought about by pressure cycles must be small, whatever its sign may be.

The values given for the *First "on"* in the strongest fields cannot claim great accuracy. The rod began to get heated before the oscillations of the magnetometer needle following the introduction of the rod into the coil had sufficiently subsided to permit of a reading being taken. The progressive increase of magnetisation was, in fact, going on most rapidly when the first pressure was applied. There had thus occurred a considerable progressive change before the next scale-reading could be taken, so that it was very difficult to deduce the true effect of the first pressure.

The values obtained in these high fields for the intensity of the induced magnetisation must also have been to some extent unduly raised by the heating, so that the curve of fig. 1 is, in its higher portions, not so flat as it ought to be.

§ 31. Table IV. gives merely certain average results, calculated from the large number of observations actually made on the residual magnetisation. On comparing the results after the conclusion of the experiments, it became apparent that the magnitudes of the several phenomena varied so little with the strength of the pre-existing field within the limits of the observations, that trustworthy deductions as to the exact modes of their variation were rendered impossible by the small variations in the distances of the pieces of the apparatus which the method of experiment was sure to introduce. The measurements of the effects of pressure are, in addition, exposed to possible irregularities in the magnitude of the pressure, but this ought not to affect sensibly the average results recorded in the Table.

Third Series of Experiments.

§ 32. In the next series of experiments, extending from January to February, the residual magnetisation alone was under investigation. Every single piece of apparatus was bolted or screwed to the supporting beam, so as to prevent any relative movement. The magnetometer needle was distant 82 cm. from the scale and 38 cm. from the middle of the rod.

First of all a set of observations were taken of the initial amount of the residual magnetisation as the strength of the pre-existing field was raised by steps from 1·7 to 79·4 C.G.S. units. The rod was apparently very nearly demagnetised by reversed currents previous to the first experiment, but no subsequent demagnetisation was

performed No pressure or tapping was applied to the rod throughout these observations The current was always in the direction making the end A the north pole

The results, as recorded in Table V, are deduced from the scale readings taken (1) with the rod in its place inside the coil immediately after the break of the current, (2) with the rod removed to a distance In general, two observations were taken for each strength of field The mean of the two observations is the $\mathfrak{I}_1'$ of the Table By $\mathfrak{H}$ is meant the strength of the field due to the current just broken, and by κ_1' the ratio of $\mathfrak{I}_1'$ to $\mathfrak{H}$ The mode of variation of $\mathfrak{I}_1'$ with $\mathfrak{H}$ is shown in curve a, fig 10, while the mode of variation of κ_1', which is subsequently termed the *residual susceptibility*, is shown in curve a, fig 11

Table V

$\mathfrak{H}$	$\mathfrak{I}_1'$	κ_1'	$\mathfrak{H}$	$\mathfrak{I}_1'$	κ_1'
1·7	22	13	14·1	12·9	91
2·6	29	11	16·1	15·8	98
3·5	56	16	18·0	21·4	1·20
4·4	95	22	22·3	27·5	1·23
5·0	1·3	25	25·0	28·5	1·14
5·6	2·0	36	27·3	31·5	1·15
6·0	2·4	40	38·0	43·0	1·13
7·0	3·8	55	45·4	50·1	1·10
9·4	6·6	70	54·6	56·4	1·03
11·5	9·1	79	70·4	61·9	88
12·2	10·2	84	79·4	63·8	80

§ 33. In continuation of these experiments a much more elaborate set of observations were taken, with the object of determining whether the amount of the residual magnetisation or its properties depended on the treatment of the rod during the flow or break of the current. In one set of experiments, spoken of in future as the L type, no pressure at all was applied during the flow of the current. In a second set, the M type, six cycles of pressure "on" and "off" were applied, the current being broken when pressure was "off" In a third set, the N type, a pressure was applied after six pressure cycles, the current being broken when pressure was "on." In these, as in all the other experiments, the rod was introduced into the coil after the scale reading showed that the current had become steady The experiments of the several types were not conducted separately, but on the following plan Starting with a certain number of storage cells and a certain resistance in circuit, complete observations were made for the corresponding field with each of the three types in the order L, M, N, say Then with a greater number of cells, or a reduced resistance, observations were taken for the next higher field, in the order M, N, L, and so on, in cyclic order Care in fact was taken that the experiments of any given type should not unduly often be either the first experiment of the day or the first

experiment with a given strength of field This variation in the order was adopted lest during the application for several hours of a succession of pressure cycles, interrupted only by the demagnetisations, the rod might become less responsive to pressure, developing a sensible amount of what may be termed *fatigue* Under the actual conditions, if this did happen, it would not affect the experiments of one type more than those of another

The strength of the pre-existing fields was raised step by step from 1 to 400 C G S units, and until fields of 350 units, or thereby were reached, the rod was demagnetised, before each single experiment, by reversed currents and vigorous tapping In the stronger fields the rod was merely exposed to a preliminary reverse current, of the same strength as that about to be used in the experiment During the actual observations the end A was invariably the north pole

In each experiment of each of the three types, with the exception of some of the weakest fields, the following operations were conducted When the spot on the scale had become stationary, after the break of the current, a reading was taken. The pin was then withdrawn and the rod carefully removed, being kept at right angles to the magnetic meridian. The scale reading when stationary having been observed, the rod was gently restored to its place and the pin secured With a little practice this operation was accomplished without shaking out any sensible amount of magnetism, as the coincidence of the scale readings before the rod's removal and after its restoration sufficiently testified Pressure was then applied once and removed in the experiments of types L and M, being simply removed in those of type N, and the corresponding scale readings observed. Twelve pressures cycles were then applied, the effects of the last six only being observed The rod was then removed and the constancy of the magnetometer zero tested

The various observations obviously supplied sufficient data for calculating the *initial residual* magnetisation, the *first "on"* or *first "off"* of pressure, the *cyclic effect* of pressure, and the *final residual* magnetisation, *i e.*, the residual surviving a definite number—13 for types L and M—of pressure cycles

§ 34. The results of the experiments of type L are given in Table VI All the headings have been already explained, except *next 'off'* This means the change in the magnetisation accompanying the removal of the first pressure applied after the break of the current In the two weakest fields the amount of the initial residual magnetisation alone was observed, and in the three next weakest fields the effects of only one pressure cycle were taken Thus no data existed for calculating the cyclic effect in these fields, and the first three results given in the last column are unduly large, because the values there assigned to $\mathfrak{I}_2'$ are the magnetisations existing after the application of only a single pressure cycle

In the fields over 90 C G S. units the changes in the properties of the residual magnetism accompanying the rise in the fields are so small that the small irregularities in the magnitudes of the pressures applied become important. Thus the

stronger fields are grouped into two sets, and the average effects for each set alone are given.

In like manner the results of the experiments of type M are given in Table VII., and those of type N in Table VIII In these, too, the cyclic effects of pressure were unfortunately not observed in the weakest fields In the six weakest fields of Table VII the value assigned to $\mathfrak{I}_2'$ is the residual magnetisation when only one pressure cycle has been applied, while in the seven weakest fields of Table VIII the value assigned it is the residual left on the removal of the pressure existing during the break of the current. The corresponding results in the last columns of these tables are consequently all unduly large. In the strongest fields average results alone are given as in Table VI

The results of Tables VI, VII, and VIII. supply the data from which the curves *b*, *c*, *d*, respectively, of figs 9 to 15 (Plate 16) are drawn

Table VI.

$\mathfrak{H}$	$\mathfrak{I}_1'$	κ_1'	First "on"	Next "off"	Cyclic "on"—"off"	Ratio of Shock-effect of first "on" to $\mathfrak{I}_1'$	Ratio of Shock-effect of next "off" to Shock-effect of first "on"	$\mathfrak{I}_2'-\mathfrak{I}_1'$
2·6	1·26	48	.	.	..			.
4·2	2·43	58		..	.			
5·6	5·0	89	−3·0	−15			05	37
8·9	8·4	95	4·7	29			06	40
11·5	10·7	93	5·8	29			05*	43*
18·0	24·4	1·36	8·5	95	0	35*	11*	57*
24·0	32·7	1·36	10·7	87	32	32	11	61
35·0	44·2	1·26	13·7	94	62	28	12	61
46·7	49·7	1·06	13·5	1·00	49	26	11	65
59·0	57·3	97	14·3	86	·57	24	10	67
74·8	61·7	82	14·4	1·22	·34	·23	11	71
93·2	64·9	·70						
109	65·7	60						
122	66·3	54						
133	68·2	51	15·2	85	53	22	09	
147	68·0	47						
173	68·5	40						
207	68·7	·33						
242	70·2	29						
292	70·5	24						
353	70·6	·20	14·1	69	49	19	08	75
385	71·4	·18						
414	71·1	17						

* In the entries above this point, in absence of experimental data, the *cyclic* "*on*"—"*off*" is neglected Thus the results above this point are not strictly comparable with those below

TABLE VII

$\mathfrak{H}$	$\mathfrak{J}_1'$	k_1'	First "on"	Next "off"	Cyclic "on"—"off"	Ratio of Shock-effect of first "on" to $\mathfrak{J}_1'$	Ratio of Shock-effect of next "off" to Shock-effect of first "on"	$\mathfrak{J}_2'-\mathfrak{J}_1'$
0 95	2 32	2 44	− 1 2	− 15		− 51	13	43
1 27	2 65	2 09	1 0	44		39	43	44
1 84	3 53	1 92	1 5	46		42	33	46
2 7	3 9	1 44	1 9	29		49	15	43
6 0	9 0	1 50	4 2	77		46	18	45
8 9	13 8	1 51	5 2	1 06		39 *	20 *	54 *
11 6	19 4	1 68	6 4	1 86	+ 53	36	19	48
18 0	30 2	1 68	7 9	2 04	+ 47	28	20	61
22 8	36 1	1 58	8 8	1 63	+ 21	25	16	62
34 9	50 4	1 45	12 8	1 72	− 27	25	16	61
46 7	53 8	1 15	12 2	1 65	− 14	23	15	67
57 5	59 7	1 04	14 2	1 27	− 32	23	11	69
76 2	65 4	86	14 3	1 12	− 54	21	12	68
94 2	66 1	70						
109	66 4	61						
121	66 9	55						
134	68 2	51	14 0	97	− 56	20	11	73
147	68 7	47						
173	68 7	40						
207	68 7	33						
242	70 2	29						
301	70 2	23	12 2	79	− 48	17	11	77
350	70 6	20						
397	70 6	18						

* In the entries above this point, in absence of experimental data, the *cyclic "on" — "off"* is neglected Thus the results above this point are not strictly comparable with those below

TABLE VIII

$\mathfrak{H}$	$\mathfrak{I}_1'$	κ_1'	First "off"	Cyclic "on" — "off"	Shock-effect of first "off" / $\mathfrak{I}_1'$	$\mathfrak{I}_2' - \mathfrak{I}_1'$
98	2·05	2·09	− 1·17		− 57	43
1·61	3·52	2·19	2·27		65	35
1·96	4·26	2·17	2·64		62	41
2·65	4·84	1·83	3·01		62	39
3·8	6·2	1·64	3·5		57	43
6·1	10·1	1·66	5·6		55	45
8·7	13·9	1·59	6·3		45	53
11·5	20·2	1·76	8·8	+ 58	41*	47*
17·3	31·4	1·82	12·1	69	36	52
22·9	37·6	1·64	11·3	54	29	62
36·0	49·4	1·38	11·4	35	23	68
46·7	53·3	1·14	12·5	36	23	69
57·5	58·1	1·01	10·6	09	18	69
79·1	61·3	76	12·4	20	20	73
93·6	63·6	68				
111	64·1	58				
124	65·3	·53				
134	65·3	49	11·6	17	18	75
147	65·6	45				
180	66·2	37				
213	66·9	32				
224	66·8	30				
299	67·4	23				
348	67·9	19	11·9	20	18	77
400	67·8	17				

Fourth Series of Experiments

§ 35 In the last series of experiments, which extended into March, the effects of pressure on the induced magnetism were more particularly under investigation Except on one or two occasions, the magnetometer was distant 50 cms from the centre of the magnetising coil, and 81 cms from the scale Previous to each experiment the rod was demagnetised by reversed currents followed by vigorous tapping

After the introduction of the rod into the coil, fourteen pressure cycles were in general applied, readings being taken of the effects of the first four and the last four cycles. From these the effect of the first pressure and the cyclic effect are calculated Then, leaving the current untouched, the rod was usually removed to a distance and replaced, scale-readings being taken The comparison of the reading taken when the rod was out with that taken prior to its original introduction into the coil, tested the constancy of the magnetometer zero From the readings taken before and after the rod's removal, we find the magnetisation $\mathfrak{I}_2$ of the rod at the end of the pressure cycles; while from the readings taken before and after the rod's re-introduction we find the magnetisation $\mathfrak{I}_3$ on the second introduction of the rod into the coil Thus, $\mathfrak{I}_2 - \mathfrak{I}_1$ is, as in former experiments, the increase in the magnetisation during the application of the pressure cycles, while $\mathfrak{I}_2 - \mathfrak{I}_3$ is the magnetisation lost during the removal of the rod from the coil and its re-introduction

* In the entries above this point, in absence of experimental data, the *cyclic* "*on*" — "*off*" is neglected. Thus the results above this point are not strictly comparable with those below

The compensating coil was intentionally placed so as to somewhat more than balance the magnetising coil, and so, in some of the strongest fields, the spot was off the scale when the rod was removed Thus, in these fields, there were no data for calculating $\mathfrak{I}_1$, and in calculating $\mathfrak{I}_2 - \mathfrak{I}_1$ and $\mathfrak{I}_2 - \mathfrak{I}_3$ it had to be assumed that the magnetometer zero had not altered in the intervals that elapsed during the application of the pressure cycles and during the absence of the rod from the coil The latter interval was always short, so that any sensible alteration during it was improbable During these observations the end A of the rod was always the north pole

There then followed in general, while the current remained unchanged, some observations intended to test the residual effects of the pressure cycles In these the rod was repeatedly removed from the coil, it might be when free from or when subjected to pressure, and while out its state, as regards pressure, might or might not be changed Not infrequently, too, the ends were changed while it was out, so that on its re-introduction the end A might be either a north or a south pole Sometimes on the re-introduction of the rod *first* "*ons*" or *first* "*offs*" of pressure were taken, with the object of determining whether their magnitude was affected by repetition or by a change in the end of the rod. The results of this last series of experiments are contained in Tables IX and X They are the results on which most of the curves in figs 1–8 (Plate 15) are based

TABLE IX

$\mathfrak{H}$	$\mathfrak{I}_1$	κ_1	$\mathfrak{I}_2-\mathfrak{I}_1$	κ_2	$\mathfrak{I}_2-\mathfrak{I}_3$	$\mathfrak{I}_A$ "on"—"off"	$\mathfrak{I}_B$ "on"—"off"	"Off" $\mathfrak{I}_A-\mathfrak{I}_B$	"On" $\mathfrak{I}_A-\mathfrak{I}_B$
1 8	4 1	2 3	2 4	3 6	.		1 4	.	
2 2 *p*	8 2	3 8	1 9	4 7	1 4			2	
*3 4	11 0	3 2	3 4	5 0	2 7		.	3 8	.
4 3 *p*	20 2	4 7	3	4 8	2 7	1 7	.		
8 1 *p*	41 5	5 1	5 0	5 8	5 2	2 4	.	9 9	.
8 6	37 3	4 3	10 2	5 5	5 1			9 7	
*12 5	61 7	4 6	5 1	6 2	..	.			10 3
23 4 *p*	147 1	6 3	3 4	6 4	13 5				
23 7	141 1	6 0	13 2	6 5	11 6			5 3	.
35 2 *p*	253 8	7 2	6 2	7 4	16 5		.		
35 7	233 4	6 5	13 3	6 9	10 4			2 9	
46 0	290 0	6 3	12 3	6 6	12 0	9 0	.		
47 5 *p*	312 3	6 6	4 7	6 7	9 2		.	5	
57 5	339 5	5 9	10 6	6 1	9 5	.			.
57 5 *p*	375 7	6 5	−4 1	6 5	11 6	8 1		5	
72 6	397 3	5 5	6 9	5 6	10 5	7 4	..	6	
79 8	405 8	5 1	8 6	5 2	10·2	6 3		1 9	.
92 0	427 0	4 6	8 1	4 7	8 4	7 3		6	
103 5	491 5	4 7	6 1	4 8	4 5	4 2		3 0	
115	510 4	4 4	3 8	4 5	4 7	1 8	2 1	2 2	.
128	530	4 1	2 4	4 2	4 4	2 1		1 8	.
144	550	3·8	3 0	3 8	..	1 4	..	.	..
158	554	3 5	2 8	3 5	2 8	5		.	.
174	..	..	1 0	.	1 1	−0 6	.	0	..
198	..	.	1 6	.	1 6	−1 6		−0 4	−1 1
224			..	..	1 4	−1 6	.	−0 7	.
270	..	.	1 2	.	0 2	−4 1	−2 2	−4 4	−5 0
351	.	..				−4 1	−3 1	−5 1	−5 6

* Current fell markedly during pressure cycles κ_1 is calculated on its initial, κ_2 on its final value.

TABLE X.

$\mathfrak{H}$.	First "on."			First "off."			Cyclic "on" — "off"	First "on" Shock-effect — $\mathfrak{I}_1$		First "off" Shock-effect — $\mathfrak{I}_1$		Cyclic "on" — "off" $\mathfrak{I}_2$
	A_1.	A_2	B_2	A_1.	A_2	B_2		A_1	A_2	A_1	A_2	
1·8	2 4						·29	51				044
2 2 *p*				1 3			34			2		034
3 4	4 8							34				035
4·3 *p*				1			72			04		036
8·1 *p*		5 8	7·0	2 3			1·7		10	098		026
8 6	10·1						1 2	24				027
12 5	9·9				— 3	1·5	1 8	13			024	017
23 4 *p*			16 1	3			2·6			·02		016
23·7	16 3		17·3				2 5	09				017
35·2 *p*				— 2·6			4 4			007		021
35·7	16 6						5 3	048				015
46·0	17 0	13 8			— 1 0		4 6	043	032		012	011
47 5 *p*		12 4	12 7	— 1 4			3 6		028	007		012
57 5	16 5	12·9	14·8		— 1 6		4·1	036	026		007	009
57 5 *p*		10 0	12 5	— 2 3	— 1 2		3 3		018	003	005	006
72 6	10 9	10 5	10 5		— 1 2		2 4	022	020		003	006
79 9	11 7	10 5			— 1 1		2 4	023	020		003	005
92 0	11·0	10·0			— 8		2 1	021	018		003	001
103 5	5 2	5 5	4·7		+ 7		7	009	009		003	001
115 0	4 7				+ 1 1		45	008			003	— 001
128	2 6	2 1	1 8		+ 1 1		— 51	006	005		001	— 002
144	1 2	0·9	0 9		+ 1 6		— 9	004	003		001	— 002
158	0 9	0 6			+ 2·0		— 1 1	004	003		002	— 0025
174	— 1 4	— 0 4			+ 2 1		— 1 4	0	002		001	— 003
198	— 2 4	— 0 5	— 0 3		+ 2 8		— 1 7	— 001	002		002	— 003
224	— 1 3		— 1 1		+ 2 6		— 1 6	001			002	— 004
270	— 2 7	— 1 8	— 2 2		+ 3 0	+ 3 0	— 2 2	— 001	001		001	— 004
351		— 2 7	— 2 4		+ 3 4	+ 3 4	— 2 7		0		001	

§ 36 In Table IX the first six headings have been already explained. The letter p after the numerical measure of the field in the first column signifies that the rod was under pressure previous to and during its introduction into the coil. In such a case the entries under $\mathfrak{I}_1$ and κ_1 refer to the magnetisation existing before this pressure was removed, and so are on a different footing from the other entries under these headings. The entries under the other headings are also in such a case doubtless indirectly affected, but to a much smaller degree.

In the last four headings the suffix indicates which end of the rod was the north pole when the magnetisation in question was measured. The results under these headings are deduced from the readings obtained by removing and replacing the rod subsequent to the application of the pressure cycles, during which application it must be remembered A was always the north pole.

The seventh column gives the excess of the magnetisation existing when the rod was under pressure when reintroduced into the coil over that existing when it was free from pressure when reintroduced, the end A being in each case the north pole of the reintroduced rod. The eighth column differs from the seventh only in that the end B was in each case the north pole of the reintroduced rod. The ninth column gives the excess of the magnetisation found when A was the north pole of the reintroduced rod over that found when B was the north pole, the rod in each of its reintroductions being free from pressure. The tenth column differs from the ninth only in that the rod during each of its reintroductions was under pressure.

§ 37. In Table X *First "on"* is, as previously, the total change in the magnetisation—cyclic and non-cyclic—due to the first application of pressure. A_1 gives the effect when the rod, immediately after being demagnetised, has been exposed for the first time to the field in question. A_2 gives the effect of the first pressure applied after the *reintroduction* of the rod in a state free from pressure, and with the end A a north pole. In the case of the results under A_2, there have intervened since demagnetisation a complete series of pressure cycles applied with A a north pole, and at least one removal and reintroduction of the rod. In some cases there were several removals and reintroductions, and the mean of the observations is recorded. Under B_2 we get the effect when, on its reintroduction, the rod has B for its north pole. In this case it must be borne in mind that the pressure cycles applied since the preceding demagnetisation occurred when the opposite end A was the north pole.

First "off" is the total effect of the removal of a pressure existing previous to and during the introduction of the rod into the coil. A_1 refers to the case when demagnetisation has immediately preceded the observation, A_2 and B_2 to the cases when a complete series of pressure cycles, with A the north pole, and at least one removal and reintroduction have intervened since demagnetisation. The distinction between A and B is the same as in the case of the *First "on"*.

Cyclic "on"—"off" has its usual meaning. Under the next two headings are given the non-cyclic portions of the effects of the first application of pressure and of the first

removal of a pre existing pressure. A_1 and A_2 refer to the results recorded in the first two columns under the headings *First "on"* and *First "off."* The last heading has been already explained. As usual . indicates that no observations were taken. When no algebraic sign is attached to a figure + is understood.

§ 38. To a casual observer the fluctuations that appear in the results of the foregoing tables may seem excessive. It should, however, be borne in mind that most of the pressure effects are very small compared to the total amount of the induced magnetisation—sometimes as little as one-thousandth part. Thus the small irregularities in the magnetising current which are produced by the slightest want of constancy in the cells, or by the vibrations communicated to the resistance wires from shakings of the floor, may produce commensurable effects. Such irregularities were frequently recognised through a continual quivering motion of the spot on the scale, while the ammeter reading appeared steady enough. Of course, when the unsteadiness was very marked, observations were suspended, but comparatively little of the work done would have been accomplished in the time if observations had been taken only when absolute steadiness prevailed.

§ 39. A separate discussion of the results of each table would involve a good deal of repetition. It has thus appeared best, as a rule, to embody the most trustworthy results in curves, and to discuss particular points in connection with the features of the individual curves. In every curve the horizontal coordinate gives the strength of the field, $\mathfrak{H}$, calculated after the manner explained in § 19. In the case of the residual magnetism, $\mathfrak{H}$ is of course the strength of the pre-existing field.

In fig. 1 (Plate 15), the ordinates give the initial magnetisation when the rod is introduced into the magnetising coil free from pressure. In fig. 3 the ordinates give the coefficient of induced magnetisation under the same conditions. The first portions of both curves, in which the individual observations are indicated by dots, are based on Table IX., while the second portions, in which the individual observations are indicated by crosses are based on Table III.

These curves are of the same general character as those obtained by many observers for iron. In fields below 8 or 9 C.G.S. units, the magnetisation increases comparatively slowly as the strength of the field is raised. There then ensues, as is most clearly shown by the steepness of the commencing portion of the curve of fig. 3, a much more rapid increase of magnetisation. The rate of increase attains a maximum, as shown in fig. 3, in a field of about 35 C.G.S. units; which, accordingly, is the *Wendepunkt* for the specimen. The rate of increase of the magnetisation then falls off, but at first in a comparatively gradual manner. There is thus no very clear indication of an approach to "saturation" until the strength of the field approaches 200 C.G.S. units. Even in the strongest experimental fields the rate of increase is by no means infinitesimal, though, as already stated, this is probably in part accounted for by the heating of the rod.

§ 40. The smallness of the scale of fig. 1, does not allow the effect of pressure on

the induced magnetisation to be shown, but the enlarged scale of fig 2 shows the effect in fields below 30 C G S units, where it is of most importance

The thick line *a*, with the individual observations indicated by dots, is merely the initial portion of fig 1 on an enlarged scale The dotted line *b*, with the individual observations indicated by circles, gives the induced magnetisation after the application of the pressure cycles. The curves, *a* and *b*, are based on the same experiments, and the difference between their ordinates answers to the values in the column $\mathfrak{J}_2 - \mathfrak{J}_1$ of Table IX in those experiments in which the rod was originally free from pressure

The third curve *c*, with the individual observations indicated by crosses, gives the induced magnetisation prior to the application of pressure cycles, in these experiments denoted by a *p* in Table IX, in which the rod was under pressure when introduced into the coil

In fig 4 the ordinates give the coefficients of magnetic induction in the three cases of fig 2 The data are taken from the same table, and the letters, &c, have the same significations

The difference between the curves *a* and *b* shows the very large effect of the pressure cycles in increasing the magnetisation in the weaker fields It should be remembered that between the corresponding pairs of observations on which these curves are based there intervened no break of the current, nor any change in the resistance, or in the position of the ammeter Thus any error in the zero or orientation of the ammeter would affect each curve alike, and leave the difference of the ordinates practically unaffected. Variations, it is true, in the strength of the current during the application of pressure cycles, would affect curve *b* without affecting curve *a* Such variations, however, could escape notice in the weaker fields only when very small, because the ammeter was then in its most sensitive position

The comparison of the curves *c* with the others is not so satisfactory During the time of most of the experiments the ammeter was being used for other purposes, and so had to be set up afresh every other day Thus, though its position was carefully adjusted, small differences were certain to occur

For these and other reasons it would be unsafe to draw any conclusions from the crossing of the curves *b* and *c* in fig 4 It may, however, be regarded as perfectly certain that in the weaker fields the curve *c* lies between the curves *a* and *b*, and that within the limits of fig. 4 the curve *c* lies very distinctly above the curve *a*

The form of all these curves of fig 4, suggests that under all conditions as to pressure the coefficient of magnetic induction becomes extremely small in very weak fields. There may, however, be a turning point in one or all of the curves in fields lower than those experimented on

§ 41. In curve *a* of fig. 5 the ordinates give the total change in the magnetisation produced by the first pressure in those experiments of Table IX. in which the rod was free from pressure when introduced into the coil. The change in magnetisation is measured with the pressure "on," and so is really the algebraic sum of the cyclic and

non-cyclic effects The individual observations are indicated by dots, whose distances from the curve are in no case serious

The curve shows that the total effect of the first pressure attains a distinct maximum in a field of about 45 C G S units—which somewhat exceeds the Wendepunkt—and then diminishes somewhat rapidly as the strength of the field is raised. An unmistakeable critical field, where the effect vanishes, appears at about 160 C G S units, and in stronger fields the total effect of the first pressure is a diminution of magnetisation From the form of the curve it would appear that the diminution does not increase indefinitely as the strength of the field is raised But the experiments leave it uncertain whether this diminution of magnetisation attains a maximum in a field of about 350 C G S. units, or whether it continually approaches an asymptotic value.

§ 42 In curve *b* of fig. 5, the ordinates give the loss in the induced magnetisation accompanying the removal of the rod from the coil, after the application of the pressure cycles, and its reintroduction The curve is based on the column $\mathfrak{J}_2 - \mathfrak{J}_3$ of Table IX.

The ordinate corresponds more or less closely to the non-cyclic effect of the sum of the pressure cycles applied during the flow of the current prior to the rod's removal There are two reasons however, why it cannot be regarded as an exact measure of this effect, more especially in the weaker fields In the first place, even when no pressures are applied it is well known that the magnetisation of iron is not exactly the same on the second exposure to a certain field as it is on the first Some occasional observations on the cobalt itself gave a somewhat greater magnetisation on a second exposure than on the first The difference was, however, unimportant, and in fields over the Wendepunkt it was, if existent, extremely small In the second place part of the effect of the pressure cycles unquestionably survives a removal of the rod from the magnetic field. As will be seen from § 65, this residual effect is of considerable importance in fields below 30 units. Owing to both these causes the effect of the pressure cycles in fields below 30 or 40 units must be decidedly greater than the curve *b* would indicate No observations exist to show whether or not the curve *b* eventually crosses the axis of abscissæ. Not improbably this would be a very difficult point to settle owing to the heating of the rod

The reason for drawing a curve for $\mathfrak{J}_2 - \mathfrak{J}_3$ in preference to one for $\mathfrak{J}_2 - \mathfrak{J}_1$, is that only a few minutes elapsed between the taking of the readings giving $\mathfrak{J}_2$ and $\mathfrak{J}_3$, while the reading which gives $\mathfrak{J}_1$ might have been taken over an hour previously. Thus under ordinary circumstances the variation of the current could not but be utterly insignificant in the one instance, whereas in the other it occasionally reached a measurable quantity

In Table IX. omitting of course the fields marked *p*, the values of $\mathfrak{J}_2 - \mathfrak{J}_1$ and $\mathfrak{J}_2 - \mathfrak{J}_3$ show a very fair agreement. In fields below the Wendepunkt the former quantity is, as the above reasoning suggests, distinctly the larger ; but in stronger

fields the one is sometimes the larger and sometimes the other In no single case was a negative value obtained for either quantity, though in fields over 200 C G S units, both become unquestionably very small

The only safe conclusion appears to be that if a critical field existed for the non-cyclic or shock-effect of the pressure cycles it exceeded 275 C G S units Considering, however, that the critical field for the cyclic effect of the pressure cycles is only about 120 C G S units, this seems an important result

§ 43 In the curve of fig. 6, which is based on Table X, the ordinates give the ratio of the increase in the induced magnetisation produced by the non-cyclic part of the first pressure relative to the amount of the induced magnetisation existing previous to the pressure The ordinates may thus be held to measure the relative importance in fields of various strengths of the non-cyclic portion of the effect of the first pressure

The curve shows in a striking manner how the relative importance of the non-cyclic portion of the effect of the first pressure continually diminishes as the strength of the field increases In the weakest field employed, viz, 1 8 C G S units, the first pressure permanently increases the magnetisation by one half its original value, whereas in a field of 160 C G.S. units the increase is certainly less than one part in two hundred

In stronger fields, as appears from the ninth and tenth columns of Table X, this shock-effect becomes extremely small, and continues to be so within the range of the experiments

The experiments cannot be said to settle conclusively the sign even of the effect, but the evidence is decidedly in favour of its remaining positive in fields of at least 270 C G S units It will, in fact, be observed that in the tenth column of Table X, a decrease of magnetisation was in no single case obtained Now in the higher fields the results in this column are calculated from the mean of two or three observations, whereas those in the preceding column answer, of course, to only one observation.

Effects of the Removal of Pressure

§ 44. It might appear at first sight that the increase in the magnetisation invariably found to accompany the application of the sum of the pressure cycles in fields up to 270 C G.S. units is in itself sufficient proof that the non-cyclic effect of the first pressure must be within the same limits an increase of magnetisation The increase, however, in the former case might be due to the removals not the applications of pressure.

The removal of pressure from a rod occasions relative displacements of its parts to pretty much the same extent as does the application of pressure, and so it too may be expected to have some permanent influence on the magnetisation. This effect was actually found to exist, and its magnitude relative to the original magnetisation is

given in the eleventh and twelfth columns of Table X. Comparing these with the two previous columns, it will be seen that in weak fields the non-cyclic effect of the removal of the original pressure, though much less important than the corresponding effect of the application of the first pressure, is by no means a negligible quantity It may, however, be easily overlooked, because in the weaker fields, where its importance is greatest, it is in the opposite direction to the cyclic effect of the removal of pressure Also in fields between 25 and 95 C G.S units, as appears from the thirteenth column of Table X, the cyclic part of the effect is numerically the larger, and so between these limits the removal of an original pressure appears to be accompanied by a diminution in the magnetisation In fields over 120 C G S units the non-cyclic effect has the same sign as the cyclic, and so the increase in magnetisation accompanying the removal of an original pressure is fairly conspicuous

Cyclic Effect of Pressure on the Induced Magnetisation

§ 45. In fig 7 the ordinates give the cyclic change in the induced magnetisation accompanying the pressure cycles. The thick line *a*, in which the individual observations are indicated by dots, is based on Table X, the broken line *b* with individual observations indicated by crosses, on Table I, and the dotted line *c* with individual observations indicated by circles, on Table III.

These curves agree in showing a critical or VILLARI field of about 120 C G S units The magnetisation is greatest or least when the rod is under pressure according as the strength of the field is less or greater than that of the critical field.

It will be seen from fig 1 that the magnetisation of the rod in the critical field is about 520 units According to both the curves, *a* and *b*, the absolute magnitude of the cyclic effect attains a maximum in a field of about 35 C G S. units This, it will be remembered, is the field found for the Wendepunkt in § 39 In general form the two curves *a* and *b* could hardly agree better than they do, and the difference in the absolute lengths of their ordinates might well be due to a difference between the standard pressures adopted at the times Both curves, it will be noticed, pass through almost all their experimental points Near the critical field all three curves would lie so close together that only the experimental points of *c* are there shown.

The highest field in Table IX. is 351 C.G.S units, and the form of the curve *a* leaves it uncertain whether the cyclic effect attains an algebraic minimum in a field somewhat higher than this and then diminishes numerically, or whether it continually approaches an asymptotic value. The former alternative is unquestionably supported by the form of the curve *c*, which extends to a field of 725 C.G S. units The deviations of the experimental points from this curve are, however, so large that it would be rash to attach much weight to its precise form. Probably all we are entitled to infer is that, if an asymptotic value exists for the cyclic effect, its numerical value cannot much exceed that found in a field of 400 C G.S. units; whereas, if a second

critical field exists, where the effect changes sign a second time, its value must very considerably exceed 700 C G S units

It should be noticed that the ordinates in fig 7 are drawn on double the scale of those in fig 5 Thus, comparing in those figures the two curves a, which are based on the same series of experiments, we see that, in fields below the VILLARI point of fig 7, the total effect of the first pressure is invariably very considerably larger than the cyclic effect of pressure The maximum value attained by the former effect is about thrice that attained by the latter

§ 46 In fig 8 the ordinates give the ratio of the cyclic increase in magnetisation accompanying pressure "on" to the amount of the induced magnetisation existing when the cyclic state is reached They thus show what proportion of the magnetisation takes part in the cyclic change

The curves a and c are derived from the same experiments as the corresponding curves a and c of fig 7 The form of a, near the vertical axis, shows how much the relative importance of the cyclic effect increases as the strength of the field is reduced

In comparing the relative magnitude of the cyclic effect and of the shock effect of the first pressure, it should be noticed that the ordinates of fig 8 are drawn on a scale ten times that on which the ordinates of fig 6 are drawn Thus, in the weakest fields, the shock effect is fully ten times the cyclic effect

Residual Magnetisation

§ 47 In figs 9 and 10 (Plate 16) the ordinates give the amount of the residual magnetisation existing immediately after the break of the current The curves b and d are shown in both figures, but on a different scale The curve a is based on Table V. The rod was here demagnetised before exposure to the weakest field, but not subsequently, and remained entirely free from pressure The curve b is based on Table VI The rod was here demagnetised by reverse currents and vigorous tapping before each introduction into the coil, but no pressures were applied while it was under the influence of the magnetising currents The curves c and d are based on Tables VII and VIII. respectively. In both cases the rod was demagnetised, as in the case of curve b, but was subjected to six pressure cycles while under the influence of the magnetising current In the case of c the pressure was "off" when the current was broken, whereas in the case of d the pressure was "on" The individual observations are indicated by dots for the curves b, by circles for the curve c, and by crosses for the curves a and d With the scale of fig 9 the curve c would be indistinguishable from the curve d in weak fields, and from the curve b in strong fields.

Effects of Pressure on the Amount of the Residual Magnetisation

§ 48. The curves a and b of fig 10 agree in showing only a small amount of residual magnetisation on the break of weak fields; but in fields below 5 C.G S. units

the ordinates of b are at least double those of a. The difference is surprisingly great, considering the similarity of the conditions under which the corresponding experiments were conducted. It must, presumably, be mainly attributed to the residual effect of the shocks employed in completing the demagnetisation in the experiments on which the curve b is based.

In weak fields, as the difference between the curves c and d on the one hand, and the curves a and b on the other abundantly proves, the effect of pressures applied during the flow of the current in increasing the residual magnetisation is simply enormous. It would thus appear *a priori* probable that the application of shocks immediately before the starting of the current should have some appreciable tendency in the same direction. Fortunately I happened to observe the induced as well as the residual magnetisation for the field 2·6 C.G.S. units in the experiments on which curve a is based. The numerical value of the induced magnetisation was 8·3, which is almost exactly the value obtained by interpolation for the same field from Table IX. Consequently, in this field, and so presumably in other weak fields, the difference between the curves a and b cannot be attributed, in any important degree, to an increase in the induced magnetisation brought about by the process of demagnetisation employed in the case of curve b.

As the strongest field in Table V is only 79 C.G.S. units, no data exist for drawing the curve a in higher fields. It is only, however, in fields below 45 or 50 C.G.S. units that there is any clear difference between the results of Tables V and VI. We are thus probably justified in concluding that the process of demagnetisation has an appreciable effect on the amount of the residual magnetisation only in fields below 50 C.G.S. units.

As already stated, the results of Table V are each the mean of two observations taken in close succession with the same current. Almost invariably the second observation showed a slightly larger amount of residual magnetisation than the first. This property has been noticed by several observers in iron.

The difference between the curves b and c of fig. 10, which are based on experiments in which the same process of demagnetisation was applied, shows how effective pressure cycles are in increasing the amount of the residual magnetisation in weak fields. In stronger fields than those of fig. 10 the curves gradually approach one another, and in fields exceeding 120 or 130 C.G.S. units they cannot with certainty be said to differ. Thus, in fields below 120 or 130 C.G.S. units, the application of pressure cycles during the flow of the current increases the amount of the residual magnetisation, but in stronger fields no effect can with certainty be said to exist.

In fields below 30, or at all events 20 C.G.S. units, the difference between the curves c and d is, though small, perfectly clear and incontestable. The existence of pressure during the break of the current is the only point in which the experiments on which curve d is based differed from those on which curve a is based. As the effect of pressure "on" during the flow of the current is an increase of induced magnetisation

in fields below 120 C G S units, an excess in the ordinates of curve *d* over those of curve *c* in weak fields might reasonably have been expected The difference, however, as will more fully be seen presently, is somewhat greater than might have been anticipated

The crossing of the curves *c* and *d* in a field of from 35 to 40 C G S units, and thus far below the VILLARI field, appears a somewhat striking fact In fields over 120 C G S units the curve *c*, if shown in fig 9, could not be distinguished from curve *b*, and the difference between its ordinates and those of curve *d* seems then truly remarkable In fields over 150 C G S units this difference remains nearly constant, and amounts to about 4 per cent of the ordinates of *c*. If it be remembered that according to the last column of Table X the percentage of the induced magnetisation, which is cyclic for the same pressure "on" and "off," is in fields between 150 and 400 C G S units never in excess of 4, the significance of this result will be more fully understood

§ 49 The ratio of the residual magnetisation existing in the rod immediately after the break of the current to the numerical value of the strength of the pre-existing field will, for shortness, be here termed the *residual susceptibility* It is denoted by κ_1' in Tables V to VIII, and its values are represented graphically by the ordinates of the curves of fig 11 In this and in the subsequent figures the letters *a*, *b*, *c*, *d* denote curves obtained from the same series of experiments as are the respective curves *a*, *b*, *c*, *d* of fig 10 In fig 11 and the subsequent figures, however, the individual observations are indicated by crosses for curves *a* and *c*, by dots for curve *b*, and by circles for curve *d*.

In the absence of pressure, as shown by curves *a* and *b*, the residual susceptibility is very small in weak fields, but increases rapidly, attaining a maximum in a field a little over 20 C G S. units This, it will be noticed, is not much over half the field which gives the Wendepunkt for the induced magnetisation The greatest value actually observed for κ_1' in the case of curve *b*, was only 1 36 After the maximum is passed there is a continuous, but gradual, diminution in the residual susceptibility

The difference between the commencing portions of curves *a* and *b* shows, even more clearly than in fig 10, the large effect in weak fields of the process of demagnetisation In fields over 50 C G S units the two curves practically coincide, and are drawn as one.

The crossing points for the various curves in fig 11 are, of course, the same as in figs. 9 and 10. Within the range of the figure the curve *c* lies distinctly above the curve *b* In fields over 55 C G S units, the curve *d* is not drawn, as it could hardly within the remaining limits of the figure be distinguished from curve *b*.

Commencing with the strongest experimental fields, the ordinates of the curves *c* and *d* increase gradually as the strength of the field is reduced, and show maxima values in somewhat lower fields than do the curves *a* and *b*. As the strength of the fields is further reduced, the ordinates of *c* and *d* pass through distinct minima values, and then increase rapidly as the strength of the fields approaches the lowest experi-

mental value The value of κ_1' cannot, of course, exceed that of κ_1, so either the ordinates of c and d must be rapidly approaching second maxima, or else the ordinates of the curves b and c of fig. 4 must possess minima values in fields lower than were experimented on.

§ 50 In comparing the amounts of the residual and induced magnetisations, the employment of the term *retentiveness* in an exact way will be found serviceable It is here used in accordance with the following definition .—

The retentiveness is the ratio of the residual to the induced magnetisation, both quantities being measured with the rod exposed to one and the same state of mechanical stress, and the stress remaining constant during the interval that elapses between the measurements

In fig 12 the ordinates give the retentiveness of the rod in the cases illustrated by the curves b, c, d As the residual magnetisation alone was observed in the experiments recorded in Tables VI, VII, and VIII, the necessary values of the induced magnetisation were derived by interpolation from Tables IX and X. In curve b the induced magnetisation is that existing prior to the application of pressure, or is the $\mathfrak{I}_1$ of Table IX. In curve c the induced magnetisation is $\mathfrak{I}_2$, for after six pressure cycles the increase in the magnetisation accompanying further pressure cycles is, for our present purpose, quite negligible. In the experiments answering to curve d, the rod, after experiencing six pressure cycles, was under pressure when the current was broken. Thus, for curve d the induced magnetisation is got by adding to $\mathfrak{I}_2$ the algebraic value of the *cyclic* "*on*"—"*off*" of Table X.

The curve b shows that, in the absence of pressure, the retentiveness attains a maximum in a field of about 15 C G.S. units, a field somewhat below that at which the maximum residual susceptibility occurs in the corresponding curve of fig 11. A great similarity exists between curve b and the curve for zero load in Professor EWING's fig. 57. The principal difference is that the field at which the maximum retentiveness appears is considerably higher in cobalt than in iron

The curves c and d of fig. 12 bear a considerable resemblance to the corresponding curves of fig 11; but there are no longer distinct maxima or minima. Comparing these curves with curve b we see that in the weakest experimental fields pressure cycles increase the retentiveness in the ratio of four or five to one The tendency to become tangential to the vertical axis shown by the curves c and d—which cannot, however, go on indefinitely as the fields are further reduced—is not exhibited by any of the curves for the retentiveness of loaded iron-wire occurring in Professor EWING's fig. 57. Possibly if he had employed lower fields he might have found similar phenomena, so it would not be safe to assume that iron and cobalt actually differ in this particular

The ordinates of curve d being greater in weak fields than those of curve c, it follows that in fields below 30 C.G.S. units, where the curves cross, the retentiveness of the rod is greatest when it is under pressure while the current is broken In fields

over 30 C G S units the retentiveness is least when the current is broken with pressure "on," and in fields over 50 units the difference is not inconsiderable In fields over the VILLARI point, of 120 C G S units, the induced magnetisation is greater for curve *c* than for curve *d*, so that in such fields the excess of the ordinates of curve *c* over those of curve *d* would, relatively to their absolute magnitude, be somewhat less in the case of fig 12 than in the case of fig 11

Similar considerations lead to the conclusion that in fields over 130 C G S units the curve *b* would be, if anything, slightly above the curve *c*. In other words, in fields over 130 C G S. units the effect of pressure cycles is, if anything, slightly to diminish the retentiveness This effect is, however, insignificant compared to that proceeding from the existence of pressure "on" during the break of the current

The curves *b* and *d* cross in a field of about 40 C G S units The curve *d* would then become the lower, but it would be for a time so close to *b* that it is not attempted to show it separately in the figure.

The conclusions which the preceding considerations lead to are the following —

In fields below 30 C G S units the application of cycles of pressure "on" and "off" during the flow of the current, and the existence of pressure "on" during the break of the current, both tend to increase the retentiveness, the former agency being in fields below 10 C G S units far the more important. In fields between 30 and 120 C G S. units, or thereby, the application of cycles of pressure still increases the retentiveness, but the effect continually diminishes as the strength of the field is raised In fields over 130 C.G S. units up to at least 300 or 400 C G S units the effect of pressure cycles is very small, but is, if anything, a diminution of the retentiveness. In fields from 30 C G S. units up to at least 300 or 400 C G S units the existence of pressure "on" during the break of the current diminishes the retentiveness, and the effect is of the same order of magnitude as that due to the same cause in fields below 30 C G S units In fields between 30 and 40 C.G S units, or thereby, the increase in the retentiveness occasioned by the pressure cycles exceeds the diminution occasioned by the existence of pressure "on," but in stronger fields the latter effect is the larger, and its superiority appears continually to increase with the strength of the field

§ 51 Curve *b* of fig. 12 is repeated in Curve I. of fig. 13, in order that its form in fields over 75 C G.S units may be seen There appears a continual but very slow diminution in the retentiveness as the field is raised to 400 C G S. units Curve II. of fig 13 will be discussed presently

§ 52. In fig 14 the ordinates give the absolute amount of the residual magnetisation which seems to disappear with the first alteration of stress after the break of the current.

Comparing the curves *b* and *c*, in which the loss of magnetisation accompanies the application of pressure, with the curve *d*, in which it accompanies the removal of the pressure existing during the break of the current, we see that the effect must be to a

large extent independent of the precise character of the change of stress, and would doubtless follow more or less any mechanical agitation of the rod

The curve *b* is drawn through nearly all the experimental points, but in the stronger fields of *c* and *d*, especially the latter, the experimental points are somewhat widely scattered These latter curves accordingly cannot be trusted in their minute details The curves are based on the data in the fourth columns of Tables VI, VII, and VIII These tables give only certain average results for fields over 90 C G S units, but they indicate that the ordinates of *b* and *c* would attain maxima values in fields somewhere between 100 and 200 C G S units. Thereafter the ordinates of *c* would apparently diminish the more rapidly of the two, so that in fields of 400 C G S units the curve *c* would have approached pretty close to the curve *d*, which remains apparently nearly horizontal

§ 53 The application of cycles of pressure causes cyclic changes in the magnetisation, whose character will presently be discussed In consequence of this the effects of the first change of stress on the residual magnetisation are only, in part, of a non-cyclic character To get the exact quantity of magnetisation whose *permanent* disappearance is secured by the first change of stress, the cyclic effect must be allowed for.

Suppose, for instance, a field of 50 C G.S units is broken when pressure is "off" The cyclic effect is then a diminution of magnetisation when pressure is "on" Consequently the loss of magnetisation accompanying the first application of pressure exceeds the quantity whose permanent disappearance is secured by the amount which takes part in the cyclic change Subtracting from the total loss the numerical value of the cyclic effect, we get what may be regarded as the true loss of residual magnetisation due to the first application of pressure. This quantity is that here termed the shock-effect of the first pressure

In the case of residual magnetism the cyclic effect is in general small compared to the shock-effect, so that a considerable error in the cyclic effect will seldom seriously affect the calculation of the shock-effect.

§ 54 In fig 15 the ordinates represent the ratio of the shock-effect of the first alteration of pressure to the amount of the pre-existing residual magnetisation They may thus be regarded as measuring the efficacy of the first change of pressure in shaking out the residual magnetisation. Unfortunately no readings were taken of the cyclic effects of pressure in the weaker fields in the series of experiments on which the curves are based. Thus up to the points where the transverse lines are drawn, the ordinates represent the ratio of the *total* effect of the first change of pressure to the initial residual magnetisation. Consequently, as will be seen presently, in the commencing portions of the curves the ordinates of *b* and *c* are probably somewhat underestimated, and those of *d* on the other hand overestimated. The corrections could, however, hardly modify the relative positions of the curves.

The curves all show that the efficacy of the first change of pressure in shaking out

the residual magnetisation, as measured by the percentage got rid of, continually diminishes as the strength of the field is increased According to the seventh columns of Tables VI and VII, and the sixth column of Table VIII, on which the curves are based, this diminution most probably goes on up to fields of 400 C.G.S units at least The difference, however, between the percentages shaken out in the case of fields of 100 and 400 C G S units is under all conditions very small

Comparing curves b and c, in figs 14 and 15, and the seventh columns of Tables VI and VII, we see that while in fields below 15 C G S units an absolutely larger amount of residual magnetisation is shaken out by the first pressure when pressure cycles have been applied during the flow of the current, still in these and in all higher fields within the experimental limits a greater percentage is removed by the first pressure when no pressures have been applied during the current

We thus see that in fields below 100 C.G.S units, pressure cycles during the flow of the current diminish simultaneously the percentage of the induced magnetisation got rid of by breaking the current, and the percentage of residual magnetisation got rid of by the subsequent application of a pressure In fields from 130 to at least 400 C.G.S units pressure cycles during the flow of the current appear slightly to increase the percentage of induced magnetisation got rid of by breaking the current, while simultaneously diminishing the percentage of residual magnetisation got rid of by the subsequent application of a pressure In these strong fields, however, both these effects are comparatively insignificant

§ 55 In the experiments on which the curves b and c are based the effects of removing the first pressure applied after the break of the current were also observed The invariable result was a loss of magnetisation, and its magnitude is recorded in the fifth columns of Tables VI and VII In considering how much of this loss is permanent, the cyclic effect of the change of pressure must be allowed for. This has been done in calculating the eighth columns of Tables VI and VII., which give the relative importance of the first pressure and its removal in permanently reducing the magnetisation

When no pressures have been applied during the flow of the current we see from Table VI that, in fields between 18 and 75 C G S units, the removal of the first pressure produces uniformly about one-ninth of the effect of its application In the higher fields the relative importance of the removal of pressure shows a distinct diminution.

Comparing Tables VI. and VII we see that in fields below 50 C G S units the importance of the removal of the first pressure after the break of the current relative to its application is largely increased by the application of pressure cycles during the flow of the current. In fields over 50 C G S. units the effect of the removal bears to that of the application of the first pressure in the case of Table VII. the apparently nearly constant ratio of 1 : 9, which still somewhat exceeds the ratios found in the corresponding fields of Table VI.

The fact that the importance of the removal of the first pressure relative to its application is greatest in weak fields must, I think, be connected with the fact that in weak fields, as shown by curves *c* and *d*, fig 15, a larger proportion of the residual magnetisation is shaken out by the removal of a pressure existing during the break of the current, than by the application of an equal pressure when the rod is free from pressure during the break of the current In the weak fields pressure cycles, which exist in the case of curves *c* and *d*, cause a large permanent increase in the induced magnetisation, and, of this, a much larger proportion is due to the application than to the removal of pressure. Thus, the phenomena suggest that the residual magnetisation arising from the portion of the induced magnetisation which the application of pressure enables the rod to acquire is more easily shaken out by the removal of pressure than is the remainder of the residual magnetisation

It may also be worth mentioning that by increasing the magnitude of the first pressure after the break of the current the effect of its application was increased, but the effect of its removal diminished absolutely as well as relatively

Cyclic effect of Pressure on the Residual Magnetisation

§ 56 The cyclic effects of pressure cycles on the residual magnetisation during the experiments on which the curves *b*, *c*, *d* are based, are shown in the sixth columns of Tables VI and VII and the fifth column of Table VIII. The cyclic effects were also observed in the December experiments, and are recorded in the fifth column of Table II. In the experiments of Table II the demagnetisation was incomplete, but otherwise the conditions were identical with those of the experiments of Table VII

The cyclic effect was found very difficult to measure accurately, partly on account of its smallness, and partly, doubtless, on account of its sensitiveness to small variations in the treatment of the rod during the flow, and more especially the break of the current. In the weaker fields the cyclic effects were observed only in the experiments of Table II., which were inferior in accuracy to the later observations

For these reasons the numerical values attributed to the cyclic effects in the tables cannot claim to be more than somewhat rough approximations Thus, while believing the results to represent correctly the general features of the phenomena, I have not embodied them in curves, as the precise forms of these might have owed too much to the imagination

§ 57. Taking first the case when pressure cycles were applied during the flow of the current but no pressure existed during the break, we see from Tables II and VII that the cyclic effect in the residual magnetisation is an increase or a decrease of magnetisation when pressure is "on," according as the pre-existing field is below or above a certain critical field. So far the parallelism between the residual and induced magnetisations seems complete. The critical field, however, for the residual magnetisation is according to both tables only about 30 C G.S. units, and so only about a quarter of

that for the induced magnetisation, and the intensity of the critical magnetisation in the former case is less than an eleventh of that in the latter Table VII shows no sensible variation in the magnitude of the effect as the field is raised from 70 to 400 C G.S. units

Taking next the case when no pressures were applied during the flow of the current, we see from Table VI that in a field of 18 C G S units—the lowest in which observations of the effect were taken—no sure cyclic effect was detected Thus a critical field must exist at or very close to this field In higher fields there is a clear diminution of magnetisation accompanying pressure "on," and in fields over 100 C G S units the results agree as closely with those of Table VII as they well could Thus, the only conspicuous difference in the cyclic effect produced by pressure cycles during the flow of the current is a rising of the critical field.

Table VIII, however, shows that a truly remarkable difference occurs in the phenomena when pressure exists during the break of the current In this case, in fields from 11 5 to 400 C G S units, the residual magnetisation in the cyclic state is invariably greater when pressure is "on" than when it is "off." The magnitude of the cyclic effect apparently diminishes at first as the field is raised, but in fields of from 70 to 400 C G S units, it is at least approximately constant

It thus appears that the sign even of the cyclic change of the residual magnetisation accompanying cycles of pressure, is altered by such a seemingly trifling circumstance as the existence of pressure during the break of the pre-existing current

It seems almost unnecessary to point out that the facts stated in this paragraph convincingly show that the effect on the magnetisation of cyclic applications of pressure is not determined solely, or in some circumstances even principally, by the measure of the rod's magnetic moment, or its so-called intensity of magnetisation

§ 58 As the conclusions of the last paragraph as to the cyclic effect seem of considerable importance, it may not be out of place to supply data by which the value of the evidence on which they are based may be fairly judged

The precautions taken in varying the order of the experiments have been already mentioned in explaining the Tables VI, VII, and VIII

Now in fields stronger than the critical field of Table VII, there were, in all, 46 separate experiments of the types L, M, and N. Answering to pressure "on," there appeared a minimum of magnetisation in each of the 31 experiments of the types L and M, a maximum in each of the 15 experiments of type N. As the difference was wholly unexpected, and the observer had no preconceived ideas on the subject it is difficult to conceive how the evidence could be stronger

§ 59. The ordinates of Curve II of fig 13, give the fraction of the residual magnetisation, which is removed by a definite number of pressure cycles in the experiments of Table VI., or type L.

The percentage removed continually diminishes as the strength of the pre existing field rises. The rate of diminution in the percentage removed is very rapid as the

strength of the field is raised to 30 or 40 C G S units, but very slow when the field is raised over 150 C G S units

A comparison of this curve with the curves of figs 4 and 6 brings out clearly the fact that the residual magnetisation, which is most easily removed by pressure cycles, is that left on the break of those fields in which pressure cycles have the greatest effect on the induced magnetisation

The data in the last columns of Tables VII and VIII would lead to curves very closely resembling Curve II Thus the preceding remarks apply equally to the experiments of types M and N.

§ 60 Comparing Curves I and II of fig 13, we see that in fields over 30 C.G S units, the increase in the percentage of the induced magnetisation removed by breaking the current, and the diminution in the percentage of the residual magnetisation removed by a definite number of pressure cycles, go hand in hand

It has been already mentioned that the application of a particularly severe first pressure after the break of a current lessened the effect of the removal of the pressure, and it also lessened the joint effect of succeeding pressure cycles, in which the pressure was of the normal intensity

Thus the close resemblance of Curves I and II in the stronger fields suggests that the sudden break of a current acts in some respects as a mechanical shock, and that the magnitude of this shock-effect continually increases as the strength of the field is raised.

Separation of the Effects of the Application and Existence of Pressure

§ 61. There are still some points connected with the fundamental character of the effect on the rod's magnetisation produced by the application of pressure, on which Tables IX. and X throw some light

We have seen that in general the first application of pressure after the rod's introduction into a given field, causes a change in the magnetisation Further, as a rule, the greater portion of this change survives the removal of the pressure, so that for its continued existence the existence of pressure is not essential It thus seems of importance to determine whether the change depends solely on the existence of pressure for a finite time during the flow of the current, or is due in whole or in part to the actual application of pressure

Taking WEBER's hypothesis for illustration, the change must consist either in a sub-permanent increase in the general mobility of the ultimate magnetic molecules, or in a swinging round of some of them into extreme positions, where they are kept by molecular friction even after the removal of the pressure The application of pressure produces doubtless molecular movements, during which the rotations of the ultimate magnets might be expected to go on more freely, so that on this theory we should *à priori* be disinclined to attribute the entire effect to the mere existence of pressure.

In practice it is difficult if not impossible wholly to separate the effects of the application of pressure from those of its mere existence. When the rod is put under pressure before its introduction into the magnetising coil, and not introduced until sufficient time has elapsed for any molecular motion to die out, the magnetisation might at first sight be supposed to differ from that possessed by the rod when the pressure is applied after its introduction by an amount which exactly measures the effect of the actual application of the pressure. It must be remembered, however, that sudden magnetisation, altering as it does the rod's length, must act in part as a mechanical shock and set up molecular movements, whose character and amplitude would naturally depend to some extent on the state of the rod as to pressure.

In the following remarks the difference between the magnetisations of the rod under pressure in the two circumstances of its introduction is entirely attributed to the effect of the application of pressure, but the previous statements should be taken as a warning against accepting this as more than an approximation to the truth.

§ 62. Certain experiments bearing on the question have been already referred to in § 40 in discussing the curves of fig. 4. These experiments were limited to fields under 60 C.G.S. units, but within that range they showed conclusively that in a freshly demagnetised bar the existence of pressure applied before the rod's introduction into the magnetising coil increases the susceptibility. In the weaker fields of fig. 4 the curves show a greater susceptibility when the application of the pressure has succeeded than when it has preceded the introduction of the rod into the coil, and the evidence for this was considered satisfactory. In fields from 30 to 60 C.G.S. units the reverse is the case according to the curves, but reasons were given for regarding the evidence on this point as insufficient.

In these experiments when pressure existed during the introduction of the rod into the coil it had always been applied some minutes previously, so that any molecular movements set a-going by the process had presumably pretty well subsided. Further, any such movements must have been inconsiderable compared to those occasioned by the vigorous tapping which accompanied the demagnetisation of the rod shortly before its introduction in both sets of experiments.

We are thus in all probability entitled to conclude from these experiments that in a freshly demagnetised rod the mere existence of pressure, apart from the consequences of its application, has, in fields up to at least 60 C.G.S. units, the effect of permanently increasing the induced magnetisation, while in the weakest fields there can be no doubt that the actual application of pressure has an independent effect in the same direction.

§ 63. The results of a more complete series of observations bearing on the same point are given in the seventh column of Table IX., under the heading $\mathfrak{J}_A$ "*on*"—"*off*." This signifies, as already stated, the excess of magnetisation possessed by the rod when introduced into the coil under pressure, over that it possesses when introduced free from pressure. The circumstances of the rod are, except as regards the pressure,

identical in the two cases. The rod was not, as in the experiments discussed in the last paragraph, in a freshly demagnetised condition, but had just been exposed to numerous pressure cycles in a field of the same strength and sign

This column agrees with fig 4 in showing in weak fields an increase of magnetisation due to the existence of pressure An unmistakeable critical field, however, ensues, the effect eventually changing sign Thus the character of the effect of the mere existence of pressure varies in the same way as does that of the joint effect of the application and existence of pressure, which we have already discussed in § 41.

In this case an exact basis of comparison exists, because in the third column of Table X, under the heading *First "on"* A_2, we have the combined effect of the application and existence of pressure when the previous treatment of the rod was the same as in the experiments of the seventh column of Table IX

A comparison of these two columns shows that the fields in which the effects attain their maxima, and the critical fields must be nearly the same, if not absolutely identical in the two cases. The magnitudes of the effects are, however, decidedly different. There can be no question that in the weaker fields the effect is very considerably greatest when the application of the pressure succeeds the introduction of the rod into the coil, and this effect of the application of the pressure clearly continues until the field of 128 units is reached. From this to somewhat over the critical field neither the application nor the existence of pressure has effects large enough for satisfactory determination. In the fields of 198 units and upwards the diminution of magnetisation due to pressure was in every case found greatest when its application preceded the introduction of the rod

Thus in all fields up to 128 C G.S. units the actual application of pressure during the flow of the current produces in the condition of the rod during these experiments an unmistakeable increase in the magnetisation, and not improbably this tendency to increase the magnetisation exists in all the fields within the range of the experiments.

In the eighth column of Table IX are a few results from experiments on the same point. The circumstances of these experiments differed from those of the preceding column, only in that the previous fields of equal strength in which pressure cycles had been applied, were of opposite sign to those in which the effects were observed The phenomena are clearly of the same general character as those of the previous column, and the existence of a critical field is unmistakeable. The observations are insufficient to determine the exact position of the critical field, but it is obviously not far from the critical field of the preceding column.

The Polar or Non-polar Character of the Effects of Pressure.

§ 64. Another very important question is whether the effects of the application and removal of pressure on the magnetisation are of a *polar* or a *non-polar* character. The meaning of these terms will be best understood from an illustration.

Let us suppose the induced magnetisation increased by a series of pressure cycles applied during the flow of a current, which makes A the north pole. The effect is non-polar if the rod be brought into the condition of an unstrained rod of a different material possessed of a higher susceptibility The effect is polar if the susceptibility, while increased in fields making A a north pole, is equally diminished in fields of opposite sign—supposing for the moment that an opposite field could be introduced without altering the rod's temporary properties If the susceptibility be simply altered by different amounts in fields of opposite sign the effect may be regarded as a combination of a polar and a non-polar effect

It is probably impossible to arrive at a complete knowledge of the character of the effect, because the introduction of an opposite field implies a break and make of the magnetised current affecting the temporary properties of the rod. It is only in so far as the effects of pressure cycles are residual that their polar or non-polar character admits of direct investigation

If, after the application of pressure cycles in a certain field, an increase of susceptibility, not otherwise accounted for, be found in fields both of the same and of opposite polarity, the existence of a non-polar effect would seem to be established. The same conclusion follows if there be a decrease of susceptibility irrespective of the sign of the field If the susceptibility be found less in fields of the same name as that existing during the pressure cycles, than in fields of opposite name, it is difficult to think of any probable explanation other than a *true* polar effect

If, however, the susceptibility be greater in fields of the same sign as that existing during the pressure cycles than in fields of opposite sign, the phenomenon may be explained without assuming a *true* polar effect.

The reasons for these statements will probably be most clearly presented by embodying them in language literally applicable only to WEBER's theory of magnetism. According to this theory, the only way in which we can imagine the application or removal of pressure to produce a permanent change in the magnetisation is either by endowing the ultimate magnetic molecules with an increased or diminished mobility, or by enabling them to swing round into extreme positions, where they stick. The change in the molecular mobility is non-polar if the molecules turn as freely in fields of one sign as in fields of the opposite sign, otherwise the change is polar

Now when the susceptibility is found increased in fields both of the same and of opposite sign to that existing during the pressure cycles, or when it is diminished irrespective of the sign of the field, or, lastly, when it is less in a field of the same sign as that during which the pressures were applied than in a field of opposite sign, it is clear that the phenomena cannot be attributed to the magnetic molecules having stuck in positions into which they swung while under the directive influence of the original field. Thus, in these cases, the phenomena must be ascribed to some change in the mobility of the magnetic molecules, and so there must be a true polar

or non-polar effect It is obvious, however, that if the magnetic molecules swung round and stuck in the way imagined, then the magnetisation would naturally be greater in a field of the same sign as the original than in a field of opposite sign This is not regarded here as a *true* polar effect, though producing in the induced magnetisation phenomena identical with those which would arise from a true polar change in the molecular mobility. The turning round of the molecules has been already discussed in §§ 22 and 28, where its occurrence was demonstrated The effect on the induced magnetisation will here be referred to as a *quasi* polar effect

§ 65. The ninth column of Table IX may be regarded as measuring the combined *true* and *quasi* polar effects of a numerous series of pressure cycles, in so far as they survive the break of the field during which the pressures were applied, and the make of an equal field of the same or opposite sign The exact quantity measured is the algebraic excess of the magnetisation in a field of the same sign as that existing during the pressure cycles over that in a field of opposite sign, the rod being in each case introduced free of pressure into the second fields

The results obtained with the weakest field of the table, 1 8 C G S units, differed so much from the others that it was inconvenient to include them in the table. They were as follows —

Circumstances of rod	End a north pole	Pressure	$\mathfrak{J}$
First introduction	A	off	4 1
After pressure cycles	,,	,,	6 5
Out, re-introduced	B	,,	4 8
Kept in	,,	on	6 5
Out, re-introduced	,,	,,	6 2
,, ,,	A	off	4 1
,, ,,	B	,,	5 5
,, ,,	A	,,	4 1
Kept in	,,	on	6 5
Out, re-introduced	,,	off	5 3
,, ,,	B	,,	5 1

There are here two or three instances of an increase of susceptibility of a non-polar character, and also clear indications of a polar effect Considering how small the total induced magnetisation is, both the polar and non-polar effects are of considerable importance.

In the next field, 2·2 C G.S. units, the rod was originally under pressure, and yet $\mathfrak{J}_2 - \mathfrak{J}_1$ exceeds $\mathfrak{J}_2 - \mathfrak{J}_3$. Thus, as $\mathfrak{J}_A - \mathfrak{J}_B$ is only 2, there occurred a very decided residual increase of susceptibility, mainly of a non-polar character.

In all the fields, from 3·4 to 35·7 C.G S units, the residual effect is largely of a polar character. Take, for instance, the field 8 6. Here the pressure cycles raise the magnetisation by 10 2 Of this, 5·1 survives the break and re-make of the current, and so only about the half of the increase is lost.

The difference, however, in this case between $\mathfrak{I}_A$ and $\mathfrak{I}_B$ amounts to no less than 9.7, and thus the susceptibility, when B is the north pole, is lowered by nearly as much as the susceptibility, when A is a north pole, is increased. The residual effect is, in fact, almost entirely of a polar character. This appears to be about the strength of field at which the polar effect is largest.

In the fields from 35.7 to 128 C.G.S. units the results show too large fluctuations to lay any claims to great accuracy. They may, however, be regarded as satisfactorily proving that within these limits the polar residual effect is small, and that it has the same sign as in the weaker fields.

In the field of 174 C.G.S. units no residual polar effect could be detected, and in stronger fields the effect changes sign. There is thus a neutral field, which if not identical with, is certainly very near that at which the total effect of the first pressure on the induced magnetisation vanishes. In weaker fields the susceptibility of the rod is greatest when the field is of the same sign as that existing during the pressure cycles, in stronger fields the susceptibility is in these circumstances least.

In fields above this neutral field there is thus, according to the previous reasoning, a *true* polar effect. In fields again, such as 12.5 C.G.S. units, a *quasi* polar effect unquestionably existed, because there could then be obtained the phenomenon of the reversal of the rod's polarity on the break of an equal reverse current described in § 22. This phenomenon could not be obtained in a field of 23.7 C.G.S. units.

There is thus positive proof that a *true* polar effect exists in strong fields and a *quasi* polar effect in weak fields, and most probably in all but certain critical fields they exist simultaneously.

§ 66. We have seen that the residual polar effect has practically the same critical field as the total effect of the first pressure, or *First "on"* of Table X. Also both these effects are relatively very large in weak fields, and it might thus be supposed they stood in some intimate relationship to one another. In fields of 30 to 50 C.G.S. units, however, the residual polar effect is very small, while the *First "on"* is about its maximum.

An explanation of the discrepancy is supplied by a consideration of the curves *b* and *c* of fig. 12. The difference between the ordinates of these curves in a field of 8 C.G.S. units is great compared to the difference in a field of 30 C.G.S. units. Thus in the former field pressure cycles are much more effective in increasing the retentiveness than in the latter. A consideration of the numerical data shows that of the additional magnetisation which the pressure cycles enable the rod to acquire the loss on the break of the current is proportionally much less in a field of 8 than in a field of 30 C.G.S. units. Thus the *quasi* polar residual effect must be much smaller in the latter field than in the former, which doubtless accounts for the rapid falling off in the value of the total polar effect as the field is raised.

In the last column of Table IX. there are a few results bearing on the same point. These are derived from experiments in which the rod was under pressure when

reintroduced into the coil after being subjected to a series of pressure cycles in a previous field of the same strength So far as they go the results are very similar to those of the previous column, from which it follows that the residual polar effect is, at least in its general character, independent of the state of the reintroduced rod as to pressure.

§ 67 Further results bearing on the polar character of the residual effects of pressure are given in the second, third, and fourth columns of Table X These give the total algebraic increase of the induced magnetisation due to the first application of pressure The headings A_1, A_2, B_2 show the condition of the rod as explained in § 37

Comparing the columns headed A_1 and A_2, we see that in fields below 50 or 60 C.G.S. units the increase in the magnetisation caused by the application of the first pressure is very considerably greater when the rod is in a freshly demagnetised condition than when it has recently been subjected to a numerous series of pressure cycles in a field of the same strength and sign. In fields between 60 and 90 C G S. units the effect of pressure is still distinctly larger in the freshly demagnetised rod, but whether considered absolutely or relatively the difference between the results in the two columns is small In stronger fields the numerical value of A_1, whether it signify an increase or a decrease of magnetisation, appears almost invariably greater than that of A_2. Both quantities, however, are then small, and in the strongest fields the effects of the first pressure were difficult to measure exactly, as the irregularities in the column headed A_1 would suggest. It would thus be unsafe to build too much upon the differences between the two columns in fields over 90 C.G S. units. The evidence, however, clearly supports the conclusion that even in the strongest fields the numerical value of the change in the magnetisation due to the first pressure is, *ceteris paribus,* greatest in a freshly demagnetised bar

The observed differences between the columns headed A_1 and A_2 are exactly of the character we should have expected from the already discussed experiments on the residual effects of pressure. In the weaker fields, we saw that the residual effect is very considerable. Thus the rod on its second exposure to a given field, under the circumstances of the experiment, is more or less in the state of a rod already subjected to pressure. Thus its condition, when the pressure A_2 is applied, is somewhat intermediate between its conditions before and after the application of the pressure A_1. Now the effect of the second pressure, after the introduction of the rod into the coil, was invariably, in the weaker fields, small compared to the effect of the first pressure Consequently, in such fields, the pressure A_2 would naturally have a smaller effect than the pressure A_1, which is precisely the phenomenon observed

Comparing the fourth column of Table X. with the previous two, we see that in fields up to 60 or 70 C.G.S. units, the increase in the magnetisation of the reintroduced rod, due to the first pressure, is distinctly greater when the field is of the

opposite sign to that existing during the pressure cycles, than when it is of the same sign In stronger fields, the sign of the field existing during the pressure cycles appears practically immaterial

A difference between the effects of the pressures A_1 and A_2, or A_1 and B_2, might be due either to a polar or a non-polar residual effect of pressure, but a difference between the effects of the pressures A_2 and B_2 must be due entirely to a polar residual effect Thus the experiments on the first pressures afford independent proof of the existence of a residual polar effect in fields up to 60 or 70 C G.S. units. This is in accordance with the results of § 65

There would also appear to be evidence of the existence of a non-polar effect in the weaker fields In such a field, we have already shown the existence of a very considerable *quasi* polar residual effect Now the application of a pressure in a field of opposite sign, would unquestionably shake out part of this *quasi* polar effect, and might be expected to produce, in addition, a polar effect of its own, equal or nearly equal to that of the first pressure in the original field Thus, if there were not a non-polar residual effect, we should expect the pressure B_2 to have, in weak fields, a larger effect than the pressure A_1 This does not, however, appear to be the case

The positions of the critical fields for the vanishing of the total effect of the first application of pressure, indicated by the second, third, and fourth columns of Table X. are in good agreement Consequently, it would appear that the position of this critical field depends but little on the treatment of the rod since its last demagnetisation.

§ 68 The results given under the heading *First "off"* in Table X , show that in the strongest fields of that table the residual polar effects, due to the application of pressure cycles in pre-existing fields of the same strength, have no appreciable influence on the magnitude of the increase of magnetisation which accompanies the removal of the pressure existing during the re-introduction of the rod into the magnetising coil

Combining this phenomenon with the close agreement in the stronger fields of the magnitudes of the effects of the first pressures A_2 and B_2, we seem entitled to conclude that the residual polar effect of pressure cycles in a strong field, of whose existence the two last columns in Table IX. give clear proof, is not sensibly diminished by the application or removal of a single pressure in a subsequent equal field, whether of the same, or of opposite sign.

Resemblances between the Phenomena shown by Iron and Cobalt.

§ 69. Most of the following resemblances between the phenomena shown by the cobalt rod and those shown by the stretched iron wires of Professor EWING's experiments, have been already incidentally noticed, but a formal statement and comparison may be useful. In the following statements, *stress* signifies *tension* in the case of the iron wires, and *pressure* in the case of the cobalt.

1 The strengths of the fields in which the changes in the magnetisation produced by the first application of a given stress, or by its cyclic repetitions, bear the greatest ratios to the pre-existing magnetisations, are very low, if indeed they possess a finite value

2 The fields in which the changes in the magnetisation produced by the first application of a given stress, or by its cyclic repetitions, are absolutely greatest, are not very far from the region of the Wendepunkt

3 The maximum cyclic effect appears, however, in a distinctly lower field than does the maximum effect of the first application of stress

4 Both the cyclic effect and the effect of the first application of stress possess critical or VILLARI points, and the critical magnetisations are both considerably greater than that answering to the Wendepunkt.

5 The critical field, however, for the cyclic effect is decidedly lower than that for the first application of stress

6 The residual magnetisation approaches "saturation" in much lower fields than does the induced

7 The *residual susceptibility*—or the ratio of the residual magnetisation to the strength of the pre-existing field—when the specimen is free from stress, attains a maximum when the pre-existing field is very considerably lower than that answering to the Wendepunkt.

8 The *retentiveness*—or the ratio of the residual to the induced magnetisation—in the absence of stress, attains a maximum in a field decidedly lower than that in which the *residual susceptibility* is a maximum

9. Stress increases largely the residual susceptibility and the retentiveness in weak fields, and decidedly lowers the fields in which these magnitudes present their maxima

§ 70. If the magnetisation answering to its Wendepunkt be taken as the unit for each individual specimen, the following Table gives approximately the induced magnetisations existing during certain critical events in the magnetisation of the cobalt rod, and three samples of stretched iron wire examined by Professor EWING.

TABLE XI.

	Maximum Retentiveness	Maximum residual susceptibility.	Critical point for cyclic effect	Critical point for first stress
Cobalt .	·40	·58	2 31	2 53
1st iron wire .	. .	. .	1 60	1 87
2nd " . .	·53	70		
3rd " .	55	70		

This will serve to show the kind of evidence on which the above statements as to the relations between cobalt and iron are based. Considering also that the intensities

of the magnetisations answering to the Wendepunkte of the stretched iron wires were on an average nearly thrice that answering to the Wendepunkt in the cobalt rod, that the wires were practically of infinite length, and that the critical fields observed in them answer, as a rule, to stresses considerably more severe than the normal stress applied to the cobalt, it seems not unlikely that by taking the position of the Wendepunkt in the absence of stress as a standard for comparison, it may be possible to trace general laws underlying the varied phenomena occurring in the different magnetic metals, and thus throw light on the material changes in which the process of magnetisation consists.

NOTE

[The referees to whom the paper was submitted having suggested the desirability of an examination into the chemical character of the rod, I add the following analysis, for which I am indebted to Mr. R. H. ADIE, of Trinity College, Cambridge, who kindly undertook the work:—

	Percentages
Cobalt	91·99
Nickel	4·36
Manganese	1·86
Iron	1·37
Carbon, &c.	·42
	100·00

The rod is thus not a pure specimen of cobalt. The principal impurity, however, is nickel, which closely resembles cobalt in its magnetic properties, though in general regarded as less strongly magnetic. The effects of pressure on the magnetisation of nickel are of the same general character as in the case of cobalt in fields below the critical, though probably more intense. One would thus expect the critical field to be slightly raised in consequence of the presence of the nickel.—April 16, 1890.]

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V. On the Theory of Free Stream Lines

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Communicated by Professor J. J. THOMSON, *F.R.S.*

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Introduction.

THE attention of mathematicians was first called to the subject of the present paper by a memoir of HELMHOLTZ'S in 1868, on "Discontinuous Fluid Motion."*

In discussing the steady motion of liquids past salient edges of fixed obstacles, it is found that the assumption of continuity of the motion leads to negative pressures in the liquid. HELMHOLTZ showed, in the paper above-mentioned, that some cases of this kind could be solved by assuming a surface of discontinuity, on one side of which the liquid is at rest, and he gave a mathematical solution of one case where the motion is in two dimensions.

The next advance in the subject was made by KIRCHHOFF who, in 1869, in a paper entitled "Zur Theorie freier Flussigkeitsstrahlen" in 'CRELLE'S Journal,' gave a generalization of the method which HELMHOLTZ had used, and obtained thereby the solution of three new interesting cases. Subsequently in his 'Vorlesungen uber mathematische Physik,' he published another method and worked out the same problems by means of it, but gave no new ones.

RAYLEIGH in the 'Philosophical Magazine,' December, 1876, discussed the solutions of KIRCHHOFF, and gave a drawing of the bounding free stream lines in one case.

As far as I know, these are the only investigations published on the mathematical side respecting a branch of hydrodynamics of great theoretical and practical interest.

In considering the method of transformation of polygons given independently by SCHWARZ and CHRISTOFFEL I have been led to a new transformation, which together with theirs, gives a general solution of the problem of free non-reentrant stream lines with plane rigid boundaries.

A considerable number of the cases of high interest prove to be of a tolerably simple nature, and I have worked out several in detail.

These problems occupy the first part of the paper. In the second part I have given some extensions of the transformation formulæ, which are applicable to problems of condensers and the form of hollow vortices in certain cases.

* 'Berlin Monatsberichte,' 1868, and 'Gesamm. Abhandl.' vol. 1.

The general Theory of Transformation

Let x, y be two conjugate functions with respect to the two variables ϕ, ψ, so that

$$x + iy = f(\phi + i\psi),$$

and write

$$x + iy = z \qquad x - iy = z',$$

$$\phi + i\psi = w \qquad \phi - i\psi = w'$$

x, y may be regarded as the rectangular coordinate of a point in a plane which we shall call the z plane, and similarly ϕ, ψ are the coordinates of a point in the w plane

Consider the functions

$$\mathrm{V} = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'},$$

$$\mathrm{W} = -i \log \frac{dz}{dw} \Big/ \frac{dz'}{dw'}.$$

Since they can be written in the form

$$\mathrm{V} = \log \frac{dz}{dw} + \log \frac{dz'}{dw'},$$

$$\mathrm{W} = -i \left[\log \frac{dz}{dw} - \log \frac{dz'}{dw'}\right],$$

they both satisfy LAPLACE's equation, and we have

$$\mathrm{V} + i\mathrm{W} = 2 \log \frac{dz}{dw},$$

so that V, W are conjugate functions with respect to x, y or ϕ, ψ

The transformations of the present paper will be deduced from the properties of the function V, so that its nature must be considered in detail. We have as alternative forms of V

$$\mathrm{V} = \log \frac{dz}{dw} \frac{dz'}{dw'} = \log \left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\} = -\log \left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\}$$

If the element of arc of ψ constant in the z plane be given by

$$ds_1 = \frac{d\phi}{h},$$

then

$$h^2 = \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2,$$

so that

$$\mathrm{V} = -\log h^2$$

And

$$\iota W = \log \frac{\dfrac{dx}{d\phi} + \iota \dfrac{dy}{d\phi}}{\dfrac{dx}{d\phi} - \iota \dfrac{dy}{d\phi}} = 2\iota\theta,$$

where θ is the angle the tangent to the curve ψ makes with the axis of x

[Mr BRILL has used the function W as a means of transformation ('Cambridge Phil Soc Proc,' vol. 6, and 'Messenger of Math,' August, 1889), and has thus anticipated me in one of the general theorems given in the latter part of this paper as I shall notice in the proper place I was not acquainted with his work when I developed the method here given]

Let k be the curvature of the curve ψ at ϕ. We have the well-known formula

$$k = \frac{dh}{d\psi}$$

which is

$$k = \frac{d}{d\psi} e^{-\frac{1}{2}V} \quad . \tag{1}$$

Now let the arc of the curve ψ be connected with k by the equation

$$s = f(k)$$

so that

$$\frac{ds}{d\phi} = \frac{d}{d\phi} f(k),$$

or from (1)

$$e^{\frac{1}{2}V} = \frac{d}{d\phi} f\left(\frac{d}{d\psi} e^{-\frac{1}{2}V}\right). \quad . \quad . \tag{2}$$

If ψ consist of parts of straight lines we have simply

$$\frac{dV}{d\psi} = 0 \tag{3}$$

The formula (2) suggests a general method for finding a transformation

$$z = f(\phi + \iota\psi),$$

such that ψ_0 is an arbitrary curve in the z plane

If the region within ψ_0 corresponds, point for point, to the part of the ω plane lying above $\psi = \psi_0$, the problem is reduced to finding a potential function V, which is continuous throughout the space bounded by a straight line, and such that

$$e^{\frac{1}{2}V} = \frac{d}{d\phi} f\left(\frac{d}{d\psi} e^{-\frac{1}{2}V}\right)$$

over that straight line

We have still to discuss the question of the singular points of V.

The function V will be finite and continuous for all points except where two branches of a ψ curve (or a ϕ curve) cut

At such a point dz/dw is either zero or infinite, and in either case V is infinite

It will be sufficient for our purpose to consider only the simplest singularity, that is, in which we have in the neighbourhood of a point $(\phi_0\ \psi_0)$ of this nature

$$V = n \log [(\phi - \phi_0)^2 + (\psi - \psi_0)^2] + C,$$

and therefore

$$\frac{dz}{dw} = A\ (w - w_0)^n,$$

where the value of n will depend on the nature of the singularity in question, and will be seen from the particular problems to which we proceed

PROBLEM I

To find the transformation $z = f(\phi + i\psi)$, which makes the area for which ψ is positive in the w plane correspond point for point to the area inside a given rectilinear polygon in the z plane (The problem of SCHWARZ* and CHRISTOFFEL†)

Consider the conditions which the function V must satisfy in the ω plane

(a) There are to be no singular points for ψ positive

(b) Along $\psi = 0$ we have $dV/d\psi = 0$

(c) At certain points of $\psi = 0$, which correspond to the angular points of the polygon in the z plane, we have V infinite

It is plain from this specification that the function V is (to a constant) merely the potential of masses at the singular points ϕ_1, ϕ_2 along $\psi = 0$, and, therefore,

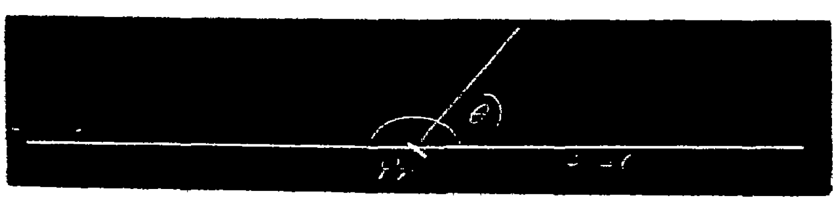
$$V = \log \Pi \{ (\phi - \phi_r)^2 + \psi^2 \}^{n_r} + C,$$

so that

$$\frac{dz}{dw} = \Pi A\ (w - \phi_r)^{n_r}$$

where Π is the product symbol

It remains to find the quantities n_r. Draw a small semicircle of radius R around the point ϕ_r,



* "Ueber einige Abbildungsaufgaben" ('CRELLE,' vol. 70, 1869)

† "Sul problema delle temperature stazionarie," &c. ('Annali di Matematica,' vol 1, 1867)

and on this semicircle let

$$w - \phi_r = R(\cos\theta + i\sin\theta),$$

so that

$$\frac{dz}{dw} = AR^{n_r}(\cos n_r\theta + i\sin n_r\theta).$$

Consequently, as we pass from $\theta = \pi$ to $\theta = 0$, dz/dw goes from

$$AR^{n_r}(\cos n_r\pi + i\sin n_r\pi)$$

to

$$AR^{n_r}.$$

The amplitude of dz/dw has, therefore, decreased by $n_r\pi$.

But the increase of amplitude is $\pi - \alpha_r$ where α_r is the internal angle of the polygon which corresponds to ϕ_r.

Therefore,

$$\pi - \alpha_r = - n_r\pi,$$

or

$$n_r = \frac{\alpha_r}{\pi} - 1,$$

so that the transformation becomes

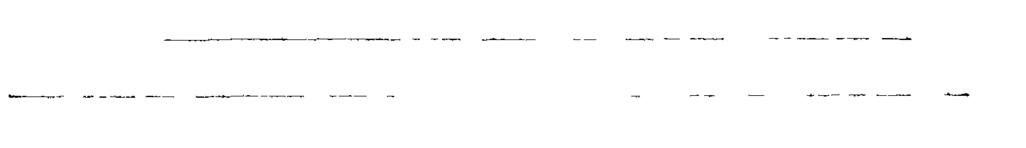
$$\frac{dz}{dw} = A\Pi(w - \phi_r)^{\alpha_r/\pi - 1},$$

which is the formula given by SCHWARZ and CHRISTOFFEL.

PROBLEM I.—SPECIAL CASE.

For the study of non-reentrant free stream lines we require a special case of this formula.

Suppose the polygon to consist of a series of straight lines infinite in one direction, all parallel to $y = 0$, so that the angles of the polygon are either 0 or 2π, and, therefore, $n_r = +1$ or -1.



Let ϕ_{1r} correspond to an angle 2π, and, therefore, to an end of a line within a finite distance of the origin, and let ϕ_{2r} correspond to an angle 0, and, therefore, to the adjacent ends of two lines at an infinite distance from the origin.

SCHWARZ's formula then becomes

$$\frac{dz}{dw} = \Pi A \frac{w - \phi_{1r}}{w - \phi_{2r}}$$

It is plain that there cannot be more than one factor more in the numerator than in the denominator, so that we can write

$$\frac{dz}{dw} = \mathrm{A}w + \mathrm{B} + \Sigma \frac{\mathrm{C}_r}{w - \phi_{2r}},$$

and, therefore, on integrating

$$z = \tfrac{1}{2}\mathrm{A}w^2 + \mathrm{B}w + \mathrm{D} + \Sigma \mathrm{C}_r \log(w - \phi_{2r}),$$

we may at once determine the distances between consecutive lines in terms of the ϕ_r, or in terms of the D_r.

For consider the passage of w around the small semicircle R above described.

We have

$$w - \phi_r = \mathrm{R}e^{i\theta},$$

therefore

$$z_1 - z_0 = \int_\pi^0 \frac{\mathrm{C}_r}{\mathrm{R}e^{i\theta}}\, d\,(\mathrm{R}e^{i\theta}),$$

$$= i\mathrm{C}_r \int_\pi^0 d\theta,$$

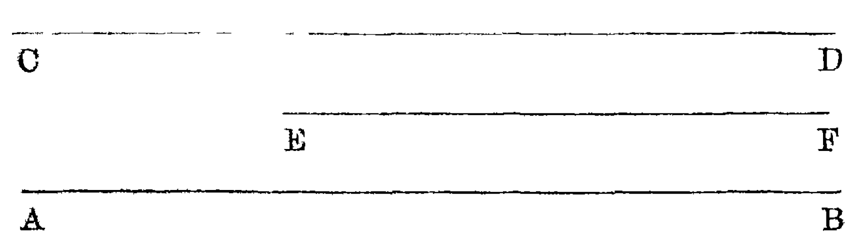
$$= -i\pi\mathrm{C}_r.$$

So that y increases by $-\pi\mathrm{C}_r$ in passing the point ϕ_{2r}, and, therefore, the distance between the parallel lines r and $r+1$ is $-\pi\mathrm{C}_r$, or, in terms of the ϕ_r, is

$$-\pi\mathrm{A}\Pi \frac{\phi_r - \phi_{1s}}{\phi_r - \phi_{2s}}.$$

After we have fixed on the angle which is to correspond to $\phi = \pm\infty$, we can in general choose the position of two other points ϕ_r, ϕ_s, and then the transformation-formula is determined to an additive constant.

For example, take the case of two doubly infinite lines AB, CD, with a semi-infinite line EF between them.



We may take the zero angle (A, C) to be $\phi = \pm\infty$, and the angles (B, F) (F, D) to be $\phi = -1$, $\phi = +1$. Then if the angle E is $\phi = c$, we have

$$\frac{dz}{dw} = \frac{\mathrm{A}(w - c)}{(w-1)(w+1)}.$$

Let the distance between AB and EF be d_1 and that between EF and CD d_2 Then integrating past the points $\omega = -1$, $\omega = 1$, we find

$$d_1 = -\pi \frac{A(1-c)}{2}$$

$$d_2 = \pi A(1+c)$$

which determine A and c, and, therefore, make the formula definite

Other examples will occur in the physical application.

PROBLEM II

The second transformation which we need may be stated most simply as an electrical problem

Let there be any number of infinitely long plane conductors, all in the same plane, and with parallel edges

It is required to find the potential at any point when these conductors are raised to given potentials.

A B C D E F

Let AB, CD, EF, be the sections of the conductors by the plane (xy)

Everything is symmetrical with regard to the line ABF which we take to be $y = 0$

Consider the specifications of the transformation-function V where ψ is the potential and ϕ are the lines of force

We must have $dV/d\psi = 0$ over the conductors, since they are straight, and. therefore, also $dV/dy = 0$

There will be infinite points at the edges A, B . of the conductors, and also at points in the field corresponding to branch points of ψ (or ϕ). These last will be distributed symmetrically with respect to $y = 0$

From these conditions it is plain that the solution is that V is the potential of masses at the singular points in question, so that we may write

$$V = \log \Pi \{(x - x_r)^2 + (y - y_r)^2\}^{n_r} + C,$$

and, therefore,

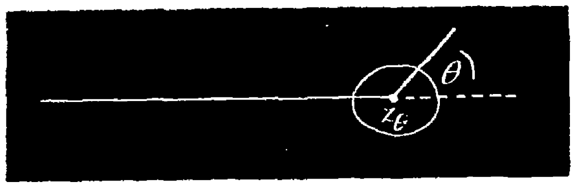
$$\frac{dz}{dw} = A\Pi (z - z_r)^{n_r}$$

It remains to find the quantities n_r.

For a singular point in the field where m branches of ψ meet we have simply

$$n_r = -m.$$

For an edge of the conductor we may proceed thus —



Let

$$z - z_0 = \mathrm{R}e^{i\theta},$$

then

$$(z - z_0)^n = \mathrm{R}^n (\cos n\theta + i \sin n\theta),$$

so that as θ goes from $-\pi$ to $+\pi$, $(z - z_0)^n$ goes from

$$\mathrm{R}^n [\cos(-n\pi) + i \sin(-n\pi)]$$

to

$$\mathrm{R}^n [\cos(n\pi) + i \sin(n\pi)],$$

and, therefore, the amplitude of dz/dw increases by $2n\pi$. Now the amplitude goes from 0 to π, therefore, $n = \frac{1}{2}$

Hence

$$\frac{dw}{dz} = \Pi \frac{(z - z_r)^{m_r}}{(z - x_s)^{\frac{1}{2}}},$$

where r refers to a point in the field and s to the edge of a conductor

If one of the conductors reduce to a line, we have two of the x_s equal, say $x_s = x_{s+1}$, and there is a factor $z - x_s$ in the denominator, and so for any number of line conductors

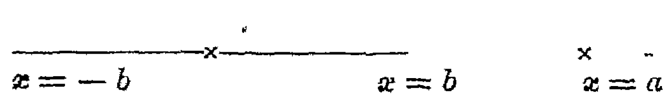
There are many other cases for which a formula like the above applies, it is not always necessary that the conductors should be in the same plane.

It is very easy to perceive such cases by considering whether the equation $d\mathrm{V}/dn = 0$ is satisfied over the conductors where dn is an element of a normal to a conductor

By combining this transformation with that of SCHWARZ and CHRISTOFFEL we get a general solution for the free-stream-line problem, as I shall presently show

It is necessary first to deduce some special formulæ, which will be continually used hereafter.

(a) Take first the case of one conductor and one line—



Let the conductor extend from $x = -b$ to $x = +b$ and the line distribution be at $x = a$.

Then the general formula reduces to

$$\frac{dw}{dz} = \frac{Ai}{(z-a)(z^2-b^2)^{\frac{1}{2}}}$$

where A is real, supposing that there is no singular point in the field

The nature of the multiplier Ai is obtained by considering that between

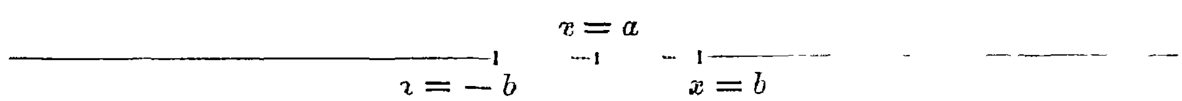
$$x = -b \text{ and } x = b \ \frac{d\psi}{dx} = 0.$$

The integral of this is

$$w = Ai \log \frac{az - b^2 + \sqrt{(a^2-b^2)}\sqrt{(z^2-b^2)}}{b(z-a)}$$

supposing the potential of the conductor is zero

(b) Now let the conductor be two semi-infinite planes, with a gap from $x = -b$ to $x = b$ between them, and let there be a line distribution at $x = a$ where $b > a > -b$



Then

$$\frac{dw}{dz} = \frac{A}{(z-a)(z^2-b^2)^{\frac{1}{2}}}$$

where A is real

And

$$w = Ai \log \frac{b^2 - az + \sqrt{(b^2-a^2)}\sqrt{(b^2-z^2)}}{b(z-a)},$$

the potential of the conductor being zero

(c) If we put $z = b + z'$ in the result of (a), and then make $b = \infty$, we get for the case of a semi-infinite conductor $x = 0$ to $x = -\infty$ with a line distribution at $x = a$,

$$w = Ai \log \frac{\sqrt{z} + \sqrt{a}}{\sqrt{z} - \sqrt{a}},$$

the potential of the conductor being 0.

These results (a), (b), (c), could of course be deduced from the known formulæ for elliptical conductors

On the Theory of Non-reentrant Free Stream Lines

The presence of sharp salient edges in a moving liquid always implies surfaces of discontinuity, but these may be closed or unclosed according to circumstances It would be difficult to give a rule as to the kind of motion in a given case, for an alteration in the relative sizes of the solids concerned will totally alter the character of the motion

As an illustration of this take the case of two parallel planes of finite breadth placed symmetrically one behind the other in a broad stream If the second plane be of less than a certain width, free stream lines will proceed from the edges of the first, and the second will be in still water

Now suppose the second plane broadened until it cuts the stream lines from the first plane

The character of the motion is changed Two vortices will appear behind the first plane, and in addition there will be free stream lines from the edge of the second plane

No method has yet been discovered which will give solutions of cases where there is motion on both sides of a surface of discontinuity. In the problems treated in the present paper there is always still water on one side of a free stream line

In the present section the motion considered is in two dimensions, the boundaries are plane, and the free stream lines are non-reentrant

Let x, y be the coordinates of a point in the liquid, ϕ, ψ the potential and stream functions respectively The region in the w plane corresponding to moving liquid in the z plane will be bounded by straight lines ψ, infinite in one direction at least and parallel to $\psi = 0$

The area in the w plane is therefore of the nature treated in Problem I. (a), that is, it is bounded by a polygon whose angles are alternately four right angles and zero.

This area, then, by means of Problem I. (a), may be transformed into the part of a new u plane in which q is positive, where $u \equiv p + iq$

In this u plane the boundaries of the liquid, both the plane boundaries and the free stream lines, are represented by the line $q = 0$

Let, as before,

$$V = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'} = -\log \left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 \right].$$

We have seen that V is a potential function, considered as a function of ϕ, ψ, and, therefore, it is also a potential function considered as a function of p, q, for ϕ, ψ are conjugate with respect to p, q.

Further, we have seen that along a straight boundary $\psi =$ constant we have $dV/d\psi = 0$, and, since all the straight boundaries correspond to portions of $q = 0$, we

must have $dV/dq = 0$ along all these portions Along a free stream line the pressure is constant, since it must be equal to the pressure on the liquid which is at rest

Now, in steady motion, we have BERNOULLI's pressure equation

$$\frac{p}{\rho} + \frac{1}{2}\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right] = \text{constant}$$

Therefore, along a free stream line $\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2$ is constant, and, therefore, V is constant

All the portions of $q = 0$ which do not correspond to plane boundaries correspond to free stream lines, for which V is constant.

Lastly $\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2$ is zero at reentrant angles of the boundary, and also at points where stream lines branch

For these points V is infinite and positive

In all the cases we shall consider, these infinite points will be along $q = 0$, so that a stream line only branches at the boundary

We have now reduced the problem to finding a function V which satisfies LAPLACE's equation, is finite and continuous in that half of the plane u for which q is positive, is constant along parts of $q = 0$, along the other parts satisfies $dV/dq = 0$; and at points along it is $+\infty$

This problem has an obvious solution

It is plain that V is merely the potential due to conductors coinciding with those parts of $q = 0$ for which V is constant, and having that constant as potential, together with masses at the points for which V is infinite.

The general solution of this has been given in Problem II

Let U be the conjugate of V, so that $U + iV = f(p + iq)$. Then, translating Problem II into the present notation we have

$$\frac{d(U + iV)}{d(p + iq)} = \Pi A \frac{(u - u_r)^{m_r}}{(u - p_s)^{\frac{1}{2}}}$$

and for a point mass two factors of the denominator coincide.

Write this

$$\frac{d(U + iV)}{du} = f(u),$$

so that

$$U + iV = \int f(u)\, du.$$

Now

$$V = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'},$$

therefore

$$U = i \log \left(\frac{dz}{dw} \Big/ \frac{dz'}{dw'}\right)$$

and

$$U + \iota V = 2\iota \log \frac{dz}{du}$$

Therefore

$$\frac{dz}{dw} = e^{-\iota 2\int f(u)\, du}$$

Now we have obtained the transformation from the w to the u plane in the form

$$\frac{dw}{du} = \phi(u) \qquad \text{[Problem I } (\alpha)].$$

Therefore

$$\frac{dz}{du} = \frac{dz}{dw}\,\frac{dw}{du} = \phi(u)\, e^{-\iota 2\int f(u)\, du},$$

which gives z as a function of u.

This is the general solution of the hydrodynamical problem before us

Near an angle of the boundary or a branching of a stream line we shall have

$$dz/du = A\,(u - u_0)^n$$

The determination of the index n rests on principles already used

If the internal angle of the boundary, or the angle between the two branches of the stream line be α, then

$$n = \frac{\alpha}{\pi} - 1$$

For example, if a stream line divide on a plane wall, $n = 0$, and the point of division is not a singular point for dz/du, although it is for dz/dw We may then lay down the following rule :—

Near a singular point

$$\frac{dz}{dw} = A\,(u - u_0)^{\frac{\alpha}{\pi} - 1}$$

where α is the internal angle of the boundary except when this point is a point of branching of a stream line, in which case

$$\frac{dz}{dw} = A\,(u - u_0)^{\frac{\alpha}{\pi} - 2},$$

for at that point

$$\frac{dz}{du} = A\,(u - u_0)^{\frac{\alpha}{\pi} - 1},$$

and

$$dw/du = B\,(u - u_0). \quad \text{[Problem I. } (\alpha)\text{]}.$$

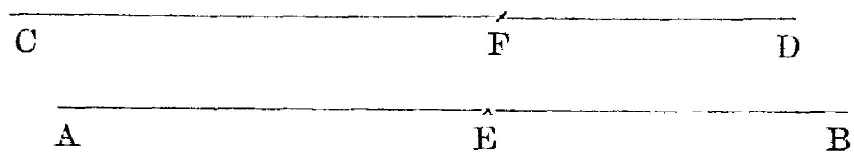
We shall now go on to consider such cases as are susceptible of tolerably simple treatment.

We shall suppose there are only two free stream lines, and throughout take the velocity along them to be 1, so that $V = 0$.

CASE I *A Single Jet from a Vessel*

There will be but two bounding stream lines, which we may take to be $\psi = 0$ $\psi = \pi$, both extending from $+\infty$ to $-\infty$

The diagram in the w plane consists merely of two parallel infinite straight lines AB, CD, at a distance π apart



A portion of each, say EB, FD, will correspond to the boundaries of the jet.

If now we transform to the u plane so that the ends D, B go to $p = \pm\infty$, we may choose the points $u = -1$, $u = 1$, to correspond to the edges of the aperture from which the jet issues

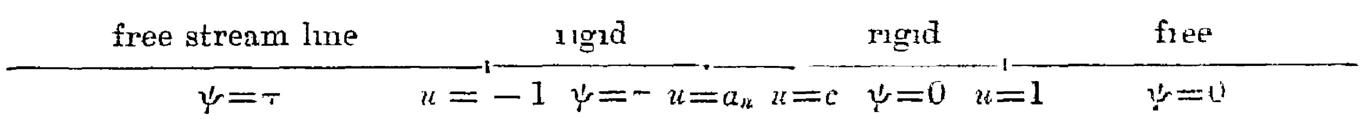
The point (C, A) will then be in a definite position, $u = c$, and the formula of transformation from w to u is

$$\frac{dw}{du} = \frac{A}{u - c}$$

Remembering that π is the distance between the two stream lines, we see that $A = 1$, and therefore

$$\frac{dw}{du} = \frac{1}{u - c}$$

Let $u = a_n$ correspond to an angle α_n of the vessel, then the points along $q = 0$ are arranged in the manner of the figure



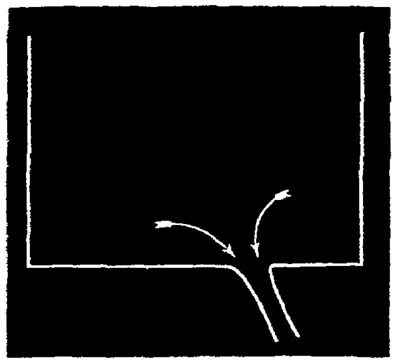
The appropriate formula is then that of Problem II (b), viz,

$$\frac{dz}{dw} = \Pi\left[\frac{1 - a_n u + \sqrt{(1 - a_n^2)}\sqrt{(1 - u^2)}}{u - a_n}\right]^{1 - \frac{\alpha_n}{\pi}}$$

and therefore

$$\frac{dz}{du} = \frac{1}{u - c}\,\Pi\left[\frac{1 - a_n u + \sqrt{(1 - a_n^2)}\sqrt{(1 - u^2)}}{u - a_n}\right]^{1 - \frac{\alpha_n}{\pi}}$$

Example I—A rectangular vessel of given width has an aperture in the bottom.



Here there are only two angles, each of which $= \frac{1}{2}\pi$, and therefore $1 - \alpha_n/\pi = \frac{1}{2}$ Let the two angles be $u = a$, $u = b$, where $1 > b > c > a > -1$

Then, observing that

$$\{1 - au + \sqrt{1-a^2}\sqrt{1-u^2}\} = \{\sqrt{\tfrac{1}{2}(1-a)}\sqrt{1+u} + \sqrt{\tfrac{1}{2}(1+a)}\sqrt{1-u}\}^2$$

we get

$$\frac{dz}{du} = \frac{1}{u-c}\,\frac{\{\sqrt{\frac{1}{2}(1-a)}\sqrt{1+u} + \sqrt{\frac{1}{2}(1+a)}\sqrt{1-u}\}\{\sqrt{\frac{1}{2}(1-b)}\sqrt{1+u} + \sqrt{\frac{1}{2}(1+b)}\sqrt{1-u}\}}{\sqrt{(u-a)(u-b)}}$$

or

$$\frac{dz}{du} = \frac{Au + B + C\sqrt{(1-u^2)}}{(u-c)\sqrt{(u-a)(u-b)}}$$

Where

$$A = \sqrt{\tfrac{1}{4}(1-a)(1-b)} - \sqrt{\tfrac{1}{4}(1+a)(1+b)},$$

$$B = \sqrt{\tfrac{1}{4}(1-a)(1-b)} + \sqrt{\tfrac{1}{4}(1+a)(1+b)},$$

$$C = \sqrt{\tfrac{1}{4}(1+a)(1-b)} + \sqrt{\tfrac{1}{4}(1-a)(1+b)}$$

Between $u = -1$ and $u = a$, that is along the bottom of the vessel from the edge of the aperture to the angle on the right, we have

$$\frac{dy}{dp} = 0,$$

$$\frac{dx}{dp} = \frac{Ap + B + C\sqrt{(1-p^2)}}{(p-c)\sqrt{(p-a)(p-b)}},$$

therefore the distance between the two points is

$$l_1 = \int_{-1}^{a} \frac{Ap + B + C\sqrt{(1-p^2)}}{(c-p)\sqrt{(p-a)(p-b)}}\,dp$$

Similarly the other piece of the bottom of the vessel is of length

$$l_2 = \int_b^1 \frac{Ap + B + C\sqrt{(1 - p^2)}}{(p - c)\sqrt{(p - a)(p - b)}}\, dp$$

When $\phi = -\infty$, that is, when $u = c$

$$\frac{dz}{dw} = -i\frac{Ac + B + C\sqrt{(1 - c^2)}}{\sqrt{(c - a)(b - c)}}$$

Therefore the velocity in the vessel at a distance from the aperture is

$$\frac{\sqrt{(c - a)(b - c)}}{Ac + B + C\sqrt{(1 - c^2)}},$$

and therefore the breadth of the vessel is

$$\pi\frac{Ac + B + C\sqrt{(1 - c^2)}}{\sqrt{(c - a)(b - c)}} = d \text{ (say)}$$

The breadth of the aperture is

$$d - l_1 - l_2$$

and the breadth of the jet is ultimately π

The question is now reduced to a matter of the integration of l_1, l_2

For the general case elliptic integrals occur, and the expressions for our purpose may as well be left in their present form If, however, the aperture be in the centre of the vessel, the integrals will work out, and we get a simple expression for the contraction of the jet To this we now proceed

Sub-Example I—Jet from an aperture in the centre of the bottom of a rectangular vessel

In this case we may take the angles of the vessel at $u = -a$, $u = a$, $a < 1$

The expression for dz/du then reduces to

$$\frac{dz}{du} = \frac{1}{u}\frac{\sqrt{(1 - a^2)} + \sqrt{(1 - u^2)}}{\sqrt{u^2 - a^2}}.$$

We now have

$$l_2 = l_1 = \int_a^1 \frac{1}{p}\frac{\sqrt{(1 - a^2)} + \sqrt{(1 - p^2)}}{\sqrt{p^2 - a^2}}\, dp.$$

Now

$$\int_a^1 \frac{1}{p}\frac{1}{\sqrt{(p^2 - a^2)}}\, dp = -\int_{\frac{1}{a}}^1 \frac{d\lambda}{\sqrt{(1 - a^2\lambda^2)}}\left(\lambda = \frac{1}{p}\right),$$

$$= -\frac{1}{a}\left[\sin^{-1} a\lambda\right]_{\frac{1}{a}}^1,$$

$$\frac{1}{a}\left[\frac{\pi}{2} \quad \sin^{-1} a\right],$$

and

$$\int_a^1 \frac{\sqrt{(1-p^2)}}{p\sqrt{(p^2-a^2)}}\,dp = -\int \frac{r^2 dr}{(1-r^2)\sqrt{(1-a^2-r^2)}} \quad (p^2 = 1 - r^2)$$

$$= \int \frac{dr}{\sqrt{(1-a^2-r^2)}} - \int \frac{dr}{(1-r^2)\sqrt{(1-a^2-r^2)}}$$

$$= \left[\sin^{-1}\frac{r}{\sqrt{(1-a^2)}} - \frac{1}{a}\tan^{-1}\frac{ar}{\sqrt{(1-a^2-r^2)}}\right]_{\sqrt{(1-a^2)}}^{0}$$

$$= \frac{\pi}{2}\frac{1-a}{a},$$

therefore

$$l_2 = l_1 = \frac{\pi}{2}\frac{1-a}{a} + \frac{\sqrt{(1-a^2)}}{a}\left[\frac{\pi}{2} - \sin^{-1} a\right]$$

When $u = 0$ that is, at $\phi = -\infty$, we have

$$\frac{dz}{dw} = -i\frac{\sqrt{(1-a^2)}+1}{a}.$$

Therefore the velocity in the vessel at a distance from the aperture is

$$\frac{a}{1+\sqrt{(1-a^2)}}$$

Therefore the breadth of the vessel is

$$\pi\frac{1+\sqrt{(1-a^2)}}{a} = d \text{ (say)}$$

So that the breadth of the aperture is

$$d - 2l_1 = \pi + 2\frac{\sqrt{(1-a^2)}}{a}\sin^{-1} a.$$

Hence, since the final breadth of the jet is π, we have as the ratio of the breadth of the jet to that of the aperture

$$\frac{\pi}{\pi + 2\frac{\sqrt{(1-a^2)}}{a}\sin^{-1} a}$$

Now

$$\frac{1+\sqrt{(1-a^2)}}{a} = \frac{d}{\pi},$$

therefore

$$\frac{1-\sqrt{(1-a^2)}}{a} = \frac{\pi}{d},$$

and

$$2\frac{\sqrt{(1-a^2)}}{a} = \frac{d}{\pi} - \frac{\pi}{d}.$$

Also

$$\sin^{-1} a = \tan^{-1}\frac{a}{\sqrt{(1-a^2)}} = \tan^{-1}\frac{2\pi d}{d^2-\pi^2}.$$

Therefore the contraction ratio is

$$\frac{\pi}{\pi + \left(\frac{d}{\pi} - \frac{\pi}{d}\right)\tan^{-1}\frac{2\pi d}{d^2-\pi^2}}.$$

Now, getting rid of the special units let d be the breadth of the vessel as before, l the breadth of the aperture, and c that of the jet, then

$$l = c\left[1 + \frac{1}{\pi}\left(\frac{d}{c} - \frac{c}{d}\right)\tan^{-1}\frac{2dc}{d^2-c^2}\right]$$

If d be very large compared with l, we get

$$l = c\left[1 + \frac{1}{\pi}\operatorname{Lim}\frac{d}{c}\tan^{-1} 2\frac{c}{d}\right] = c\,\frac{\pi+2}{\pi}.$$

This is the result obtained by RAYLEIGH* from KIRCHHOFF's solution for the case of an aperture in an infinite plane bounding wall

As d/l decreases from infinity the contraction also continually decreases, until when $d = l$ the contraction is zero

In order to get some idea of how soon the finite breadth of the vessel affects the contraction ratio perceptibly, consider the case when

$$d^2 - c^2 = 2cd$$

or

$$d = (1+\sqrt{2})\,c.$$

The contraction ratio is then

$$\frac{1}{1+\frac{2}{\pi}\,\frac{\pi}{4}} = \tfrac{2}{3}$$

and

$$d = (1+\sqrt{2})\,\tfrac{2}{3}l,$$

so that the finiteness of the vessel has very little effect on the jet if the breadth is more than twice that of the aperture.

The equations to the free stream line $\psi = \pi$ are

$$\left.\begin{aligned}\frac{dx}{dp} &= -\sqrt{(1-a^2)}\,\frac{1}{p}\,\frac{1}{\sqrt{(p^2-a^2)}}\\ \frac{dy}{dp} &= -\frac{1}{p}\,\frac{\sqrt{(p^2-1)}}{\sqrt{(p^2-a^2)}}\end{aligned}\right\}$$

* 'Phil. Mag.,' Dec., 1876.

The former gives

$$x = \frac{\pi}{2} - \frac{\sqrt{(1-a^2)}}{a}\sin^{-1}\frac{a}{p}.$$

Now

$$p = -e^{\phi} = -e^{s},$$

where s is the arc Therefore

$$x = \frac{\pi}{2} + \frac{\sqrt{(1-a^2)}}{a}\sin^{-1} ae^{-s}$$

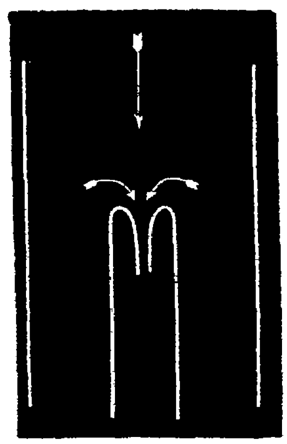
When $a = 0$ we have

$$x = \frac{\pi}{2} + e^{-s},$$

which is the form given by RAYLEIGH ('Phil. Mag,' 1876) for KIRCHHOFF'S case.

Example II—Tube projecting far into the bottom of a vessel of given breadth.

The tube is supposed to project so far into the vessel that the motion at its bottom may be neglected.



There are here two singular points only, $u = -a$, $u = a$, corresponding to the coalescing angles of the bottom of the vessel

The appropriate transformation is, therefore,

$$\frac{dz}{dw} = \mathrm{A}\,\frac{1 - au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u-a}\cdot\frac{1 + au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u+a}$$

$$= \mathrm{A}\,\frac{2 - a^2 - u^2 + 2\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u^2 - a^2}.$$

Since, when $u = \infty$, $dz/dw = -i$, we have $\mathrm{A} = i$

So that

$$\frac{dz}{dw} = i\,\frac{2 - a^2 - u^2 + 2\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u^2 - a^2}$$

and

$$\frac{dw}{du} = \frac{1}{u} \text{ as before},$$

therefore

$$\frac{dz}{du} = \frac{i}{u}\,\frac{2 - a^2 - u^2 + 2\sqrt{(1 - a^2)}\sqrt{(1 - u^2)}}{u^2 - a^2}\,.$$

When $u = 0$, that is, at a distance upwards from the tube,

$$\frac{dz}{dw} = -\frac{i}{a^2}\left(2 - a^2 - 2\sqrt{1 - a^2}\right)$$

therefore the breadth of the vessel is

$$\frac{\pi a^2}{2 - a^2 - 2\sqrt{(1 - a^2)}} = d \text{ (say)}$$

In passing over $u = -a$, x increases by

$$2\frac{\pi}{a^2}(1 - a^2)$$

therefore the breadth of the tube is

$$\frac{\pi a^2}{2 - a^2 - 2\sqrt{(1 - a^2)}} - \frac{4\pi}{a^2}(1 - a^2) = \frac{\pi}{a^2}\left\{3a^2 - 2 + 2\sqrt{(1 - a^2)}\right\} = k \text{ (say)}$$

Hence

$$d - k = \frac{4\pi}{a^2}(1 - a^2),$$

therefore

$$\frac{d - k + 4\pi}{4\pi} = \frac{1}{a^2}$$

Substituting in the value of d we have

$$d + \pi = \pi\frac{d - k + 4\pi}{2\pi} + \sqrt{\left(\frac{d - k}{d - k + 4\pi}\right)}\ \frac{d - k + 4\pi}{2}$$

or

$$d + k - 2\pi = \sqrt{\{(d - k)(d - k + 4\pi)\}}$$

On squaring this gives

$$dk - 2\pi d + \pi^2 = 0\,,$$

or, getting rid of the special unit of length, if c be the breadth of the jet, we have

$$dk - 2cd + c^2 = 0\,.$$

or

$$(d - c)^2 = d(d - k)\,.$$

When d is very great we get

$$k = 2c\,,$$

and the ratio of contraction is $\frac{1}{2}$.

This particular case was the first solution of free stream lines given, and by HELMHOLTZ

Lord RAYLEIGH* has given the equation

$$\frac{2}{k} = \frac{1}{c} + \frac{1}{d}$$

for the coefficient of contraction when there is a tube projecting inwards in a vessel of finite breadth

The assumption made is, however, not that of this example, for he has taken the velocity along the bottom of the vessel to be the same as that at a distance upwards from the tube

This is one of the very few cases in which the contraction can be determined accurately from elementary principles

It is scarcely worth while to put the proof of this down, but it is worth remarking that the corresponding case in three dimensions, can also be worked out, viz, the case of one circular cylinder projecting far into another

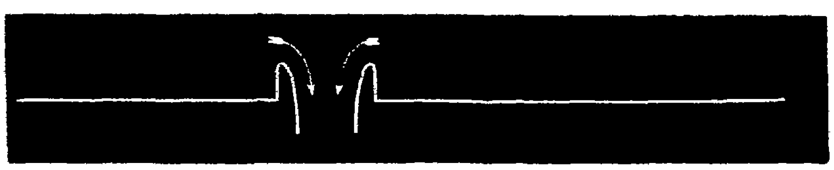
Let r_1, r_2 be the radii of the cylinders $r_1 > r_2$, and r_3 the radius of the jet, then

$$r_1^2 r_2^2 - 2 r_1^2 r_3^2 + r_3^4 = 0$$

This includes the case first noticed, I believe, by BORDA,† of a cylindrical tube in an infinite plane wall, the proof being just the same

Example III—Tube projecting into a vessel of great breadth

If we take the case of a tube projecting into a vessel of great breadth, we do not get such simple expressions



The transformation to the u plane is the same as before, but there is now a singular point at $u = 0$, corresponding to $\phi = -\infty$ We arrive at the correct result by first supposing the vessel of finite breadth, so that there are points at $u = -a$, $u = -b$, $u = b$, $u = a$. for each of which $\alpha/\pi - 1 = -\frac{1}{2}$, and then making b vanish

We have, therefore,

$$\frac{dz}{dw} = \frac{i}{\sqrt{(u^2 - a^2)}} \{\sqrt{(1 - a^2)} + \sqrt{(1 - u^2)}\} \frac{1 + \sqrt{(1 - u^2)}}{u},$$

and

$$\frac{dz}{du} = \frac{i}{u^2\sqrt{(u^2 - a^2)}} \{\sqrt{(1 - a^2)} + 1 - u^2 + (1 + \sqrt{1 - a^2}) \sqrt{(1 - u^2)}\}$$

* "The Contracted Vein," 'Phil Mag,' Dec. 1876

† 'Mém. de l'Acad,' 1766, Paris I owe this reference to the kindness of Lord RAYLEIGH

Between $u = a$ and $u = 1$ we have

$$\frac{dx}{dp} = 0$$

$$\frac{dy}{dp} = \frac{1}{p^2\sqrt{(p^2-a^2)}}\{\sqrt{(1-a^2)} + 1 - p^2 + \sqrt{(1-p^2)}(1+\sqrt{1-a^2})\},$$

therefore the length of the pipe is

$$(1+\sqrt{1-a^2})\int_a^1 \frac{dp}{p^2\sqrt{(p^2-a^2)}} - \int_a^1 \frac{dp}{\sqrt{(p^2-a^2)}} + (1+\sqrt{1-a^2})\int_a^1 \frac{\sqrt{(1-p^2)}}{p^2\sqrt{(p^2-a^2)}}\,dp$$

Now

$$\int_a^1 \frac{dp}{p^2\sqrt{(p^2-a^2)}} = \frac{1}{a^2}[\sin\theta]_0^{\cos^{-1}a} \qquad (p = a\sec\theta) = \frac{1}{a^2}\sqrt{(1-a^2)}$$

$$\int_a^1 \frac{dp}{\sqrt{(p^2-a^2)}} = \log\frac{1+\sqrt{(1-a^2)}}{a} = \cosh^{-1}\frac{1}{a}$$

$$\int_a^1 \frac{\sqrt{(1-p^2)}\,dp}{p^2\sqrt{(p^2-a^2)}} = \mathrm{L}(a)\ \text{(say)},$$

an elliptic integral which will be brought to the standard form below

Therefore the length of the pipe is

$$\frac{(1+b)\,b}{a^2} - \log\frac{1+b}{a} + (1+b)\,\mathrm{L}(a),$$

where

$$b \equiv \sqrt{(1-a^2)}.$$

From

$$u = 1 \text{ to } u = \infty$$

we have

$$\int_1^\infty \frac{dx}{dp}\,dp = (1+b)\int_1^\infty \frac{\sqrt{(p^2-1)}}{p^2\sqrt{(p^2-a^2)}}\,dp = (1+b)\,\mathrm{M}(a)\ \text{(say)},$$

where M is another elliptic integral to be reduced below.

Summing up, the breadth of jet is π, that of the aperture is

$$\pi + 2(1+\sqrt{1-a^2})\,\mathrm{M}(a),$$

so that the contraction ratio is

$$\frac{\pi}{\pi + 2(1+\sqrt{1-a^2})\,\mathrm{M}(a)},$$

where the length of the pipe is

$$\frac{1+\sqrt{(1-a^2)}}{a}\sqrt{(1-a^2)} - \log\frac{1+\sqrt{(1-a^2)}}{a} + (1+\sqrt{1-a^2})\,\mathrm{L}(a).$$

To reduce the elliptic integrals, we have

$$\int_a^1 \frac{1}{p} \frac{\sqrt{(1-p^2)}}{\sqrt{(p^2-a^2)}}\, dp = \int_1^{1/a} \frac{\sqrt{(r^2-1)}}{\sqrt{(1-a^2r^2)}}\, dr \qquad rp = 1$$

$$= \int_1^{1/a} \frac{r^2-1}{\sqrt{(r^2-1\cdot 1-a^2r^2)}}$$

Now (CAYLEY, 'Elliptic Functions,' p 315)

$$\frac{dr}{\sqrt{(r^2-1\cdot 1-a^2r^2)}} = \frac{dx}{\sqrt{(1-x^2\quad 1-k^2x^2)}} \text{ where } k^2 = \frac{1-a^2}{a^2} \text{ and } r^2 = \frac{1}{1-(1-a^2)x^2}$$

Thus

$$\mathrm{L}(a) = \int_0^1 \left[\frac{1}{1-(1-a^2)x^2} - 1\right] \frac{dx}{\sqrt{(1-x^2\quad 1-k^2x^2)}}$$

$$= \Pi_1(k,\ a^2-1) - \mathrm{F}_1(k),$$

where Π and F are the third and first elliptic integrals And

$$\mathrm{M}(a) = \int_1^\infty \frac{\sqrt{(p^2-1)}}{p^2\sqrt{(p^2-a^2)}}\, dp = \int_0^1 \frac{\sqrt{(1-r^2)}}{\sqrt{(1-a^2r^2)}}\, dr$$

$$= \left(1-\frac{1}{a^2}\right)\int_0^1 \frac{dr}{\sqrt{(1-r^2\cdot 1-a^2r^2)}} + \frac{1}{a^2}\int_0^1 \frac{dr\sqrt{(1-a^2r^2)}}{\sqrt{(1-r^2)}}$$

$$= \left(1-\frac{1}{a^2}\right)\mathrm{F}_1(a) + \frac{1}{a^2}\mathrm{E}_1(a),$$

where E is the second elliptic integral.

Suppose now the length of the pipe is small, so that a is nearly unity.

We have

$$\mathrm{L}(a) = \int_a^1 \frac{\sqrt{(1-p^2)}}{p^2\sqrt{(p^2-a^2)}}\, dp$$

If we put $p^2 = x$, $2pdp = dx$, we may put the factor p in the integral equal to unity, and so get

$$\mathrm{L}(a) = \tfrac{1}{2}\int_{a^2}^1 \frac{\sqrt{(1-x)}}{\sqrt{(x-a^2)}}\, dx$$

$$= \tfrac{1}{4}\pi(1-a^2)$$

So that the length of pipe becomes

$$l = \left(\frac{\pi}{4}+1\right)(1-a^2) \quad \text{to the first order;}$$

while the aperture becomes

$$l = \pi + 2\left(1 + \sqrt{1 - a^2}\right)\left[\left(1 - \frac{1}{a^2}\right)\log\frac{4}{\sqrt{(1-a)}} + 1\right]$$
$$= \pi + 2 + 2\sqrt{(1 - a^2)}$$

to the first order, and the breadth of jet is $c = \pi$.

So that

$$\frac{l}{k} = \frac{\pi + 4}{4(\pi + 2)}(1 - a^2),$$

and

$$\frac{c}{k} = \frac{\pi}{\pi + 2}\left(1 - \frac{2}{\pi + 2}\sqrt{1 - a^2}\right),$$

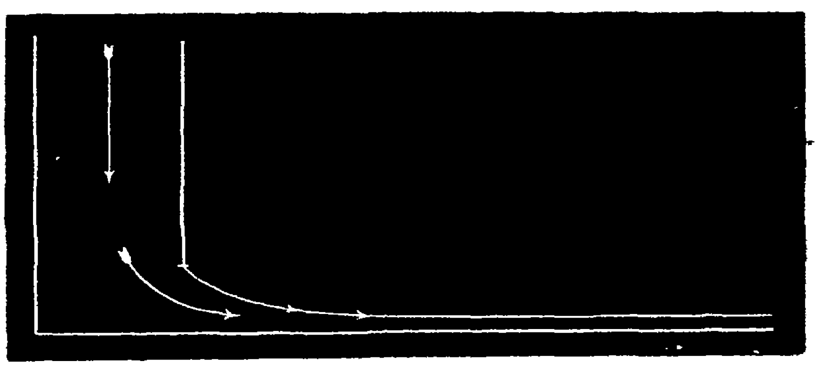
therefore

$$\frac{c}{l} = \frac{\pi}{\pi + 2}\left(1 - \frac{4}{\pi + 2}\sqrt{\frac{\pi + 2}{\pi + 4}\frac{l}{k}}\right),$$

which is the approximate expression required.

Example IV.— Flow from an aperture in a pipe in which the water is at rest

By considering the symmetry of the motion this can be reduced to the case of flow from a rectangular vessel in which the aperture extends from the bottom up, and the bottom is continued into a horizontal plane, as in the figure



The w diagram consists of two doubly infinite straight lines, $\psi = 0$, $\psi = \pi$, as before

But if in the u plane we make $u = -\infty$ correspond to $\phi = \infty$, the arrangement of points is not the same as before and the notation must be altered

We take, then, $u = 0$ for the edge of the aperture, $u = 1$ the right angle of the vessel, and then put $u = a$ for $\phi = -\infty$.

We then have

$$\frac{dw}{du} = \frac{1}{u - a}$$

and the arrangement of points on $q = 0$ is as in the figure.

free rigid rigid rigid

$\psi = \pi$ $u = 0$ $\psi = \pi$ $u = a$ $\psi = 0$ $u = 1$

The only singular point for dz/dw is at $u = 0$, and, therefore, from Problem II (c),

$$\frac{dz}{dw} = \left(\frac{\sqrt{u} + 1}{\sqrt{u} - 1}\right)^{\frac{1}{2}}$$

$$\frac{dz}{du} = \frac{1}{u - a}\left(\frac{\sqrt{u} + 1}{\sqrt{u} - 1}\right)^{\frac{1}{2}}$$

Along the free stream line

$$\frac{dz}{dp} = \frac{1}{p - a}\left(\frac{p + 1 + 2i\sqrt{(-p)}}{p - 1}\right)^{\frac{1}{2}}$$

where p lies between 0 and $-\infty$

If

$$\frac{dy}{dx} = \tan\psi$$

we have

$$\frac{p + 1}{p - 1} = \cos 2\psi$$

$$\frac{2\sqrt{(-p}}{p - 1} = \sin 2\psi$$

$$p = -\cot^2\psi$$

Therefore

$$\frac{dx}{d\psi} = 2\,\frac{\cos^2\psi\,\operatorname{cosec}^3\psi}{\cot^2\psi + a}$$

$$\frac{dy}{d\psi} = 2\,\frac{\cos\psi\,\operatorname{cosec}^2\psi}{\cot^2\psi + a}.$$

If we put $\cos\psi = \lambda$ in the first we get

$$\begin{aligned}
x &= -2\int\frac{\lambda^2}{1 - \lambda^2}\,\frac{1}{a + (1 - a)\lambda^2}\,d\lambda,\\
&= -2\int\frac{d\lambda}{1 - \lambda^2} - 2\,\frac{a}{1 - 2a}\int\frac{d\lambda}{a + (1 - a)\lambda^2},\\
&= 2\log\frac{1 - \lambda}{1 + \lambda} - 2\,\frac{\sqrt{a}}{1 - 2a}\,\frac{1}{\sqrt{(1 - a)}}\tan^{-1}\frac{\lambda}{\sqrt{a}}\sqrt{(1 - a)} + \mathrm{C},
\end{aligned}$$

and, in the second, putting

$$\sin\psi = \mu$$

we have

$$\begin{aligned}
y &= 2\int\frac{d\mu}{1 - (1 - a)\mu^2},\\
&= -\frac{1}{\sqrt{(1 - a)}}\log\frac{1 - \sqrt{(1 - a)}\,\mu}{1 + \sqrt{(1 - a)}\,\mu} + \pi.
\end{aligned}$$

Hence when $\psi = \frac{1}{2}\pi$, i e, at the edge of the aperture

$$y = \frac{1}{\sqrt{(1-a)}} \log \frac{1+\sqrt{(1-a)}}{1-\sqrt{(1-a)}} + \pi,$$

therefore the ratio of the jet to the aperture is

$$\frac{c}{l} = \frac{\pi}{\pi + \dfrac{1}{\sqrt{(1-a)}} \log \dfrac{1+\sqrt{(1-a)}}{1-\sqrt{(1-a)}}}.$$

The velocity at $\phi = -\infty$ is

$$\left(\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)^{\frac{1}{2}},$$

therefore the width of the vessel is

$$d = \pi \left(\frac{1+\sqrt{a}}{1-\sqrt{a}}\right)^{\frac{1}{2}},$$

therefore

$$\frac{d}{c} = \left(\frac{1+\sqrt{a}}{1-\sqrt{a}}\right)^{\frac{1}{2}}$$

From which we obtain

$$\frac{1+\sqrt{a}}{2} = \frac{1}{\dfrac{c^2}{d^2}+1},$$

$$\frac{1-\sqrt{a}}{2} = \frac{1}{\dfrac{d^2}{c^2}+1},$$

therefore

$$k = c\left[1 + \frac{1}{2\pi}\left(\frac{c}{d}+\frac{d}{c}\right)\log\frac{\frac{1}{2}\left(\dfrac{c}{d}+\dfrac{d}{c}\right)+1}{\frac{1}{2}\left(\dfrac{c}{d}+\dfrac{d}{c}\right)-1}\right],$$

or

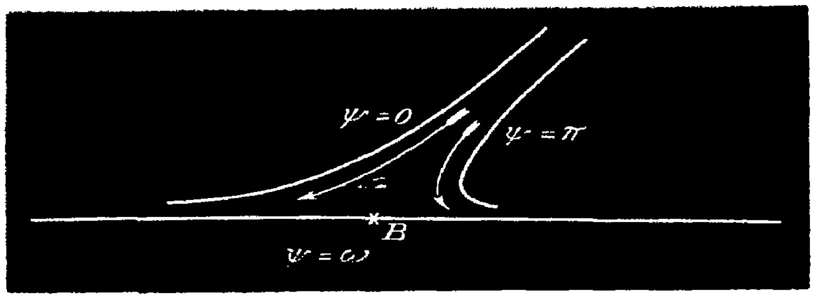
$$k = c\left[1 + \frac{1}{\pi}\left(\frac{d}{c}+\frac{c}{d}\right)\log\frac{d+c}{d-c}\right]$$

This is for the vessel considered For the pipe

$$k = c\left[1 + \frac{1}{\pi}\left(\frac{2d}{c}+\frac{c}{2d}\right)\log\frac{2d+c}{2d-c}\right].$$

Case II.—*Impact of a stream against a plane*

A stream of given breadth impinges at a given angle against a plane.



The impinging stream is bounded by the stream lines $\psi = 0$, $\psi = \pi$

The stream line which branches at the point B on the plane is $\psi = \varpi$

The diagram in the w plane consists of two infinite straight lines with a semi-infinite one between them, as in the figure.

$\psi = \pi$

$\psi = \varpi$

$\psi = 0$

In transforming to the u plane we suppose that $\phi = -\infty$ corresponds to $u = \pm\infty$, and that $u = -1$, $u = 1$ are the extremities of the plane

We must then take $u = a$, an unknown constant for the point B, where the stream line $\psi = \varpi$ divides, and observe that $a < 1 > -1$

We then have

$$\frac{dw}{du} = \mathrm{A}\,\frac{u-a}{(u-1)(u+1)}$$

where

$$\left.\begin{aligned} -\pi\mathrm{A}\,\frac{1+a}{2} &= \varpi \\ -\pi\mathrm{A}\,\frac{1-a}{2} &= \pi - \varpi \end{aligned}\right\},$$

therefore

$$\mathrm{A} = -1 \quad \text{and} \quad a = \frac{2\varpi}{\pi} - 1.$$

Along $q = 0$ we have, therefore, the following arrangement of points —

free			rigid		rigid	free	
$\phi = -\infty$	$\psi = 0$	$u = -1$	$\psi = \varpi$	$u = a$	$u = 1$	$\psi = \pi$	$\phi = -\infty$

The only singular point of dz/dw is at $u = a$

Hence

$$\frac{dz}{dw} = \frac{1 - au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u - a},$$

and, therefore,

$$\frac{dz}{du} = -\frac{1 - au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{(u-1)(u+1)}$$

$$= \frac{1 - au}{1 - u^2} + \sqrt{(1-a^2)}\frac{1}{\sqrt{(1-u^2)}}$$

When $u = \pm\infty$, we have

$$\frac{dx}{d\phi} + \imath\frac{dy}{d\phi} = -a - \imath\sqrt{(1-a^2)}.$$

Therefore, if θ be the inclination of the stream to the plane, we have

$$\tan\theta = \frac{\sqrt{(1-a^2)}}{a} \quad \text{or} \quad \cos\theta = a.$$

Since we have before found

$$a = \frac{2\varpi}{\pi} - 1,$$

this gives

$$\pi\cos\theta = 2\varpi - \pi$$

This equation merely expresses that the momentum parallel to the plane is unaltered by impact

Along the stream line $\psi = \pi$ we have

$$\left.\begin{aligned} \frac{dx}{dp} &= -\frac{1 - ap}{p^2 - 1} \\ \frac{dy}{dp} &= \sqrt{(1-a^2)}\frac{1}{\sqrt{(p^2-1)}}, \end{aligned}\right\}$$

therefore

$$x = \frac{a}{2}\log(p^2 - 1) + \tfrac{1}{2}\log\frac{p+1}{p-1} + \mathrm{A}$$

$$\left.\begin{aligned} &= \frac{1+a}{2}\log(p+1) - \frac{1-a}{2}\log(p-1) + \mathrm{A} \\ y &= \sqrt{(1-a^2)}\log\left(p + \sqrt{p^2-1}\right) + \pi - \varpi. \end{aligned}\right\}$$

where p lies between $+1$ and ∞; and for $\psi = 0$ we have

$$\frac{dx}{dp} = -\frac{1 - ap}{p^2 - 1} \qquad \frac{dy}{dp} = -\sqrt{(1-a^2)}\frac{1}{\sqrt{(p^2-1)}},$$

therefore

$$\left.\begin{aligned} x &= \frac{1+a}{2}\log -(1+p) - \frac{1-a}{2}\log(1-p) + \mathrm{A}' \\ y &= \sqrt{(1-a^2)}\log\left(\sqrt{p^2-1}-p\right) + \varpi \end{aligned}\right\}$$

where p lies between -1 and $-\infty$

If we put $p = \frac{1}{\cos\theta}$ in the first and $p = -\frac{1}{\cos\theta}$ in the second, we get for $\psi = \pi$

$$\left.\begin{aligned} x_1 &= (1+a)\log\cos\frac{\theta}{2} - (1-a)\log\sin\frac{\theta}{2} - a\log\cos\theta + \mathrm{A} \\ y_1 &= \sqrt{(1-a^2)}\log\cot\tfrac{1}{2}\left(\frac{\pi}{2}-\theta\right) + \pi - \varpi, \end{aligned}\right\}$$

and for $\psi = 0$

$$\left.\begin{aligned} x_2 &= (1+a)\log\sin\frac{\theta}{2} - (1-a)\log\cos\frac{\theta}{2} - a\log\cos\theta + \mathrm{A}' \\ y_2 &= \sqrt{(1-a^2)}\log\cot\tfrac{1}{2}\left(\frac{\pi}{2}-\theta\right) + \varpi \end{aligned}\right\}$$

where in both cases θ lies between 0 and $\frac{1}{2}\pi$ When $\theta = \frac{1}{2}\pi$ in both we get

$$\left.\begin{aligned} x_1 - x_2 &= \mathrm{A} - \mathrm{A}' \\ y_1 - y_2 &= \pi - 2\varpi \end{aligned}\right\}$$

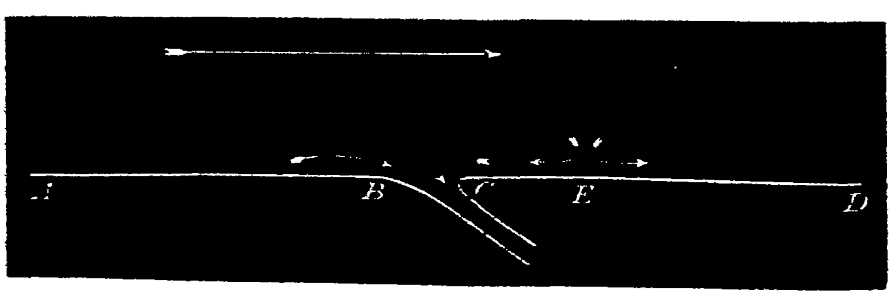
therefore

$$\mathrm{A} - \mathrm{A}' = \frac{\pi + (\pi - 2\varpi)a}{\sqrt{(1-a^2)}} = \pi\sqrt{(1-a^2)},$$

and the equations of the bounding stream lines are now completely determined

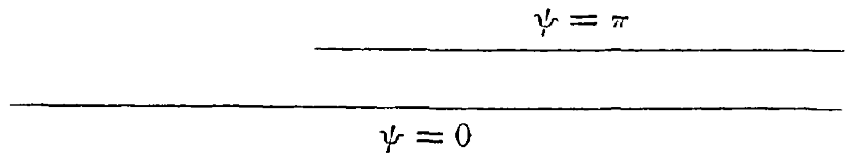
CASE III.—*Flow of a broad stream past a plane wall in which there is an aperture.*

Let BC be the aperture in a plane wall ABCD, and let the stream flow from left to right.



The left boundary of the issuing jet will be the continuation of the stream line AB. The right boundary will be one branch of a stream line which divides on the plane CD at some point E.

The diagram in the w plane will then consist of one infinite straight line and one semi-infinite, as in the figure



We suppose $\psi = 0$, $\psi = \pi$ to be the bounding stream lines. In transforming to the u plane we take the point $\psi = 0$, $\phi = -\infty$ to be $u = -\infty$, $\psi = 0$, $\phi = \infty$ to be $u = a$, and then use the two arbitrary constants at our disposal by making the edges of the aperture to be $u = -b$, $u = b$, and the branch point of $\psi = \pi$ to be $u = 1 + a$, where $1 + a > b > a$

We then have

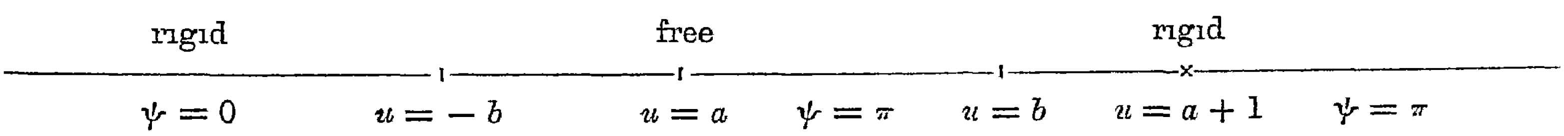
$$\frac{dw}{du} = A\frac{u - a - 1}{u - a},$$

and, remembering that the bounding stream lines are $\psi = 0$, $\psi = \pi$, we obtain $A = 1$, so that

$$\frac{dw}{du} = \frac{u - a - 1}{u - a}.$$

There is only one singular point for dz/dw, viz, $u = a + 1$, and at this point $\alpha/\pi - 1 = 0$, so that no factor $u - a - 1$ appears in dz/du

The arrangement of points on $q = 0$ is as in the figure



and hence

$$\frac{dz}{dw} = \frac{(a + 1)u - b^2 + \sqrt{(a + 1)^2 - b^2}\sqrt{(u^2 - b^2)}}{b(u - a - 1)},$$

so that

$$\frac{dz}{du} = \frac{(a + 1)u - b^2 + \sqrt{(a + 1)^2 - b^2}\sqrt{(u^2 - b^2)}}{b(u - a)}$$

When $u = \pm\infty$, we have

$$\frac{dz}{dw} = \frac{a + 1 + \sqrt{(a + 1)^2 - b^2}}{b},$$

and, therefore, the velocity of the stream is

$$\frac{b}{(a + 1) + \sqrt{(a + 1)^2 - b^2}}.$$

When $u = a$, that is at a great distance along the jet,

$$\frac{dz}{dw} = -\frac{a(a+1) - b^2 + i\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}}{b},$$

so that the inclination of the jet to the wall is ultimately

$$\cos^{-1}\frac{b^2 - a(a+1)}{b},$$

the final breadth of the jet being π.

There appear to be two constants here, whereas there ought only to be one.

The explanation is that we have not yet expressed that the two parts of the boundary are in the same plane This will give a relation between a and b

Between $u = -b$ and $u = a$ we have

$$\frac{dx}{dp} = \frac{1}{b}\frac{(a+1)p - b^2}{p - a},$$

$$\frac{dy}{dp} = \frac{\sqrt{(a+1)^2 - b^2}}{b}\quad\frac{\sqrt{(b^2 - p^2)}}{p - a}.$$

Now

$$\int\frac{\sqrt{(b^2 - p^2)}}{p - a} = \int\frac{1}{\sqrt{(b^2 - p^2)}}\left[\frac{b^2 - a^2}{p - a} - a - p\right],$$

and

$$\int\frac{dp}{(p-a)\sqrt{(b^2 - p^2)}}$$

$$= -\int\frac{dr}{\sqrt{(b^2 - a^2)r^2 + 2ar - 1}},\qquad \text{where } p - a = -\frac{1}{r},$$

$$= -\int\frac{dr}{\sqrt{\left(\sqrt{b^2 - a^2}\,r + \frac{a}{\sqrt{(b^2 - a^2)}}\right)^2 - \frac{b^2}{b^2 - a^2}}},$$

$$= -\frac{1}{\sqrt{(b^2 - a^2)}}\log\left[\sqrt{(b^2 - a^2)}\,r + \frac{a}{\sqrt{(b^2 - a^2)}} + \sqrt{(b^2 - a^2)r^2 + 2ar - 1}\right],$$

$$= -\frac{1}{\sqrt{(b^2 - a^2)}}\log\frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{a - p}.$$

Thus along $\psi = 0$, measuring from the edge A, we have

$$x - x_A = \frac{a+1}{b}(p + b) + \frac{a(a+1) - b^2}{b}\log\frac{a - p}{a + b},$$

$$y - y_A = \frac{\sqrt{(a + 1^2) - b^2}}{b}\left[-\sqrt{(b^2 - a^2)}\log\frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{(a - p)b}\right.$$

$$\left. + a\sin^{-1}.\frac{-p}{b} - a\frac{\pi}{2} + \sqrt{(b^2 - p^2)}\right].$$

In like manner between $u = a$ and $u = b$ on $\psi = \pi$, we have

$$x - x_B = \frac{a+1}{b}(p - b) + \frac{a(a+1) - b^2}{b} \log \frac{p-a}{b-a},$$

$$y - y_B = \frac{\sqrt{(a+1)^2 - b^2}}{b}\left[- \sqrt{(b^2 - a^2)} \log \frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{(p-a)b} - a \sin^{-1}\frac{p}{b} + a\frac{\pi}{2} + \sqrt{(b^2 - p^2)}\right]$$

To find $x_A - x_B$ and $y_A - y_B$ we must put $p = a - \epsilon$ and $p = a + \epsilon$ respectively in the above, and then decrease ϵ indefinitely

If we then remember that, in passing the point $p = a$, $x + iy$ increases by

$$-\frac{i\pi}{b}\left\{(a+1)a - b^2 + i\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}\right\}$$

we get

$$x_B - x_A = 2(a+1) + \frac{a(a+1) - b^2}{b} \log \frac{b-a}{b+a} + \frac{\pi}{b}\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}$$

and

$$y_B - y_A = -\pi a \frac{\sqrt{(a+1)^2 - b^2}}{b} + \frac{\pi}{b}\{b^2 - a(a+1)\}$$

In the case we are considering $y_B = y_A$, and, therefore,

$$b^2 - a(a+1) = a\sqrt{(a+1)^2 - b^2},$$

or

$$b^2 = (a+1)^2 - 1,$$

so that

$$\left.\begin{aligned}(a+1)^2 - b^2 &= 1\\ a(a+1) - b^2 &= -a.\end{aligned}\right\}$$

Using these equalities, we now get for the breadth of the aperture

$$x_B - x_A = 2(a+1) + \frac{a}{b}\log\frac{b+a}{b-a} + \pi\frac{\sqrt{2a}}{b}$$

where $b/(a+2)$ is the velocity of the stream, π the breadth of the jet, and $\cos^{-1}(a/b)$ the inclination of the jet to the plane.

To get rid of the special units let v_1 be the velocity of the stream, v_2 of the jet, k the breadth of the aperture, c that of the jet

Then

$$\frac{v_1}{v_2} = \frac{b}{a+2},$$

therefore

$$\left(\frac{v_1}{v_2}\right)^2 = \frac{b^2}{(a+2)^2} = \frac{a}{a+2}$$

So that

$$a = \frac{2v_1^2}{v_2^2 - v_1^2},$$

and

$$b = \frac{2v_2 v_1}{v_2^2 - v_1^2},$$

therefore

$$a + b = \frac{2v_1(v_2 + v_1)}{v_2^2 - v_1^2}$$

$$-a + b = \frac{2v_1(v_2 - v_1)}{v_2^2 - v_1^2}$$

Hence

$$\frac{h}{c}\pi = 2\frac{v_2^2 + v_1^2}{v_2^2 - v_1^2} + \frac{v_1}{v_2}\log\frac{v_2 + v_1}{v_2 - v_1} + \pi\sqrt{\frac{v_2^2 - v_1^2}{v_2^2}},$$

and the inclination of the jet to the plane is

$$\cos^{-1}\frac{v_1}{v_2}$$

An interesting element not yet calculated is the distance of the branching of the stream line $\psi = \pi$ from the edge of the aperture.

This distance is

$$\int_b^{a+1}\left[\frac{1}{b}\,\frac{(a+1)p - b^2 + \sqrt{(p^2 - b^2)}}{p - a}\right]dp$$

$$= \int_b^{a+1}\left[\frac{a+1}{b} - \frac{a}{b}\,\frac{1}{p-a} + \frac{1}{b}\left\{p + a - \frac{2a}{p-a}\right\}\frac{1}{\sqrt{(p^2 - b^2)}}\right]dp,$$

where we have used the relation between a and b to simplify the expression,

$$= \left[\frac{a+1}{b}p - \frac{a}{b}\log(p-a) + \frac{1}{b}\sqrt{(p^2 - b^2)} + \frac{a}{b}\log\left(p + \sqrt{p^2 - b^2}\right) + \frac{\sqrt{2a}}{b}\sin^{-1}\left\{\frac{b^2 - ap}{b(p-a)}\right\}\right]_b^{a+1}$$

$$= \frac{a+1}{b}(a + 1 - b) + \frac{a}{b}\log(b - a) + \frac{1}{b} + \frac{a}{b}\log\frac{a+2}{b} + \frac{\sqrt{2a}}{b}\left(\sin^{-1}\frac{a}{b} - \frac{\pi}{2}\right).$$

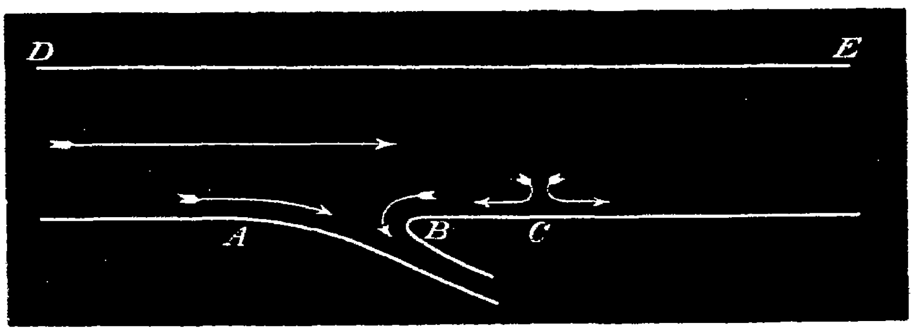
Therefore, in terms of v_1/v_2 and c, this distance is l where

$$\frac{l}{c}\pi = \left[\frac{v_2^2 + v_1 v_2 + v_1^2}{v_1 v_2} \times \frac{v_2 - v_1}{v_2 + v_1} + \frac{v_1}{v_2}\log\frac{2v_2}{v_2 + v_1} + \sqrt{\frac{v_2^2 - v_1^2}{v_2^2}}\left(\sin^{-1}\frac{v_1}{v_2} - \frac{\pi}{2}\right)\right]$$

For example, let the stream have half the velocity of the issuing jet, so that $v_1 = \frac{1}{2} v_2$ Then the jet makes an angle of 60° with the plane, and its breadth is $\frac{1}{2} k$ approximately, while l is about $\frac{1}{4} k$

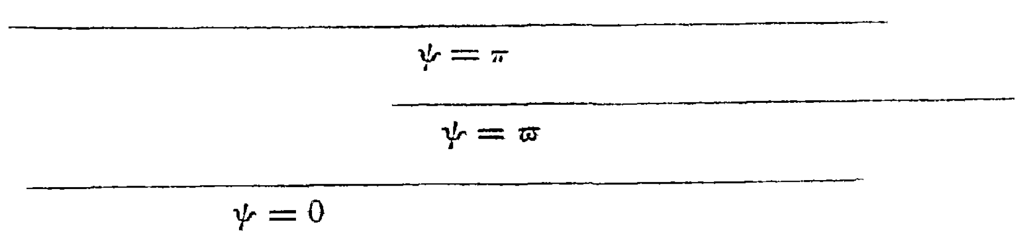
CASE IV —*Jet from a pipe along which liquid is flowing*

The liquid is flowing along a pipe bounded by the plane walls AC, DE, and there is an aperture AB in the former



The left boundary of the jet is $\psi = 0$, the right boundary is a stream line $\psi = \varpi$, which branches at a point C on BC, and DE is $\psi = \pi$

The w diagram is as in the figure, consisting of two infinite lines, $\psi = 0$, $\psi = \pi$, with a semi-infinite line $\psi = \varpi$ between them



In transforming to the u plane, we suppose that $\phi = -\infty$ corresponds to $u = -\infty$, and that $u = -1$, $u = +1$ are the edges of the aperture.

The constants of the transformation are then determined, and we take $u = a$ for the branch point, $u = b$ for the jet at an infinite distance, $u = c$ for $\phi = \infty$ in the pipe

Then

$$\frac{dw}{du} = \mathrm{A}\,\frac{u - a}{(u - b)(u - c)} \qquad b < a < c,$$

where

$$-\mathrm{A}\,\frac{b - a}{b - c}\,\pi = \varpi$$

$$-\mathrm{A}\,\frac{c - a}{b - c}\,\pi = \varpi - \pi,$$

and therefore

$$A = -1$$

$$\frac{\varpi}{\pi} = \frac{b-a}{b-c}$$

$$\frac{\pi - \varpi}{\pi} = -\frac{c-a}{b-c}.$$

The arrangement of points on $q = 0$ is as in the figure

rigid		free		free		rigid		rigid
$\psi=0$	$u=-1$	$\psi=0$	$u=b$	$\psi=\varpi$	$u=1$	$u=a$	$u=c$	$\psi=\pi$

and from Problem II (a) we have

$$\frac{dz}{dw} = \frac{au - 1 + \sqrt{(a^2-1)}\ \sqrt{(u^2-1)}}{u-a},$$

so that

$$\frac{dz}{du} = -\frac{au - 1 + \sqrt{(a^2-1)}\ \sqrt{(u^2-1)}}{(u-b)\ (u-c)}.$$

Hence, in the pipe, the velocity at $\phi = -\infty$ is

$$\frac{1}{a + \sqrt{(a-1)}} = a - \sqrt{(a^2-1)},$$

and at $\phi = \infty$ is

$$\frac{c-a}{ac - 1 + \sqrt{(a^2-1)}\sqrt{(c^2-1)}}$$

The breadth of the pipe is, therefore,

$$\pi\left(a + \sqrt{a^2-1}\right) = (\pi - \varpi)\,\frac{ac - 1 + \sqrt{(a^2-1)}\sqrt{(c^2-1)}}{c-a}$$

Now

$$\pi - \varpi = \pi\,\frac{c-a}{c-b},$$

therefore

$$(c-b)\left(a + \sqrt{a^2-1}\right) = ac - 1 + \sqrt{(a^2-1)}\ \sqrt{(c^2-1)}.$$

This is the relation between a, b, and c To find the size of the aperture we must integrate between $u = -1$ and $u = 1$. Within these limits we have

$$\frac{dx}{dp} = \frac{1-ap}{(b-p)\,(c-p)} = \frac{1-ab}{c-b}\,\frac{1}{b-p} + \frac{ac-1}{c-b}\,\frac{1}{c-p}$$

$$\frac{dy}{dp} = -\sqrt{(a^2-1)}\,\frac{\sqrt{(1-p^2)}}{(b-p)\,(c-p)} = -\frac{\sqrt{(a^2-1)}}{c-b}\sqrt{1-p^2}\left[\frac{1-ab}{b-p} + \frac{ac-1}{c-p}\right].$$

The former gives

$$x - x_A = -\frac{1-ab}{c-b}\log\frac{b-p}{b+1} - \frac{ac-1}{c-b}\log\frac{c-p}{c+1} \quad \text{between } p = -1 \text{ and } p = b,$$

and

$$x - x_B = \frac{ab-1}{c-b}\log\frac{p-b}{1-b} - \frac{ac-1}{c-b}\log\frac{c-p}{c-1} \quad \text{between } p = b \text{ and } p = 1$$

Also in passing the point $u = b$, x changes by

$$\pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b},$$

therefore, putting $p = b - \epsilon$ in $x - x_A$, $p = b + \epsilon$ in $x - x_B$, and proceeding to the limit, we get as the breadth of the aperture,

$$\frac{1-ab}{c-b}\log\frac{1+b}{1-b} + \frac{ac-1}{c-b}\log\frac{c+1}{c-1} + \pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b}.$$

The direction of the jet makes an angle

$$\cos^{-1}\left[\frac{1-ab}{a-b}\right]$$

with the bounding planes. The breadth of the jet is

$$\pi\frac{a-b}{c-b}$$

To sum up Let d be the breadth of the pipe, k of the aperture, l of the jet. Let, further, v_1 be the velocity at $\phi = -\infty$, v_2 that at $\phi = \infty$, v_3 that of the jet.

$$\frac{v_3}{v_1} = a + \sqrt{(a^2-1)} \quad . \quad . \qquad (1)$$

$$\frac{v_1}{v_2} = \frac{c-b}{c-a} \quad . \quad . \qquad (2)$$

and therefore

$$\frac{v_1}{v_1 - v_2} = \frac{c-b}{a-b} \quad . \qquad (3)$$

$$\frac{d}{l} = \frac{v_3}{v_1 - v_2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

$$\frac{k}{l} = \frac{c-b}{\pi(a-b)}\left\{\frac{1-ab}{c-b}\log\frac{1+b}{1-b} + \frac{ac-1}{c-b}\log\frac{1+c}{-1+c} + \pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b}\right\} \quad . \qquad (5)$$

with

$$(c-b)(a+\sqrt{a^2-1}) = ac - 1 + \sqrt{(a^2-1)}\sqrt{(c^2-1)} \quad . \quad . \quad . \qquad (6)$$

The equation (6) may be written

$$(c-a)\frac{v_3}{v_2} = (ac-1) + \sqrt{(a^2-1)}\sqrt{(c^2-1)}.$$

Now

$$(c-a)^2 \equiv (ac-1)^2 - (a^2-1)(c^2-1),$$

therefore

$$(c-a)\frac{v_2}{v_3} = (ac-1) - \sqrt{(a^2-1)}\sqrt{(c^2-1)},$$

and, by addition and subtraction,

$$(c-a)\left(\frac{v_3}{v_2}+\frac{v_2}{v_3}\right) = 2(ac-1),$$

$$(c-a)\left(\frac{v_3}{v_2}-\frac{v_2}{v_3}\right) = 2\sqrt{(a^2-1)}\sqrt{(c^2-1)}.$$

From (1)

$$a = \tfrac{1}{2}\left(\frac{v_3}{v_1}+\frac{v_1}{v_3}\right)$$

$$\sqrt{(a^2-1)} = \tfrac{1}{2}\left(\frac{v_3}{v_1}-\frac{v_1}{v_3}\right)$$

Therefore

$$c\left[\frac{v_3}{v_2}+\frac{v_2}{v_3}-\frac{v_3}{v_1}-\frac{v_1}{v_3}\right] = \tfrac{1}{2}\left(\frac{v_3}{v_2}+\frac{v_2}{v_3}\right)\left(\frac{v_3}{v_1}+\frac{v_1}{v_3}\right) - 2,$$

or,

$$c(v_1-v_2)(v_3^2-v_1v_2) = \tfrac{1}{2}(v_1-v_2)^2 v_3 + \tfrac{1}{2}(v_3^2-v_1v_2)^2\frac{1}{v_3},$$

so that

$$\frac{1+c}{-1+c} = \frac{[(v_1-v_2)v_3 + (v_3^2-v_1v_2)]^2}{[-(v_1-v_2)v_3 + (v_3^2-v_1v_2)]^2}$$

$$= \frac{(v_3-v_2)^2(v_3+v_1)^2}{(v_3+v_2)^2(v_3-v_1)^2}$$

And

$$\frac{ac-1}{a-b} = \tfrac{1}{2}\left(\frac{v_3}{v_2}+\frac{v_2}{v_3}\right)\frac{v_2}{v_1-v_2}.$$

Also

$$(c-a)\frac{v_1}{v_2} = c-b \quad \text{from (2)},$$

therefore

$$b = \tfrac{1}{2}\frac{v_3^2+v_1^2}{v_2v_3} - \tfrac{1}{2}\frac{(v_1-v_2)^2v_3^2 + (v_3^2-v_1v_2)^2}{v_2v_3(v_3^2-v_1v_2)}$$

$$= \tfrac{1}{2}\frac{(v_1^2+v_1v_2)(v_3^2-v_1v_2) - (v_1-v_2)^2v_3^2}{v_2v_3(v_3^2-v_1v_2)},$$

so that

$$1+b=\tfrac{1}{2}\,\frac{(v_1^2+v_1v_2+2v_2v_3)(v_3^2-v_1v_2)-(v_1-v_2)^2v_3^2}{v_2v_3(v_3^2-v_1v_2)}$$

and

$$1-b=\tfrac{1}{2}\,\frac{(2v_2v_3-v_1^2-v_1v_2)(v_3^2-v_1v_2)+(v_1-v_2)^2v_3^2}{v_2v_3(v_3^2-v_1v_2)}.$$

These may be reduced to the form

$$1+b=\tfrac{1}{2}\,\frac{(v_3+v_1)^2}{v_3^2-v_1v_2}\;\frac{2v_3-v_1-v_2}{v_3},$$

$$1-b=\tfrac{1}{2}\,\frac{(v_3-v_1)^2}{v_3^2-v_1v_2}\;\frac{2v_3+v_1+v_2}{v_3}$$

Further,

$$a-b=\tfrac{1}{2}\,\frac{(v_3^2v_2-v_1^3)(v_3^2-v_1v_2)+(v_1-v_2)^2v_3^2v_1}{v_1v_2v_3(v_3^2-v_1v_2)},$$

$$=\tfrac{1}{2}\,\frac{(v_3^2-v_1^2)^2}{v_1v_3(v_3^2-v_1v_2)},$$

$$1-ab=\tfrac{1}{4}\,\frac{\{4v_3^2v_1v_2-(v_3^2+v_1^2)(v_1^2+v_1v_2)\}\{v_3^2-v_1v_2\}+(v_3^2+v_1^2)(v_1-v_2)^2v_3^2}{v_1v_2v_3^2(v_3^2-v_1v_2)},$$

$$=\tfrac{1}{4}\,\frac{(v_1+v_2)(v_3^2-v_1^2)^2}{v_1v_3^2(v_3^2-v_1v_2)}.$$

Whence, finally,

$$\frac{1-ab}{a-b}=\tfrac{1}{2}\,\frac{v_1+v_2}{v_3},$$

$$\frac{1+b}{1-b}=\frac{(v_3+v_1)^2}{(v_3-v_1)^2}\;\frac{2v_3-v_1-v_2}{2v_3+v_1+v_2},$$

$$\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b}=\tfrac{1}{2}\,\frac{\sqrt{4v_3^2-(v_1+v_2)^2}}{v_3}.$$

So that we get for k/l

$$\frac{k}{l}\pi=\tfrac{1}{2}\,\frac{v_1+v_2}{v_3}\log\frac{(v_3+v_1)^2}{(v_3-v_1)^2}\;\frac{2v_3-v_1-v_2}{2v_3+v_1+v_2}$$

$$+\frac{v_3^2+v_2^2}{v_3(v_1-v_2)}\log\frac{(v_3-v_2)(v_3+v_1)}{(v_3+v_2)(v_3-v_1)}$$

$$+\tfrac{1}{2}\pi\left\{\frac{4v_3^2-(v_1+v_2)^2}{v_3^2}\right\}^{\frac{1}{2}}.$$

An interesting special case is the flow from an aperture in the side of a rectangular vessel, of which the bottom is at a considerable distance from the aperture.

For this case $v_2 = 0$, and, therefore,

$$\frac{d}{l} = \frac{v_3}{v_1},$$

expressing the equality of inflow and outflow.

So that

$$\frac{h}{l}\pi = \tfrac{1}{2}\frac{v_1}{v_3}\log\left(\frac{v_3+v_1}{v_3-v_1}\right)^2 \cdot \frac{2v_3-v_1}{2v_3+v_1} + \frac{v_3}{v_1}\log\frac{v_3+v_1}{v_3-v_1} + \frac{\pi}{2}\left\{4-\frac{v_1^2}{v_3^2}\right\}^{\frac{1}{2}}$$

$$= \left(\frac{v_1}{v_3}+\frac{v_3}{v_1}\right)\log\frac{v_3+v_1}{v_3-v_1} + \tfrac{1}{2}\frac{v_1}{v_3}\log\frac{2v_3-v_1}{2v_3+v_1} + \frac{\pi}{2}\left\{4-\frac{v_1^2}{v_3^2}\right\}^{\frac{1}{2}},$$

or

$$\frac{h}{l}\pi = \left(\frac{d}{l}+\frac{l}{d}\right)\log\frac{d+l}{d-l} + \tfrac{1}{2}\frac{l}{d}\log\frac{2d-l}{2d+l} + \frac{\pi}{2}\left(4-\frac{l^2}{d^2}\right)^{\frac{1}{2}},$$

which is the simplest expression for this case.

The angle the jet makes with the bounding wall is

$$\cos^{-1}\frac{1-ab}{a-b},$$

or

$$\cos^{-1}\tfrac{1}{2}\frac{v_1+v_2}{v_3}$$

in the general case, while in the particular case it is simply

$$\cos^{-1}\tfrac{1}{2}\frac{v_1}{v_3}.$$

PART II.

We will now go on to consider problems in which the region of (x, y) is not simply connected, and consequently SCHWARZ's transformation does not apply.

First consider the area outside a closed polygon.

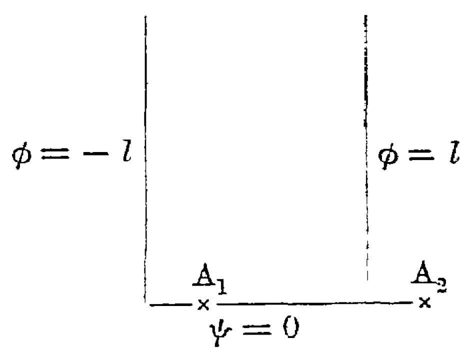
We may state the problem electrically thus —

PROBLEM III

To find the potential due to a polygonal prismatic conductor at a given potential which we may take to be zero, the field at infinity being at an infinite potential

Let ψ be the potential, ϕ the lines of force, and let ϕ increase by $2l$ in going round the polygon

Then the area in the ω plane is a rectangle bounded by $\psi = 0$, $\psi = \infty$, $\phi = -l$, $\phi = l$.



The conditions which the transformation function V satisfy are

(a) $$\frac{dV}{d\psi} = 0 \text{ over } \psi = 0$$

(b) V finite and continuous at all points within a finite distance in the rectangle.

(c) V periodic in ϕ so that

$$V(\phi + 2l) = V(\phi).$$

(d) V infinite at points A_1 .. along $\psi = 0$.

We can determine V from these specifications by means of W. THOMSON's method of images For if we repeat the points $A_1 \ldots$ at equal distances $2l$ along $\psi = 0$, and make V the potential of these points, the conditions will clearly be satisfied.

Hence

$$V = A\Sigma_r \log \prod_{-\infty}^{+\infty} \{(\phi - \phi_r - 2nl)^2 + \psi^2\}^m,$$

and

$$\frac{dz}{dw} = A\Pi_r \prod_{n\,-\infty}^{\;+\infty} \{w - \phi_r - 2nl\}^m$$

$$= B\Pi_r \left\{\sin (w - \phi_r) \frac{\pi}{2l}\right\}^m,$$

and exactly as in the treatment of SCHWARZ's formula we have $m = \alpha_r/\pi - 1$, where α_r is the exterior angle of the polygon corresponding to ϕ_r

So that, finally,*

$$\frac{dz}{dw} = \mathrm{B}\Pi\left\{\sin\left(w - \phi_r\right)\frac{\pi}{2l}\right\}^{\alpha_r/\pi - 1}$$

As an example, take the case of a rectangle

The four singular points may be taken to be

$$\phi = -a, \qquad \phi = a, \qquad \phi = l - a, \qquad \phi = l + a,$$

and we have taking for simplicity $l = \frac{1}{2}\pi$,

$$\frac{dz}{dw} = \mathrm{A}\sqrt{-\sin(w - a)\sin(w + a)\cos(w - a)\cos(w + a)},$$

$$= \tfrac{1}{2}\mathrm{A}\sqrt{(\sin^2 2a - \sin^2 2w)}$$

Hence

$$z = \tfrac{1}{2}\mathrm{A}\int\sqrt{(\sin^2 2a - \sin^2 2w)}\,dw + \mathrm{B},$$

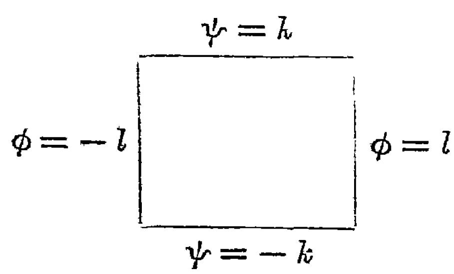
and z is an elliptic integral of w

PROBLEM IV.

Suppose now there are two polygonal prismatic conductors, one at potential $\psi = -k$, the other at $\psi = k$, and at first (A) suppose that one of the conductors is within the other

Let ϕ increase by $2l$ in going round either polygon

The area in the w plane is now a finite rectangle bounded by $\phi = -l$, $\phi = l$, $\psi = -k$, $\psi = k$.



In order to satisfy the condition $d\mathrm{V}/d\psi = 0$ over $\psi = -k$ and $\psi = k$ we must have a double system of images of the singular points, viz., at

$$\phi_0 + 2ml, \qquad \psi_0 + 4nk,$$
$$\phi_0 + 2ml, \qquad \psi_0 + (2n + 1)\,2k.$$

* Mr BRILL has already given this formula ('Messenger of Math,' August, 1889), and I only insert it here for the sake of completeness.

NOTE.—April 29. Since the above paper was read, Mr. BRILL has given the next transformation in the same journal.

Thus each singular point gives a factor

$$\prod_{-\infty}^{+\infty}\prod_{-\infty}^{+\infty}\{w - w_0 - 2ml - 4nk\}^{\mathrm{M}}\{w - w_0 - 2ml - (2n + 1)\,2k\}^{\mathrm{M}},$$

or, what is the same thing,

$$\Theta\,[a\,(w - w_0)]^{\mathrm{M}}\,\mathrm{H}\,[a\,(w - w_0)]^{\mathrm{M}},$$

where Θ, H are JACOBI's functions so indicated, and, therefore,

$$\frac{dz}{dw} = \Pi\,\{\Theta\,[a\,(w - w_0)]\,\mathrm{H}\,[a\,(w - w_0)]\}^{\mathrm{M}},$$

where

$$\left.\begin{aligned} al &= \mathrm{K} \\ 2ak &= \mathrm{K}' \end{aligned}\right\}$$

K, K′ being the complete elliptic integrals usually so denoted

If α_0 be the internal angle of the figure corresponding to w_0, we have, as before,

$$\mathrm{M} = \frac{\alpha_0}{\pi} - 1,$$

so that, finally,

$$\frac{dz}{dw} = \Pi_r\,\{\Theta\,[a\,(w - w_r)]\,\mathrm{H}\,[a\,(w - w_r)]\}^{\alpha_r/\pi - 1}$$

(B) Suppose now that one conductor is outside the other, and that the potential at infinity is zero, that of the conductors being $-k$ and $+k$. We suppose equal and opposite quantities of electricity on the conductors, $2l$ being the cyclic constant, as before.

The terms corresponding to angles of the polygon will be the same as in (A).

But there is now in addition a singular point in the field which we proceed to determine.

At a great distance from the prisms the potential will be the same as for two line distributions at the centres of mass, say at $z = a$, $z = -a$.

So that

$$w = \mathrm{M}\log\frac{z - a}{z + a} = -\,2\,\frac{\mathrm{M}a}{z}$$

ultimately, and

$$\frac{dw}{dz} = 2\mathrm{M}az^{-2} = \frac{w^2}{2\mathrm{M}a}$$

or

$$\frac{dz}{dw} = \frac{2\mathrm{M}a}{w^2},$$

and, therefore, there is a point of order -2 at the point in the w rectangle corresponding to the potential at infinity.

Let this point be ϕ_0, ψ_0. Then we have

$$\frac{dz}{dw} = \frac{\Pi_r \{\Theta[a(w - w_r)]\, \mathrm{H}[a(w - w_r)]\}^{\alpha_r/\pi - 1}}{\{\Theta[a(w - w_0)]\, \mathrm{H}[a(w - w_0)]\}^2}$$

as the general expression for the potential of two polygonal prismatic conductors exterior to each other

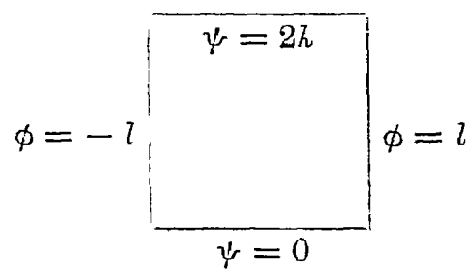
Hollow Vortices

In conclusion I shall show how the methods of this paper may be applied to find the form of hollow vortices.

Inside a vessel bounded by plane walls, let there be a hollow vortex in steady motion

Let $\psi = 0$ be the free steam line of the vortex, $\psi = 2k$ the rigid boundary.

In the w plane the area is bounded by $\psi = 0$, $\psi = 2k$, $\phi = -l$, $\phi = l$, $2l$ being the circulation round the vortex.



The function V satisfies the following conditions —

(a.) $\mathrm{V} = 0$ over $\psi = 0$, if the velocity along the free steam line be unity.

(b) $d\mathrm{V}/d\psi = 0$ over $\psi = 2k$.

(c.) V is periodic with respect to ϕ, so that $\mathrm{V}(\phi + 2l) = \mathrm{V}(\phi)$.

These conditions are to be satisfied by taking equal singular points at distances $2l$ along $\psi = 2k$, and then continually reflecting these points in the two planes $\psi = 0$, $\psi = 2k$, but in reflecting in $\psi = 0$ the image is of opposite sign to the object

Corresponding, then, to a point M at $(\phi_0, 2k)$ we have positive images at

$$\phi_0 + 2ml, \quad 2k + 2m \cdot 4k,$$

and negative images at

$$\phi_0 + 2ml, \quad -2k + 2m \cdot 4k.$$

Therefore, corresponding to this point M, we have a factor

$$\frac{\mathrm{H}^{\mathrm{M}} a(w - \phi_0 - 2ik)}{\Theta^{\mathrm{M}} a(w - \phi_0 - 2ik)} \quad \text{in} \quad \frac{dz}{dw},$$

where

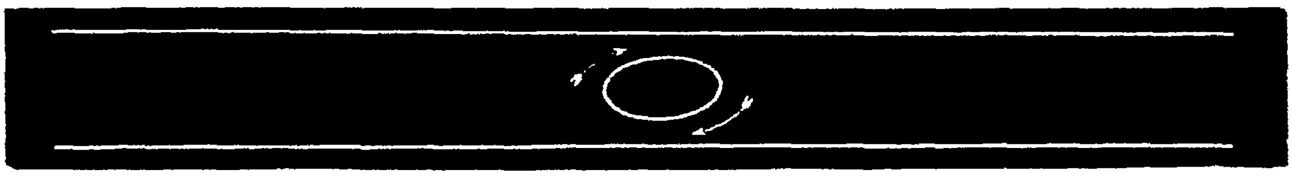
$$\left.\begin{aligned} al &= \mathrm{K} \\ 4ak &= \mathrm{K}' \end{aligned}\right\}$$

and

$$M = \frac{\alpha_0}{\pi} - 1$$

It is quite clear that all but the simplest cases will be of quite unmanageable complexity

One of the very simplest cases may be taken as an example



Example —Hollow vortex between two parallel planes

Here for the two singular points $\alpha = 0$

If for simplicity we put $a = 1$, and, therefore, $4k = K'$, the singular points are at $\phi = 0$, $\phi = K$.

Hence

$$\frac{dz}{dw} = C \frac{\Theta\left(w - \frac{\iota K'}{2}\right)}{H\left(w - \frac{\iota K'}{2}\right)} \frac{\Theta\left(w - \frac{\iota K'}{2} - K\right)}{H\left(w - \frac{\iota K'}{2} - K\right)}$$

$$= C' \frac{1}{sn\left(w - \frac{\iota K'}{2}\right) sn\left(w - \frac{\iota K'}{2} - K\right)}$$

$$= C'' \frac{dn\left(w - \frac{\iota K'}{2}\right)}{sn\left(w - \frac{\iota K'}{2}\right) cn\left(w - \frac{\iota K'}{2}\right)}$$

This integrates at once, giving

$$z = C'' \log \frac{sn\,(w - \frac{1}{2}\iota K')}{cn\,(w - \frac{1}{2}\iota K')} + C'''$$

$$\equiv C'' \log \mathrm{tn}\,(w - \tfrac{1}{2}\iota K') + C'''$$

The equation to the vortex is, therefore,

$$z = C'' \log \mathrm{tn}\,(\phi - \tfrac{1}{2}i K') + C'''.$$

VI. On the Extension and Flexure of Cylindrical and Spherical Thin Elastic Shells.

By A. B. BASSET, *M.A.*, *F.R.S.*

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1. THE various theories of thin elastic shells which have hitherto been proposed have been discussed by Mr. LOVE* in a recent memoir, and it appears that most, if not all of them, depend upon the assumption that the three stresses which are usually denoted by R, S, T are zero; but, as I have recently pointed out,† a very cursory examination of the subject is sufficient to show that this assumption cannot be rigorously true. It can, however, be proved that, when the external surfaces of a plane plate are not subjected to pressure or tangential stress, these stresses depend upon quantities proportional to the square of the thickness, and whenever this is the case they may be treated as zero in calculating the expression for the potential energy due to strain, because they give rise to terms proportional to the fifth power of the thickness, which may be neglected, since it is usually unnecessary to retain powers of the thickness higher than the cube. It will also, in the present paper, be shown by an indirect method that a similar proposition is true in the case of cylindrical and spherical shells, and, therefore, the fundamental hypothesis upon which Mr. LOVE has based his theory, although unsatisfactory as an assumption, leads to correct results. A general expression for the potential energy due to strain in curvilinear coordinates has also been obtained by Mr. LOVE, and the equations of motion and the boundary conditions have been deduced therefrom by means of the Principle of Virtual Work, and if this expression and the equations to which it leads were correct, it would be unnecessary to propose a fresh theory of thin shells; but although those portions of Mr. LOVE's results which depend upon the thickness of the shell are undoubtedly correct, yet, for reasons which will be more fully stated hereafter, I am of opinion that the terms which depend upon the cube of thickness are not strictly accurate, inasmuch as he has omitted to take into account several terms of this order, both in the expression for the potential energy and elsewhere. His preliminary analysis is also of an exceedingly complicated character.

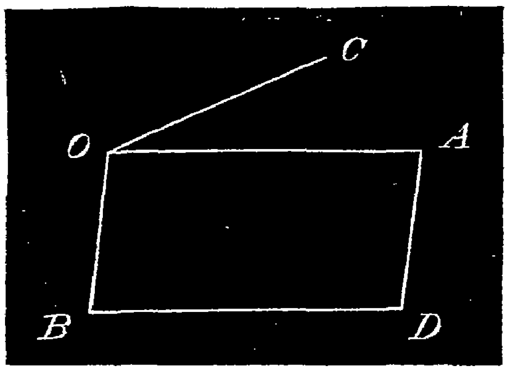
2. Throughout the present paper the notation of THOMSON and TAIT's "Natural Philosophy" will be employed for stresses and elastic constants, but, for the purpose

* 'Phil. Trans.,' A, 1888, p. 491.

† 'London Math. Soc. Proc.,' vol. 21, p. 33.

of facilitating comparison, Mr LOVE's notation will be employed for strains and directions It will also be convenient to denote the values of the various quantities involved, at a point P on the middle surface of the shell by *unaccented* letters, and the values of the same quantities at a point P′ on the normal at P, whose distance from P is h', by *accented* letters The radius of the shell will also be denoted by a, and its thickness by $2h$

The theory which it is proposed to develop for cylindrical and spherical shells is identical except in matters of detail, with the theory of plane plates which I recently communicated to the London Mathematical Society,* but for the sake of completeness a short outline will be given



In the figure let OADB be a small curvilinear rectangle described on the middle surface of the shell, of which the sides are lines of curvature; and let us consider a small element of the shell bounded by the external surfaces, and the four planes passing through the sides of this rectangle, which are perpendicular to the middle surface

The resultant stresses per unit of length which act upon the element, and which are due to the action of contiguous portions of the shell, are completely specified by the following quantities; viz, across the section AD,

$T_1 =$ a tension across AD parallel to OA,
$M_2 =$ a tangential shearing stress along AD,
$N_2 =$ a normal shearing stress parallel to OC,
$G_2 =$ a flexural couple from C to A, whose axis is parallel to AD,
$H_1 =$ a torsional couple from B to C, whose axis is parallel to OA

Similarly the resultant stresses per unit of length which act across the section BD are,

$T_2 =$ a tension across BD parallel to OB,
$M_1 =$ a tangential shearing stress along BD,
$N_1 =$ a normal shearing stress parallel to OC,
$G_1 =$ a flexural couple from B to C, whose axis is parallel to BD,
$H_2 =$ a torsional couple from C to A, whose axis is parallel to OB.

* 'London Math Soc Proc,' vol 21, p 33

If the edges AD, BD were of finite length, there would also be a couple whose axis is parallel to the normal, but since this couple is proportional to the cube of the edge, it vanishes in comparison with the other stresses when the rectangle OADB is indefinitely diminished

We shall denote the components of the bodily forces per unit of mass in the directions OA, OB, OC by X, Y, Z, but for reasons which will be more fully explained hereafter, we shall suppose that these forces arise solely from external causes, such as gravity and the like All forces arising from pressures or tangential stresses applied to the surface of the shell will be expressly excluded

The first step is to write down the equations of motion of an element of the shell in terms of the sectional stresses,* which can be done by the usual methods, we shall thus obtain six equations, three of which are formed by resolving the forces parallel to OA, OB, OC, and three more by taking moments about these lines

These equations will not, however, enable us to solve any statical or dynamical problems, in order to do this we require the equations of motion in terms of the displacement of a point on the middle surface and their space variations with respect to the coordinates of that point

3. The values at P′ of all the quantities with which we are concerned are functions of the position of P′, and are, therefore, functions of (r, z, ϕ) or (r, θ, ϕ), according as the shell is cylindrical or spherical If, therefore, $\mathfrak{Q}'$ be the value of any such quantity at P′, and $\mathfrak{Q}$ the value of the same quantity at the point P, which is the projection of P′ on the middle surface, it follows that

$$\begin{aligned}\mathfrak{Q}' &= \mathrm{F}(r) = \mathrm{F}(a + h') \\ &= \mathfrak{Q} + h'\left(\frac{d\mathfrak{Q}}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\mathfrak{Q}}{dr^2}\right) + \dots \qquad (1)\end{aligned}$$

by TAYLOR's theorem, where the brackets are employed, as will be done throughout this paper, to denote the values of the differential coefficients at the middle surface where $r = a$

Objections have been raised by SAINT-VENANT and endorsed by Mr LOVE, to the supposition that the first few terms of the expansion by TAYLOR's theorem of the quantities involved may be taken as a sufficient approximation If, however, this objection were valid, it would appear to me to upset the greater part of most physical investigations, inasmuch as it is always assumed as a general principle, that when a quantity is known to be a function of the position of a point P its value at a neighbouring point P′ may be obtained by TAYLOR's theorem, unless some physical discontinuity exists in passing from P to P′. If, therefore, we put $\mathfrak{Q} = \mathrm{R}$, we may write

$$\mathrm{R}' = \mathrm{A} + \mathrm{A}_1 h' + \tfrac{1}{2}\mathrm{A}_2 h'^2 + \dots \qquad (2)$$

where the A's are functions of the position of P and also of the thickness of the shell.

* See BESANT "On the Equilibrium of a Bent Lamina," 'Quart Journ Math,' vol 4, p 12.

A question which is of fundamental importance in the theory now arises, as to the way in which the A's depend upon h

If R′ were of the order of the square of the thickness, it is evident that A and A_1 could not contain any powers of h lower than the second and first respectively, whilst A_2 could not contain any negative power of h The A's are entirely unknown quantities, and as there appears to be no possibility of determining them by an *a priori* method, it seems hopeless to attempt to construct any theory of thin shells without the aid of some assumption which will enable us to get rid of them *If, however, we assume, as has been practically done by previous writers, that, when the surfaces of the shell are not subjected to any surface forces such as pressures or tangential stresses,* R′ *and also* S′ *and* T′, *so far as they depend on h and h′, are capable of being expressed in the form*

$$u_2 + u_3 + \ldots u_n +$$

where u_n is a homogeneous n-tic function of h and h′, the problem can be completely solved without attempting to determine by any a priori *method the values of any unknown quantities, and upon this fundamental hypothesis the theory of the present paper will be based*

There is some direct evidence of the truth of this hypothesis In the case of a plane plate of infinite extent, it can be proved to be true by means of the general equations of motion of an elastic solid ;* and for the purpose of testing the hypothesis in the case of a curved shell, I have recently investigated to a second approximation, so as to obtain the term in h^2, the period of the radial vibrations of an indefinitely long cylindrical shell, by means of the general equations, and also by means of the theory of thin shells, and both results agree † But far the most conclusive evidence in favour of the truth of the hypothesis is furnished by the results to which it leads, and I have, therefore, conducted the following investigation in such a manner as to furnish a test of the correctness of the final results, and consequently of the fundamental hypothesis by means of which they are deduced.

Having obtained the equations of motion of a cylindrical and a spherical shell in terms of the sectional stresses, all these stresses are then calculated by a direct method, with the exception of the tensions T_1, T_2, which cannot be calculated directly, since they involve the unknown quantities A and A_2 After that the potential energy and the other constitutents of the variational equation are calculated, and the variation worked out by the usual methods The final result, as is always the case in such investigations, consists of a line integral and a surface integral, the former of which determines the values of the sectional stresses in terms of the displacements, and the latter of which determines in the same form the three equations of motion. Now, if the work and the fundamental hypothesis upon which the theory is based are correct,

* Lord RAYLEIGH, 'London Math Soc Proc ,' vol 20, p 225 , see also vol 21, p 33

† 'London Math. Soc Proc ,' vol 21, p 53.

the variational equation will give the correct values of the tensions T_1, T_2, which are unknown, and will also reproduce the values of the other stresses which have been obtained directly. This is the first test The second test is furnished by the consideration that, if we substitute the values of the sectional stresses which we have obtained from the variational equation, in the first three of our original equations of motion in terms of these stresses, we ought to reproduce the equations of motion in terms of the displacements, which have been obtained from the variational equation. This is found to be the case both when the shell is cylindrical and when it is spherical; and I therefore think that the fundamental hypothesis is sufficiently established Having obtained the values of the sectional stresses, the boundary conditions can be deduced by means of STOKES' theorem, which enables us to prove that it is possible to apply a certain distribution of stress to the edge of a thin shell, without producing any alteration in the potential energy due to strain

The fundamental hypothesis that R', S', T' may be treated as zero is not true when the surfaces of the shell are subjected to external pressures or tangential stresses, for if the convex and concave surfaces of the shell were subjected to pressures Π_1, Π_2, the value of R' as we pass through the substance of the shell from its exterior to its interior surface, must vary from $-\Pi_1$ to $-\Pi_2$, and consequently (excepting in very special cases) R will contain a term independent of the thickness Hence the theory developed in the present paper is not applicable to problems relating to the collapse of boiler flues, or to the communication of the vibrations of a vibrating body to the atmosphere In order to obtain a theory which would enable such questions to be mathematically investigated, it would be necessary to find the values of the additional terms in the variational equation of motion, which depend upon the external pressures, and this is a problem which awaits solution.

It will be convenient briefly to state the notation employed

In the case of a cylindrical shell, OA is measured along a generating line, and OB along a circular section In the case of a spherical shell, OA is measured along a meridian, and OB along a parallel of latitude

The three extensional strains along OA, OB, OC are denoted by σ_1, σ_2, σ_3, and the three shearing strains about those lines by ϖ_1, ϖ_2, ϖ_3 We shall also use the letters λ, μ, p, λ', μ', p' to denote the first and second differential coefficients of σ_1, σ_2, ϖ_3 with respect to r, when $r = a$ We shall also write

$$\begin{aligned} \mathrm{E} &= (m-n)/(m+n), & \mathrm{K} &= \sigma_1 + \sigma_2, \\ \mathfrak{A} &= \sigma_1 + \mathrm{E}\,(\sigma_1 + \sigma_2), & \mathfrak{B} &= \sigma_2 + \mathrm{E}\,(\sigma_1 + \sigma_2), \\ \mathfrak{E} &= \lambda + \mathrm{E}\,(\lambda + \mu), & \mathfrak{F} &= \mu + \mathrm{E}\,(\lambda + \mu), \\ \mathfrak{E}' &= \lambda' + \mathrm{E}\,(\lambda' + \mu'), & \mathfrak{F}' &= \mu' + \mathrm{E}\,(\lambda' + \mu') \end{aligned}$$

Cylindrical Shells

4 Before we can obtain the equations of motion or the potential energy, it will be necessary to ascertain the values of the first and second differential coefficients of the displacements with respect to r when $r = a$ We shall, therefore, proceed to calculate these quantities

Putting λ, μ, λ', μ' for the values of $(d\sigma_1/dr)$, $(d\sigma_2/dr)$; $(d^2\sigma_1/dr^2)$, $(d^2\sigma_2/dr^2)$ when $r = a$, we have

$$\begin{aligned} R' &= (m+n)\sigma'_3 + (m-n)(\sigma'_1 + \sigma'_2) \\ &= (m+n)\sigma_3 + (m-n)(\sigma_1+\sigma_2) \\ &\quad + \left\{(m+n)\left(\frac{d\sigma_3}{dr}\right) + (m-n)(\lambda+\mu)\right\}h' \\ &\quad + \tfrac{1}{2}\left\{(m+n)\left(\frac{d^2\sigma_3}{dr^2}\right) + (m-n)(\lambda'+\mu')\right\}h'^2 + \quad . \quad (3) \end{aligned}$$

But from (2),

$$R' = A + A_1 h' + \tfrac{1}{2}A_2 h'^2 + \quad . \quad (4),$$

whence

$$\left.\begin{aligned} A &= (m+n)\sigma_3 + (m-n)(\sigma_1+\sigma_2) \\ A_1 &= (m+n)\left(\frac{d\sigma_3}{dr}\right) + (m-n)(\lambda+\mu) \\ A_2 &= (m+n)\left(\frac{d^2\sigma}{dr^2}\right) + (m-n)(\lambda'+\mu') \end{aligned}\right\} \quad . \quad (5),$$

where A, A_1 do not contain any lower powers of h, than h^2 and h respectively, and A_2 does not contain any negative powers of h

If u', v', w' be the component displacements of any point of the substance of the shell in the direction z, ϕ, r, the equations connecting the displacements and strains are

$$\left.\begin{aligned} \sigma'_1 &= \frac{du'}{dz} \\ \sigma'_2 &= \frac{1}{r}\left(\frac{dv'}{d\phi} + w'\right) \\ \sigma'_3 &= \frac{dw'}{dr} \\ \varpi'_1 &= \frac{dv'}{dr} - \frac{v'}{r} + \frac{1}{r}\frac{dw'}{d\phi} \\ \varpi'_2 &= \frac{dw'}{dz} + \frac{du'}{dr} \\ \varpi'_3 &= \frac{1}{r}\frac{du'}{d\phi} + \frac{dv'}{dz} \end{aligned}\right\} \quad . \quad . \quad (6),$$

whence if

$$E = \frac{m-n}{m+n}, \quad K = \sigma_1 + \sigma_2 \quad . \quad . \quad . \quad . \quad . \quad (7),$$

we obtain

$$\left(\frac{du}{dr}\right) = \varpi_2 - \frac{dw}{dz}, \quad \left(\frac{dv}{dr}\right) = \varpi_1 + \frac{v}{a} - \frac{1}{a}\frac{dw}{d\phi}, \quad \left(\frac{dw}{dr}\right) = \frac{A}{m+n} - EK \qquad (8),$$

and

$$\left.\begin{aligned} \left(\frac{d^2u}{dr^2}\right) &= \left(\frac{d\varpi_2}{dr}\right) - \left(\frac{d^2w}{dz\,dr}\right) = \left(\frac{d\varpi_2}{dr}\right) - \frac{1}{m+n}\frac{dA}{dz} + E\frac{dK}{dz} \\ \left(\frac{d^2v}{dr^2}\right) &= \left(\frac{d\varpi_1}{dr}\right) + \frac{\varpi_1}{a} - \frac{1}{a(m+n)}\frac{dA}{d\phi} + \frac{E}{a}\frac{dK}{d\phi} \\ \left(\frac{d^2w}{dr^2}\right) &= \frac{A_1}{m+n} - E(\lambda + \mu) \end{aligned}\right\} \quad . \quad . \quad (9)$$

5 We can now obtain the equations of motion in terms of the sectional stresses

Let dS be an element of the middle surface whose coordinates are (a, z, ϕ), and dS' an element of a layer of the shell whose coordinates are $(a + h', z, \phi)$, then $dS' = (1 + h'/a)\,dS$ If we consider a small element of volume bounded by the two external surfaces of the shell, and the four planes passing through the sides of dS, which are perpendicular to the middle surface, we obtain by resolving parallel to OA,

$$\frac{d}{dz}(T_1 a\,\delta\phi)\,\delta z + \frac{d}{d\phi}(M_1\,\delta z)\,\delta\phi = \rho\,dS\int_{-h}^{h}(\ddot{u}' - X)(1 + h'/a)\,dh' \qquad (10)$$

But

$$u' = u + h'\left(\frac{du}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2u}{dr^2}\right),$$

accordingly if we substitute the values of (du/dr) and (d^2u/dr^2) from (8) and (9) and recollect that all terms which vanish with h may be omitted when multiplied by h^3, the right hand side of (10) becomes

$$\rho\,dS\left\{2h(\ddot{u} - X) + \tfrac{1}{3}h^3 E\frac{d\ddot{K}}{dz} - \frac{2h^3}{3a}\frac{d\ddot{w}}{dz}\right\}$$

Resolving parallel to OB, OC, and then taking moments about OA, OB, OC, we shall obtain in a similar way five other equations, which, together with (10) may be written

$$\left.\begin{aligned} \frac{dT_1}{dz} + \frac{1}{a}\frac{dM_1}{d\phi} &= \rho\left\{2h(\ddot{u} - X) + \tfrac{1}{3}h^3E\frac{d\ddot{K}}{dz} - \frac{2h^3}{3a}\frac{d\ddot{w}}{dz}\right\} \\ \frac{1}{a}\frac{dT_2}{d\phi} + \frac{N_1}{a} + \frac{dM_2}{dz} &= \rho\left\{2h(\ddot{v} - Y) + \frac{h^3}{3a}E\frac{d\ddot{K}}{d\phi} + \frac{2h^3}{3a^2}\left(\ddot{v} - \frac{d\ddot{w}}{d\phi}\right)\right\} \\ \frac{dN_2}{dz} + \frac{1}{a}\frac{dN_1}{d\phi} - \frac{T_2}{a} &= \rho\left\{2h(\ddot{w} - Z) - \tfrac{1}{3}h^3E(\ddot{\lambda} + \ddot{\mu}) - \frac{2h^3}{3a}E\ddot{K}\right\} \\ \frac{1}{a}\frac{dG_1}{d\phi} + \frac{dH_1}{dz} + N_1 &= \frac{2\rho h^3}{3a}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + Y\right) \\ \frac{dG_2}{dz} + \frac{1}{a}\frac{dH_2}{d\phi} - N_2 &= -\tfrac{2}{3}\rho h^3\left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{X}{a}\right) \end{aligned}\right\} \qquad (11)$$

$$(M_2 - M_1)a - H_2 = 0.$$

These equations will not enable us to solve the problem in hand; in order to do this we require the equations of motion in terms of the displacements, and also the values of the sectional stresses in terms of the same quantities.

6. The values of the couples, and also the values of M_1, M_2, can be obtained by direct calculation; but the values of T_1, T_2 cannot be so obtained, since they involve the quantities Ah and A_2h^3, which are unknown, and which cannot be neglected. We shall, therefore, be compelled to find the expression for the potential energy, and employ the Calculus of Variations.

The following results will, however, be necessary hereafter. If P′, Q′, R′, S′, T′, U′, be the stresses at the point $a + h'$, z, ϕ, we have

$$\begin{aligned} T_1 a\,\delta\phi &= \int_{-h}^{h} P'(a + h')\,\delta\phi\,dh' \\ &= \int_{-h}^{h} \left\{P + h'\left(\frac{dP}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2P}{dr^2}\right)\right\}\left(1 + \frac{h'}{a}\right) a\,\delta\phi\,dh', \end{aligned}$$

whence

$$\left.\begin{aligned} T_1 &= 2hP + \tfrac{1}{3}h^3\left(\frac{d^2P}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{dP}{dr}\right) \\ T_2 &= 2hQ + \tfrac{1}{3}h^3\left(\frac{d^2Q}{dr^2}\right) \\ M_2 &= 2nh\varpi_3 + \tfrac{1}{3}nh^3\left(\frac{d^2\varpi_3}{dr^2}\right) + \frac{2nh^3}{3a}\left(\frac{d\varpi_3}{dr}\right) \\ M_1 &= 2nh\varpi_3 + \tfrac{1}{3}nh^3\left(\frac{d^2\varpi_3}{dr^2}\right) \\ G_1 &= -\tfrac{2}{3}h^3\left(\frac{dQ}{dr}\right) \\ G_2 &= \tfrac{2}{3}h^3\left\{\left(\frac{dP}{dr}\right) + \frac{P}{a}\right\} \\ H_1 &= -\tfrac{2}{3}nh^3\left\{\left(\frac{d\varpi_3}{dr}\right) + \frac{\varpi_3}{a}\right\} \\ H_2 &= \tfrac{2}{3}nh^3\left(\frac{d\varpi_3}{dr}\right) \end{aligned}\right\} \qquad (12)$$

From the third, fourth, and last of these we see that $(M_2 - M_1)\,a = H_2$, as ought to be the case.

Let

$$\left.\begin{aligned} \mathfrak{A} &= \sigma_1 + E(\sigma_1 + \sigma_2), & \mathfrak{B} &= \sigma_2 + E(\sigma_1 + \sigma_2) \\ \mathfrak{E} &= \lambda + E(\lambda + \mu), & \mathfrak{F} &= \mu + E(\lambda + \mu) \end{aligned}\right\} \quad . \quad . \qquad (13).$$

Then, in the terms multiplied by h^3, we may put

$$P = 2n\mathfrak{A}, \qquad Q = 2n\mathfrak{B},$$
$$\left(\frac{dP}{dr}\right) = 2n\mathfrak{E}, \qquad \left(\frac{dQ}{dr}\right) = 2n\mathfrak{F},$$

whence, if $p = (d\varpi_3/dr)$, the last four of (12) become

$$\left.\begin{aligned} G_1 &= -\tfrac{4}{3} nh^3 \mathfrak{F}, & G_2 &= \tfrac{4}{3} nh^3 \left(\mathfrak{E} + \frac{\mathfrak{A}}{a}\right) \\ H_1 &= -\tfrac{2}{3} nh^3 \left(p + \frac{\varpi_3}{a}\right), & H_2 &= \tfrac{2}{3} nh^3 p \end{aligned}\right\} \quad . \qquad (14)$$

Since the couples are proportional to the cube of the thickness, it follows from the fourth and fifth of (11), that the normal shearing stresses N_1, N_2 are also proportional to the cube of the thickness, and therefore the terms of lowest order in the expressions for the shearing strains ϖ_1', ϖ_2' are quadratic functions of h and h', since such functions when integrated through a section of the shell, give rise to quantities proportional to the cube of the thickness This is consistent with the fundamental hypothesis.

The next thing is to calculate the values of the quantities λ, μ, p

From the first and fifth of (6) we obtain

$$\lambda = \left(\frac{d\sigma_1}{dr}\right) = \frac{d\varpi_2}{dz} - \frac{d^2 w}{dz^2},$$

and, since the terms in λ are all multiplied by h^3, we may put

$$\lambda = -\frac{d^2 w}{dz^2} \quad . \qquad (15)$$

Similarly from the second and fourth of (6) we obtain

$$\mu = -\frac{1}{a^2}\left(\frac{d^2 w}{d\phi^2} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2) \quad . \qquad (16)$$

Lastly,

$$p = \left(\frac{d\varpi_3}{dr}\right) = \frac{1}{a}\left(\frac{d^2 u}{dr\, d\phi}\right) + \left(\frac{dv}{dr\, dz}\right) - \frac{1}{a^2}\frac{du}{d\phi},$$

or

$$p = -\frac{2}{a}\frac{d^2 w}{dz\, d\phi} + \frac{1}{a}\frac{dv}{dz} - \frac{1}{a^2}\frac{dv}{d\phi} \qquad (17).$$

We have, therefore, completely determined the values of the couples in terms of known quantities

We shall also require the values of $(d^2\sigma_1/dr^2)$, $(d^2\sigma_2/dr^2)$, $(d^2\varpi_3/dr^2)$, the first two of which we have denoted by λ', μ'; and the last of which we shall denote by p'. The values of these quantities can, by a similar process, be shown to be

$$\left.\begin{aligned} \lambda' &= E\frac{d^2 K}{dz^2} \\ \mu' &= -\frac{2\mu}{a} + \frac{E}{a^2}\frac{d^2 K}{d\phi^2} - \frac{E}{a}(\lambda + \mu) \\ p' &= -\frac{p}{a} + \frac{\varpi_3}{a^3} + \frac{2E}{a}\frac{d^2 K}{dz\, d\phi} \end{aligned}\right\} \qquad (18).$$

Equation (17) and the last of (18), combined with the third and fourth of (12), determine the values of M_1, M_2

It will be desirable to point out at this stage of the investigation, that we have obtained materials for the complete solution of any problem in which T_1 and u are zero, and none of the quantities are functions of z The boundary conditions at a free edge will be discussed in § 11, and the reader who does not wish to be troubled with the long analytical process of finding the potential energy and working out the variational equation of motion, may pass at once to § 10, and the following sections where certain problems of a fairly simple kind are discussed

7 We must now find the potential energy due to strain.

By the ordinary formula, the potential energy of a portion of the shell is

$$W = \tfrac{1}{2}\iiint_{-h}^{h} [(m+n)\Delta'^2 + n\{\varpi_1'^2 + \varpi_2'^2 + \varpi_3'^2 - 4(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1')\}](1 + h'/a)\,dh'dS \quad (19),$$

where the integration with respect to z and ϕ extends over the middle surface of the portion considered In evaluating this expression we may at once omit ϖ_1', ϖ_2', for since they are quadratic functions of h and h', they will give rise to terms which are proportional to h^5, which are to be neglected.

Since

$$\Delta' = \Delta + h'\left(\frac{d\Delta}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\Delta}{dr^2}\right) + \ldots$$

it follows that

$$\tfrac{1}{2}(m+n)\int_{-h}^{h}\Delta'^2\left(1+\frac{h'}{a}\right)dh'$$
$$= (m+n)\left\{h\Delta^2 + \tfrac{1}{3}h^3\left(\frac{d\Delta}{dr}\right)^2 + \tfrac{1}{3}h^3\Delta\left(\frac{d^2\Delta}{dr^2}\right) + \frac{2h^3}{3a}\Delta\left(\frac{d\Delta}{dr}\right)\right\},$$

from which it is seen that W is expressible in a series of odd powers of h

From (5) we obtain

$$\Delta = (1-E)(\sigma_1+\sigma_2) + \frac{A}{m+n},$$

$$\left(\frac{d\Delta}{dr}\right) = (1-E)(\lambda+\mu) + \frac{A_1}{m+n},$$

$$\left(\frac{d^2\Delta}{dr^2}\right) = (1-E)(\lambda'+\mu') + \frac{A_2}{m+n},$$

and, therefore, the portion of W per unit of area of the middle surface, which depends upon Δ', is

$$\frac{4n^2}{m+n}\Big\{h\,(\sigma_1+\sigma_2+A/2n)^2+\tfrac{1}{3}h^3(\lambda+\mu+A_1/2n)^2$$
$$+\tfrac{1}{3}h^3(\sigma_1+\sigma_2+A/2n)(\lambda'+\mu'+A_2/2n)+\frac{2h^3}{3a}(\sigma_1+\sigma_2+A/2n)(\lambda+\mu+A_1/2n)\Big\} \quad (20)$$

in which in the last three terms we may omit the A's since they are multiplied by h^3

Again

$$\sigma_1'\sigma_2'(1+h'/a)=\sigma_1\sigma_2+\lambda\mu h'^2+\tfrac{1}{2}(\lambda'\sigma_2+\mu'\sigma_1)h'^2+(\lambda\sigma_2+\mu\sigma_1)h'^2/a+$$

whence

$$2n\int_{-h}^{h}\sigma_1'\sigma_2'(1+h'/a)\,dh'=4nh\sigma_1\sigma_2+\tfrac{4}{3}nh^3\lambda\mu+\tfrac{2}{3}nh^3(\lambda'\sigma_2+\mu'\sigma_1)$$
$$+\frac{4nh^3}{3a}(\lambda\sigma_2+\mu\sigma_1) \qquad (21)$$

Also

$$(\sigma_1'+\sigma_2')\sigma_3'(1+h'/a)=(\sigma_1+\sigma_2)\sigma_3+h'^2(\lambda+\mu)\left(\frac{d\sigma_3}{dr}\right)+\tfrac{1}{2}h'^2(\lambda'+\mu')\sigma_3$$
$$+\tfrac{1}{2}h'^2(\sigma_1+\sigma_2)\left(\frac{d^2\sigma_3}{dr^2}\right)+\frac{h'^2}{a}\left\{(\lambda+\mu)\sigma_3+(\sigma_1+\sigma_2)\left(\frac{d\sigma_3}{dr}\right)\right\},$$

whence

$$2n\int_{-h}^{h}(\sigma_1'+\sigma_2')\sigma_3'(1+h'/a)\,dh'=4nh(\sigma_1+\sigma_2)\left\{\frac{A}{m+n}-E(\sigma_1+\sigma_2)\right\}$$
$$-\tfrac{4}{3}nh^3E(\lambda+\mu)^2-\tfrac{4}{3}nh^3E(\sigma_1+\sigma_2)(\lambda'+\mu')-\frac{8nh^3}{3a}E(\lambda+\mu)(\sigma_1+\sigma_2) \qquad (22)$$

Lastly

$$\tfrac{1}{2}n\int_{-h}^{h}\varpi_3'^2(1+h'/a)\,dh'=nh\varpi_3^2+\tfrac{1}{3}nh^3p^2+\tfrac{1}{3}nh^3\varpi_3p'+\frac{2nh^3}{3a}\varpi_3p \qquad (23)$$

Substituting from (20), (21), (22), (23) in (19), it will be found that the term Ah, which is (or at any rate may be) proportional to h^3, disappears; and thus the value of the potential energy per unit of the area of the middle surface is

$$\begin{aligned}W=\;&2nh\,\{\sigma_1^2+\sigma_2^2+E(\sigma_1+\sigma_2)^2+\tfrac{1}{2}\varpi^2\}\\&+\tfrac{2}{3}nh^3\{\lambda^2+\mu^2+E(\lambda+\mu)^2+\tfrac{1}{2}p^2\}\\&+\tfrac{2}{3}nh^3(\mathfrak{A}\lambda'+\mathfrak{B}\mu'+\tfrac{1}{2}\varpi p')\\&+\tfrac{4}{3}\frac{nh^3}{a}(\mathfrak{A}\lambda+\mathfrak{B}\mu+\tfrac{1}{2}\varpi p)\quad .\quad .\end{aligned} \qquad (24),$$

in which ϖ is written for ϖ_3, the suffix being no longer required.

This is the expression for the potential energy as far as the term involving h^3. The first term depends solely upon the extension of the middle surface; the second term depends principally upon the quantities by which the bending is specified, and the

third and fourth consist of the products of the extensions and the quantities which principally depend upon the bending

8. Having obtained the value of the potential energy, we must in the next place form the variational equation of motion This equation may symbolically be written

$$\delta W + \delta\mathfrak{T} = \delta U + \delta\mathfrak{L} \tag{25}$$

where $\delta\mathfrak{T}$ is the term which depends upon the time variations of the displacements, δU is the work done by the bodily forces, and $\delta\mathfrak{L}$ represents the work done upon the edges of the portion of the shell considered, in producing the displacements, δu, δv, δw, by the forces arising from the action of contiguous portions of the shell It, therefore, follows that $\delta\mathfrak{L}$ is a line integral taken round the edge of the portion of the shell which is being considered, and as one of our objects is to calculate the values of the sectional stresses in terms of the displacements by means of (25), it will be convenient to apply the variational equation to a curvilinear rectangle bounded by four lines of curvature.

We must now calculate $\delta\mathfrak{T}$ We have

$$\delta\mathfrak{T} = \rho\iiint_{-h}^{h} (u'\,\delta u' + v'\,\delta v' + w'\,\delta w')\,(1 + h'/a)\,dh'\,dS$$

Now

$$\int_{-h}^{h} u'\,\delta u'\,(1 + h'/a)\,dh' = 2hu\,\delta u + \tfrac{2}{3}h^3\frac{du}{dr}\frac{d\delta u}{dr} + \tfrac{1}{3}h^3\left(u\frac{d^2\delta u}{dr^2} + \frac{d^2\dot{u}}{dr^2}\delta u\right)$$
$$+ \frac{2h^3}{3a}\left(u\frac{d\delta u}{dr} + \frac{du}{dr}\delta u\right)$$
$$= 2h\dot{u}\,\delta u + \tfrac{2}{3}h^3\frac{dw}{dz}\frac{d\delta w}{dz} + \tfrac{1}{3}h^3E\left(u\frac{d\delta K}{dz} + \frac{dK}{dz}\delta u\right)$$
$$- \frac{2h^3}{3a}\left(u\frac{d\delta w}{dz} + \frac{dw}{dz}\delta u\right)$$

by (8) and (9). Treating the other terms in a similar way, we shall find that the value of $\delta\mathfrak{T}$ is

$$\delta\mathfrak{T} = 2\rho h\iint(\ddot{u}\,\delta u + \dot{v}\,\delta v + \dot{w}\,\delta w)\,dS$$
$$+ \tfrac{2}{3}\rho h^3\iint\left\{\frac{d\dot{w}}{dz}\frac{d\delta w}{dz} + \frac{1}{a^2}\left(\frac{d\dot{w}}{d\phi} - \ddot{v}\right)\left(\frac{d\delta w}{d\phi} - \delta v\right) + E^2\dot{K}\,\delta K\right\}dS$$
$$+ \tfrac{1}{3}\rho h^3E\iint\left\{\dot{u}\frac{d\delta K}{dz} + \frac{\dot{v}}{a}\frac{d\delta K}{d\phi} - \ddot{w}\,(\delta\lambda + \delta\mu) + \frac{dK}{dz}\delta u + \frac{1}{a}\frac{dK}{d\phi}\delta v - (\dot{\lambda} + \dot{\mu})\,\delta w\right\}dS$$
$$- \frac{2\rho h^3}{3a}\iint\left\{\ddot{u}\frac{d\delta w}{dz} + \frac{d\ddot{w}}{dz}\delta u + \frac{\ddot{v}}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) + \frac{1}{a}\left(\frac{d\ddot{w}}{d\phi} - \dot{v}\right)\delta v\right.$$
$$\left. + E\,(w\,\delta K + \ddot{K}\,\delta w)\right\}dS \quad . \quad . \quad (26).$$

We must, in the next place, calculate $\delta\mathfrak{L}$. We have

$$\delta\mathfrak{L} = \iint_{-h}^{h} (\mathrm{P}'\,\delta u' + \mathrm{U}'\,\delta v')\,(d + h')\,dh'\,d\phi + \iint_{-h}^{h} (\mathrm{Q}'\,\delta v' + \mathrm{U}'\,\delta u')\,dh'\,dz$$
$$+ \int \mathrm{N}_2\,\delta w/a\,d\phi + \int \mathrm{N}_1\,\delta w\,dz \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From the way in which $\delta\mathfrak{T}$ has been calculated, we see from (8), (9), and (12), that

$$\int_{-h}^{h} \mathrm{P}'\,\delta u'\,(1 + h'/a)\,dh' = \mathrm{T}_1\,\delta u + \mathrm{G}_2\,\frac{d\delta u}{dr} + \tfrac{1}{3}\,h^3\mathrm{P}\,\frac{d^2\delta u}{dr^2}$$
$$= \mathrm{T}_1\,\delta u - \mathrm{G}_2\,\frac{d\delta w}{dz} + \tfrac{2}{3}\,nh^3\mathrm{E}\mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz}$$

Treating the other terms in a similar way, we find

$$\delta\mathfrak{L} = \int\Big\{\mathrm{T}_1\,\delta u + \mathrm{M}_2\,\delta v + \mathrm{N}_2\,\delta w - \mathrm{G}_2\,\frac{d\delta w}{dz} + \frac{\mathrm{H}_1}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right)$$
$$+ \tfrac{2}{3}\,nh^3\mathrm{E}\mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz} + \frac{nh^3\mathrm{E}\varpi}{3a}\,\frac{d\delta\mathrm{K}}{d\phi}\Big\}a\,d\phi$$
$$+ \int\Big\{\mathrm{M}_1\,\delta u + \mathrm{T}_2\,\delta v + \mathrm{N}_1\,\delta w + \frac{\mathrm{G}_1}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) - \mathrm{H}_2\,\frac{d\delta w}{dz}$$
$$+ \frac{2nh^3}{3a}\,\mathrm{E}\mathfrak{B}\,\frac{d\delta\mathrm{K}}{\delta\phi} + \tfrac{1}{3}\,nh^3\mathrm{E}\varpi\,\frac{d\delta\mathrm{K}}{dz}\Big\}dz \qquad (28).$$

Lastly, since the shell is supposed to be so thin that X, Y, Z, may be treated as constants during the integration with respect to h',

$$\delta\mathrm{U} = \rho\iiint_{-h}^{h} (\mathrm{X}\,\delta u' + \mathrm{Y}\,\delta v' + \mathrm{Z}\,\delta w')\,(1 + h'/a)\,dh'\,d\mathrm{S}$$
$$= 2\rho\,h\iint(\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w)\,d\mathrm{S}$$
$$+ \tfrac{1}{3}\rho\,h^3\mathrm{E}\iint\Big\{\mathrm{X}\,\frac{d\delta\mathrm{K}}{dz} + \frac{\mathrm{Y}}{a}\,\frac{d\delta\mathrm{K}}{d\phi} - \mathrm{Z}\,(\delta\lambda + \delta\mu)\Big\}\,d\mathrm{S}$$
$$- \frac{2\rho h^3}{3a}\iint\Big\{\mathrm{X}\,\frac{d\delta w}{dz} + \frac{\mathrm{Y}}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) + \mathrm{ZE}\,\delta\mathrm{K}\Big\}d\mathrm{S} \quad . \quad . \quad (29).$$

9. We have now obtained all the materials for the complete solution of the problem, and we shall proceed to work out the variation in the ordinary way.

Let us denote the four terms of the expression for W given in (24), when integrated over a curvilinear rectangle bounded by four lines of curvature, by W_1, W_2, W_3, W_4. Then

$$\delta\mathrm{W}_1 = 4nh\iint(\mathfrak{A}\delta\sigma_1 + \mathfrak{B}\delta\sigma_2 + \tfrac{1}{2}\,\varpi\delta\varpi)\,a\,dz\,d\phi$$

Substituting the values of σ_1, σ_2, ϖ from the first, second, and sixth of (6), and integrating by parts we shall obtain

$$\delta W_1 = 4nh \int (\mathfrak{A}\delta u + \tfrac{1}{2}\varpi\delta v)\, a\, d\phi + 4nh \int (\tfrac{1}{2}\varpi\delta u + \mathfrak{B}\delta v)\, dz$$
$$- 4nh \iint \left\{ \left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right)\delta u + \left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \frac{1}{2}\frac{d\varpi}{dz}\right)\delta v - \frac{\mathfrak{B}}{a}\delta w \right\} a\, dz\, d\phi \qquad (30)$$

Now δW_2, δW_3, δW_4 depend upon h^3, if therefore we substitute in (25) the value of δW_1 from (30), and the portions of $\delta\mathfrak{T}$, δU, and $\delta\mathfrak{L}$, which depend upon h, we shall obtain the approximate equations

$$\left.\begin{aligned} T_1 &= 4nh\mathfrak{A}, \qquad T_2 = 4nh\mathfrak{B} \\ M_1 &= M_2 = 2nh\varpi \end{aligned}\right\} \qquad (31),$$

and

$$\left.\begin{aligned} \rho\ddot{u} &= 2n\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right) + \rho X \\ \rho\ddot{v} &= 2n\left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \frac{1}{2}\frac{d\varpi}{dz}\right) + \rho Y \\ \rho\ddot{w} &= -2n\mathfrak{B}/a + \rho Z \end{aligned}\right\} \qquad (32)$$

These equations are the same as those obtained by Mr LOVE,* and which are employed by him in discussing the vibrations of a cylindrical shell The complete equations giving $\rho\ddot{u}$, $\rho\ddot{v}$, $\rho\ddot{w}$ in terms of the displacements and their space variations contain certain additional terms involving h^2 (since the common factor h disappears) which it is our object to determine, but, since we do not retain terms higher than h^3, we may, if convenient, substitute the above approximate values in all terms of (25) which are multiplied by h^3.

Again

$$\delta W_2 = \tfrac{4}{3} nh^3 \iint (\mathfrak{E}\,\delta\lambda + \mathfrak{F}\,\delta\mu + \tfrac{1}{2}p\,\delta p)\, a\, dz\, d\phi$$

Substituting the values of λ, μ, p from (15), (16), and (17), we obtain

$$\iint \mathfrak{E}\,\delta\lambda\, dz\, d\phi = -\iint \mathfrak{E}\frac{d^2\delta w}{dz^2}\, dz\, d\phi$$
$$= \int \left(\frac{d\mathfrak{E}}{dz}\delta w - \mathfrak{E}\frac{d\,\delta w}{dz}\right) d\phi - \iint \frac{d^2\mathfrak{E}}{dz^2}\delta w\, dz\, d\phi \qquad (33),$$

also

* 'Phil. Trans,' A., 1888, pp 538 and 540. Equations (32) correspond to LOVE's equations (86), (87), and (88); and (31) to (101).

$$\iint \mathfrak{F}\,\delta\mu\,dz\,d\phi = -\frac{1}{a^2}\iint \mathfrak{F}\left\{\frac{d^2\,\delta w}{d\phi^2}+\delta w+\mathrm{E}\left(a\frac{d\,\delta u}{dz}+\frac{d\,\delta v}{d\phi}+\delta w\right)\right\}dz\,d\phi$$

$$= -\frac{\mathrm{E}}{a}\int \mathfrak{F}\,\delta u\,d\phi - \frac{1}{a^2}\int\left(\mathrm{E}\mathfrak{F}\,\delta v - \frac{d\mathfrak{F}}{d\phi}\,\delta w + \mathfrak{F}\frac{d\delta w}{d\phi}\right)dz$$

$$+\frac{1}{a^2}\iint\left\{\mathrm{E}a\frac{d\mathfrak{F}}{dz}\,\delta u+\mathrm{E}\frac{d\mathfrak{F}}{d\phi}\,\delta v-\left(\frac{d^2\mathfrak{F}}{d\phi^2}+\mathfrak{F}+\mathrm{E}\mathfrak{F}\right)\delta w\right\}dz\,d\phi \qquad (34),$$

and

$$\tfrac{1}{2}\iint p\,\delta p\,dz\,d\phi = \frac{1}{2a^2}\iint p\left(a\frac{d\delta v}{dz}-\frac{d\delta u}{d\phi}-2a\frac{d^2\delta w}{dz\,d\phi}\right)dz\,d\phi$$

$$= \frac{1}{2a}\int p\,\delta v\,d\phi - \frac{1}{2a^2}\int p\,\delta u\,dz + \frac{1}{2a^2}\iint\left(\frac{dp}{d\phi}\,\delta u - a\frac{dp}{dz}\,\delta v\right)dz\,d\phi$$

$$-\frac{1}{a}\iint p\,\frac{d^2\delta w}{dz\,d\phi}\,dz\,d\phi.$$

The last integral can be evaluated in two different ways, according as we integrate, first, with respect to ϕ, and secondly, with respect to z; or first with respect to z, and secondly with respect to ϕ. The proper way to deal with such a term is, to evaluate the integral in both ways, and then multiply the two values by β and $1-\beta$, and add, where β is a quantity which must be determined from the conditions of the problem in hand. We shall thus find that the value of β is $\frac{1}{2}$, we therefore obtain

$$\tfrac{1}{2}\iint p\,\delta p\,dz\,d\phi = \frac{1}{2a}\int\left(p\,\delta v+\frac{dp}{d\phi}\,\delta w - p\frac{d\delta w}{d\phi}\right)d\phi$$

$$-\frac{1}{2a^2}\int\left(p\,\delta u - a\frac{dp}{dz}\,\delta w + ap\frac{d\delta w}{dz}\right)dz$$

$$+\frac{1}{2a^2}\iint\left(\frac{dp}{d\phi}\,\delta u - a\frac{dp}{dz}\,\delta v - 2a\frac{d^2p}{dz\,d\phi}\,\delta w\right)dz\,d\phi \qquad (35)$$

Collecting all the terms together from (33), (34), and (35), we finally obtain

$$\delta \mathrm{W}_2 = \tfrac{4}{3}nh^3\int\left\{-\frac{\mathrm{E}\mathfrak{F}}{a}\,\delta u + \left(\frac{d\mathfrak{E}}{dz}+\frac{1}{2a}\frac{dp}{d\phi}\right)\delta w - \mathfrak{E}\frac{d\delta w}{dz} - \frac{p}{2a}\left(\frac{d\delta w}{d\phi}-\delta v\right)\right\}a\,d\phi$$

$$+\tfrac{4}{3}nh^3\int\left\{-\frac{p}{2a}\,\delta u - a\frac{(1+\mathrm{E})\mathfrak{F}}{a}\,\delta v + \left(\frac{1}{a}\frac{d\mathfrak{F}}{d\phi}+\frac{1}{2}\frac{dp}{dz}\right)\delta w\right.$$

$$\left. -\frac{\mathfrak{F}}{a}\left(\frac{d\delta w}{d\phi}-\delta v\right)-\tfrac{1}{2}p\frac{d\delta w}{dz}\right\}dz$$

$$+\frac{4nh^3}{3a}\iint\left[\left(\mathrm{E}\frac{d\mathfrak{F}}{dz}+\frac{1}{2a}\frac{dp}{d\phi}\right)\delta u + \left(\frac{\mathrm{E}}{a}\frac{d\mathfrak{F}}{d\phi}-\frac{1}{2}\frac{dp}{dz}\right)\delta v\right.$$

$$\left.-\left\{a\frac{d\,\mathfrak{E}}{dz^2}+\frac{1}{a}\left(\frac{d^2\mathfrak{F}}{d\phi^2}+\mathfrak{F}+\mathrm{E}\mathfrak{F}\right)+\frac{d^2p}{dz\,d\phi}\right\}\delta w\right]a\,dz\,d\phi \qquad (36).$$

Let

$$\mathfrak{E}' = \lambda' + E(\lambda' + \mu'), \qquad \mathfrak{F}' = \mu' + E(\lambda' + \mu') \tag{37},$$

then

$$\delta W_3 = \tfrac{2}{3} nh^3 \iint (\mathfrak{E}'\delta\sigma_1 + \mathfrak{F}'\delta\sigma_2 + \tfrac{1}{2} p'\delta\varpi + \mathfrak{A}\,\delta\lambda' + \mathfrak{B}\delta\mu' + \tfrac{1}{2}\varpi\delta p')\, dS\,,$$

from this result, together with (24), it is seen that δW_3 and δW_4 each consist of two parts, which may be denoted by $\delta W_3'$, $\delta W_3''$ and $\delta W_4'$, $\delta W_4''$ respectively The values of $\delta W_3'$ and $\delta W_4'$ may at once be written down from (30), by changing $\mathfrak{A}$, $\mathfrak{B}$, ϖ into $\mathfrak{E}'$, $\mathfrak{F}'$, p' and $\mathfrak{E}$, $\mathfrak{F}$, p respectively, and by altering the coefficients from $4nh$ into $\frac{2}{3}nh^3$ and $4nh^3/3a$ respectively. With regard to $\delta W_3''$ we have

$$\delta W_3'' = \tfrac{2}{3} nh^3 \iint (\mathfrak{A}\,\delta\lambda' + \mathfrak{B}\,\delta\mu' + \tfrac{1}{2}\varpi\delta p')\, dS.$$

Substituting the value of λ' from (18) and integrating once by parts, we obtain

$$\begin{aligned}\iint \mathfrak{A}\,\delta\lambda' dS &= \iint E\,\mathfrak{A}\frac{d^2\delta K}{dz^2}\, dS \\ &= E\int \mathfrak{A}\frac{d\delta K}{dz}\, a\,d\phi - E\iint \frac{d\mathfrak{A}}{dz}\frac{d\delta K}{dz}\, dS\end{aligned}$$

Treating the other terms in a similar way, we shall finally obtain

$$\begin{aligned}\delta W_3'' = &\int\left(\tfrac{2}{3} nh^3 E\mathfrak{A}\frac{d\delta K}{dz} + \frac{nh^3}{3a}E\varpi\frac{d\delta K}{d\phi}\right) a\,d\phi + \int\left(\frac{2nh^3}{3a}E\mathfrak{B}\frac{d\delta K}{d\phi} + \tfrac{1}{3}nh^3 E\varpi\frac{d\delta K}{dz}\right)dz \\ &- \tfrac{2}{3}nh^3\iint\left\{E\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right)\frac{d\delta K}{dz} + \frac{E}{a}\left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d\varpi}{dz}\right)\frac{d\delta K}{d\phi} - \frac{E\mathfrak{B}}{a}(\delta\lambda + \delta\mu)\right\}dS \\ &+ \tfrac{2}{3}\frac{nh^3}{a}\iint\left\{-2\mathfrak{B}\delta\mu + \frac{\varpi}{2a}(\delta\varpi - a\delta p)\right\}dS \quad . \quad . \quad . \quad . \qquad (38)\end{aligned}$$

If in the first surface integral in this equation, we substitute the approximate values of the coefficients of $d\delta K/dz$, &c , from (32). which we may do, since this integral is multiplied by h^3, and then substitute the values of $\delta W_3''$, $\delta\mathfrak{T}$, δU, and $\delta\mathfrak{L}$ in (25), it will be found that all the terms involving $d\delta K/dz$, $d\delta K/d\phi$, and $\delta\lambda + \delta\mu$ cut out; we are, therefore, no longer concerned with them, and the value of $\delta W_3''$ reduces to the last line. On this understanding we may, therefore, write

$$\begin{aligned}\delta W_3'' + \delta W_4'' &= \frac{4nh^3}{3a}\iint\left(\mathfrak{A}\,\delta\lambda + \frac{\varpi}{4a}\delta\varpi + \tfrac{1}{4}\varpi\delta p\right) dS \\ &= -\frac{4nh^3}{3a}\int\left\{\mathfrak{A}\frac{d\delta w}{dz} + \frac{\varpi}{2a}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}a\,d\phi \\ &\quad + \frac{4nh^3}{3a}\iint\left\{\frac{d\mathfrak{A}}{dz}\frac{d\delta w}{dz} + \frac{1}{2a}\frac{d\varpi}{dz}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}dS \quad . \quad . \qquad (39).\end{aligned}$$

We are now in a position to test the correctness of some of our work for picking out the terms involving $d\delta w/d\phi - \delta v$, $d\delta w/dz$ in the line integrals in (36) and (39), and equating them to the corresponding terms in the value of $\delta\mathfrak{L}$ which is given by (28), we see that we have reproduced the values of the couples, which we have already obtained in equation (14). We may therefore leave the couple terms out henceforth

Collecting all our results from (26), (28), (29), and (39) the variational equation becomes

$$\begin{aligned}
&\delta W_1 + \delta W_2 + \delta W_3' + \delta W_4' + \tfrac{4}{3}\frac{nh^3}{a}\iint\left\{\frac{d\mathfrak{A}}{dz}\frac{d\delta w}{dz} + \frac{1}{2a}\frac{d\varpi}{dz}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}dS \\
&\quad + 2\rho h\iint(u\delta u + v\delta v + w\delta w)\,dS \\
&\quad + \tfrac{2}{3}\rho h^3\iint\left\{\left(\frac{dw}{dz} - \frac{u}{a}\right)\frac{d\delta w}{dz} + \frac{1}{a^2}\left(\frac{dw}{d\phi} - 2v\right)\left(\frac{d\delta w}{d\phi} - \delta v\right) + E\,(EK - w/a)\,\delta K\right\}dS \\
&\quad + \tfrac{1}{3}\rho h^3 E\iint\left\{\frac{dK}{dz}\delta u + \frac{1}{a}\frac{dK}{d\phi}\delta v - (\lambda + \mu)\,\delta w\right\}dS \\
&\quad - \frac{2\rho h^3}{3a}\iint\left\{\frac{dw}{dz}\delta u + \frac{1}{a}\left(\frac{dw}{d\phi} - v\right)\delta v + EK\,\delta w\right\}dS \\
&= 2\rho h\iint(X\,\delta u + Y\,\delta v + Z\,\delta w)\,dS - \frac{2\rho h^3}{3a}\iint\left\{X\frac{d\delta u}{dz} + \frac{Y}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) + ZE\,\delta K\right\}dS \\
&\quad + \int(T_1\,\delta u + M_2\,\delta v + N_2\,\delta w)\,a\,d\phi + \int(M_1\,\delta u + T_2\,\delta v + N_1\,\delta w)\,dz \qquad . \qquad (40)
\end{aligned}$$

where the values of δW_1, δW_2 are given by (30) and (36), and the values of $\delta W_3'$, $\delta W_4'$ are obtained from (30) by changing certain letters as we have explained above

We have now got rid of all the terms involving the second differential coefficients of δu, δv, δw, and all that now remains to be done is to integrate by parts the terms which involve the first differential coefficients Putting

$$\alpha = \frac{2n}{\rho a}\frac{d\mathfrak{A}}{dz} + \frac{du}{dz} - \frac{v}{a} + \frac{X}{a},$$

$$\beta = \frac{n}{\rho a}\frac{d\varpi}{dz} + \frac{1}{a}\left(\frac{dw}{d\phi} - 2v\right) + \frac{Y}{a},$$

$$\gamma = E\,(EK - w/a + Z/a) \qquad . \quad . \quad . \qquad (41),$$

we have

$$\tfrac{2}{3}\rho h^3 \iint\left(\alpha \frac{d\,\delta v}{dz} + \frac{\beta}{a}\frac{d\,\delta u}{d\phi} + \gamma\,\delta K\right) dS$$
$$= \tfrac{2}{3}\rho h^3 \int(\gamma\,\delta u + \alpha\,\delta w)\, a\, d\phi + \tfrac{2}{3}\rho h^3 \int(\gamma\,\delta v + \beta\,\delta w)\, dz$$
$$- \tfrac{2}{3}\rho h^3 \iint\left\{\frac{d\gamma}{dz}\delta u + \frac{1}{a}\frac{d\gamma}{d\phi}\delta v + \left(\frac{d\alpha}{dz} + \frac{1}{a}\frac{d\beta}{d\phi} - \frac{\gamma}{a}\right)\delta w\right\} dS \qquad (42)$$

Substituting the values of δW_1, δW_2, $\delta W_3'$, $\delta W_4'$, and the right hand side of (42) in (40), and picking out the line integral terms, we obtain the following equations for the sectional stresses, viz ,

$$\left.\begin{aligned}
T_1 &= 4nh\mathfrak{A} - \frac{4nh^3}{3a}E\mathfrak{F} + \tfrac{2}{3}nh^3\mathfrak{E}' + \frac{4nh^3}{3a}\mathfrak{E} + \frac{2\rho h^3}{3a}E\,(aEK - w + Z) \\
M_2 &= 2nh\varpi + \tfrac{1}{3}nh^3p' + \frac{2nh^3}{3a}p \\
N_2 &= \tfrac{4}{3}nh^3\left(\frac{d\mathfrak{E}}{dz} + \frac{1}{2a}\frac{dp}{d\phi}\right) + \frac{4nh^3}{3a}\frac{d\mathfrak{A}}{dz} + \tfrac{2}{3}\rho h^3\left(\frac{dw}{dz} - \frac{u}{a} + \frac{X}{a}\right) \\
G_2 &= \tfrac{4}{3}nh^3\left(\mathfrak{E} + \frac{\mathfrak{A}}{a}\right) \\
H_1 &= -\tfrac{2}{3}nh^3\left(p + \frac{\varpi}{a}\right)
\end{aligned}\right\} \qquad (43)$$

which give the values of the sectional stresses across a circular section , and

$$\left.\begin{aligned}
M_1 &= 2nh\varpi + \tfrac{1}{3}nh^3p' \\
T_2 &= 4nh\mathfrak{B} - \frac{4nh^3}{3a}E\mathfrak{F} + \tfrac{2}{3}nh^3\mathfrak{F}' + \frac{2\rho h^3}{3a}E\,(aEK - w + Z) \\
N_1 &= \tfrac{4}{3}nh^3\left(\frac{1}{a}\frac{d\mathfrak{F}}{d\phi} + \tfrac{1}{2}\frac{dp}{dz}\right) + \frac{2nh^3}{3a}\frac{d\varpi}{dz} + \frac{2\rho h^3}{3a}\left(\frac{dw}{d\phi} - 2\dot{v} + Y\right) \\
G_1 &= -\tfrac{4}{3}nh^3\mathfrak{F} \\
H_2 &= \tfrac{2}{3}nh^3p
\end{aligned}\right\} \qquad (44)$$

which give the values of the sectional stresses across a meridian

If we compare these equations with the third and fourth of (12), with (14), and with the fourth and fifth of (11), we see that we have reproduced (i) the values of M_1, M_2 given by (12); (ii.) the values of the couples given by (14); (iii) the values of the normal shearing stresses which are obtained from the fourth and fifth of (11), by substituting the values of the couples from (14) We have thus subjected our fundamental hypothesis to a fairly searching test. It is, however, in our power to subject it to a still further test ; for if we equate the coefficients of δu, δv, δw in the surface integrals in (40) and (42), we shall obtain the equations of motion in terms of the displacements,

and on substituting the values of the sectional stresses from (43) and (44), in the first three of (11), we ought to reproduce the equations of motion in terms of the displacements which we have obtained from the variational equation

From (30), (36), (40), and (42) it follows that these equations are

$$\rho\left\{2(\ddot{u}-X)+\tfrac{1}{3}h^2\mathrm{E}\frac{d\mathrm{K}}{dz}-\frac{2h^2}{3a}\frac{d\ddot{u}}{dz}\right\}=4n\left(\frac{d\mathfrak{A}}{dz}+\frac{1}{2a}\frac{d\varpi}{d\phi}\right)-\frac{4nh^2}{3a}\mathrm{E}\frac{d\mathfrak{F}}{dz}$$
$$+\tfrac{2}{3}nh^2\left(\frac{d\mathfrak{E}'}{dz}+\frac{1}{2a}\frac{dp'}{d\phi}\right)+\frac{4nh^2}{3a}\frac{d\mathfrak{E}}{dz}+\tfrac{2}{3}\rho h^2\frac{d\gamma}{dz}\qquad(45)$$

$$\rho\left\{2(\ddot{v}-Y)+\frac{h^2}{3a}\mathrm{E}\frac{d\mathrm{K}}{d\phi}+\frac{2h^2}{3a}\left(\ddot{v}-\frac{d\ddot{w}}{d\phi}\right)\right\}$$
$$=4n\left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi}+\tfrac{1}{2}\frac{d\varpi}{dz}\right)+\frac{4nh^2}{3a^2}\left\{(1-\mathrm{E})\frac{d\mathfrak{F}}{d\phi}+a\frac{dp}{dz}+\tfrac{1}{2}\frac{d\varpi}{dz}\right\}$$
$$+\tfrac{2}{3}nh^2\left(\frac{1}{a}\frac{d\mathfrak{F}'}{d\phi}+\tfrac{1}{2}\frac{dp'}{dz}\right)+\frac{2\rho h^2}{3a}\frac{d\gamma}{d\phi}+\frac{2\rho h^2}{3a^2}\left(\frac{d\ddot{u}}{d\phi}-2\ddot{v}+Y\right)\qquad(46).$$

$$\rho\left\{2(\ddot{w}-Z)-\tfrac{1}{3}h^2\mathrm{E}(\lambda+\mu)-\frac{2h^2}{3a}\mathrm{EK}\right\}$$
$$=-\frac{4n\mathfrak{B}}{a}+\frac{4nh^2}{3a^2}\left\{a^2\frac{d^2\mathfrak{E}}{dz^2}+\frac{d^2\mathfrak{F}}{d\phi^2}+\mathrm{E}\mathfrak{F}+a\frac{d^2p}{dz\,d\phi}\right\}$$
$$-\frac{2nh^2}{3a}\mathfrak{F}'+\frac{4nh^2}{3a}\frac{d}{dz}\left(\frac{d\mathfrak{A}}{dz}+\frac{1}{2a}\frac{d\varpi}{d\phi}\right)+\tfrac{2}{3}\rho h^2\frac{d}{dz}\left(\frac{d\ddot{w}}{dz}-\frac{\ddot{u}}{a}+\frac{X}{a}\right)$$
$$+\frac{2\rho h^2}{3a^2}\frac{d}{d\phi}\left(\frac{d\ddot{w}}{d\phi}-2\dot{v}+Y\right)-\frac{2\rho h^2}{3a^2}\mathrm{E}(a\mathrm{EK}-\dot{w}+Z)\qquad(47)$$

If we compare these equations with the equations obtained by substituting the values of the sectional stresses in the first line of (11), it will be found that they agree in every respect

10. It will hereafter be necessary to consider certain problems in which the middle surface is supposed to experience no extension or contraction throughout the motion; and it will, therefore, be necessary to obtain the requisite equations when this is supposed to be the case.

The conditions of inextensibility are

$$\sigma_1=0,\quad\sigma_2=0,\quad\varpi=0,$$

or

$$\frac{du}{dz}=0,\qquad\frac{dv}{d\phi}+w=0,\qquad\frac{du}{d\phi}+a\frac{dv}{dz}=0\qquad(48),$$

which require that

$$u = \mathrm{U}a, \qquad v = -z\frac{d\mathrm{U}}{d\phi} + \mathrm{V}, \qquad w = z\frac{d^2\mathrm{U}}{d\phi^2} - \frac{d\mathrm{V}}{d\phi} \tag{49},$$

where U and V are functions of ϕ alone

In this case the potential energy reduces to the second line alone, and from (15), (16) and (17) we obtain

$$\left.\begin{aligned} \lambda &= 0 \\ \mu &= -\frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right) \\ p &= -\frac{2}{a}\left(\frac{d^2w}{dzd\phi} - \frac{dv}{dz}\right) \end{aligned}\right\} \tag{50}$$

and from (24)

$$\mathrm{W} = \frac{4nh^3}{3a^2}\left\{\frac{m}{a^2(m+n)}\left(\frac{d^2w}{d\phi^2} + w\right)^2 + \left(\frac{d^2w}{dzd\phi} - \frac{dv}{dz}\right)^2\right\} \tag{51},$$

which agrees with the expression obtained by Lord RAYLEIGH *

Also from (14)

$$\left.\begin{aligned} \mathrm{G}_1 &= \frac{4nh^3}{3a^2}(1 + \mathrm{E})\left(\frac{d^2w}{d\phi^2} + w\right) \\ \mathrm{G}_2 &= -\frac{4nh^3}{3a^2}\mathrm{E}\left(\frac{d^2w}{d\phi^2} + w\right) \\ \mathrm{H}_1 &= -\mathrm{H}_2 = -\tfrac{2}{3}nh^3p \end{aligned}\right\} \tag{52}$$

The values of the stresses M_1, M_2 may be obtained either from (43) and (44), or from (12) combined with (15), (16), (17), and (18) by introducing the conditions of inextensibility, and the values of T_1, T_2 might be calculated by taking the variation subject to the conditions of inextensibility, and using indeterminate multipliers This process would not, however, be of much assistance, inasmuch as it would introduce two undetermined quantities into the values of T_1, T_2, which depend upon the boundary conditions, whereas in this case the values of T_1, T_2 can be obtained directly from the first and third of (11) combined with (49). The values of N_1, N_2 are given by the fourth and fifth of (11) combined with (50) and (52).

11 We must lastly consider the boundary conditions.

Equations (43) and (44) determine the stresses on the line elements $ad\phi$ and dz respectively, which are produced by the action of contiguous portions of the shell; and it might at first sight appear, as was supposed by POISSON,† that when a shell

* 'Roy. Soc Proc.,' vol 45, p 116

† 'Paris, Acad des Sciences, Mémoires,' 1829, vol. 8, p. 357

of finite dimensions is under the influence of forces and couples applied to its edges, these equations would give the values of such forces or couples, and that the conditions to be satisfied at a free edge would require that each of the above five stresses should vanish at a free edge KIRCHHOFF* has, however, shown that this is not the case, but that the boundary conditions are only *four* in number, and the reason of this is, that it is possible to apply a certain distribution of stress to the edge of a shell, without producing any alteration in the potential energy.

By STOKES' theorem,

$$\int\left(\frac{d\mathrm{H}'}{d\phi}\,\delta w+\mathrm{H}'\,\frac{d\delta w}{d\phi}\right)d\phi+\int\left(\frac{d\mathrm{H}'}{dz}\,\delta w+\mathrm{H}'\,\frac{d\delta w}{dz}\right)dz=0\,,$$

the integration extending round any curvilinear rectangle bounded by four lines of curvature OA, AD, DB, BA If, therefore, we apply to the side AD the stresses

$$\mathrm{M}_2'=\mathrm{H}'/a,\qquad \mathrm{N}_2'=\frac{1}{a}\,\frac{d\mathrm{H}'}{d\phi},\qquad \mathrm{H}_1'=\mathrm{H}'\,,$$

to the side DB the stresses

$$\mathrm{N}_1'=\frac{d\mathrm{H}'}{dz},\qquad \mathrm{H}_2'=-\,\mathrm{H}',$$

and to the sides BO, OA, corresponding and opposite stresses respectively, the preceding integral becomes

$$\int\left\{\mathrm{M}_2'\,\delta v+\mathrm{N}_2'\,\delta w+\frac{\mathrm{H}_1'}{a}\left(\frac{d\delta w}{d\phi}-\delta v\right)\right\}a\,d\phi+\int\left(\mathrm{N}_1'\,\delta w-\mathrm{H}_2'\,\frac{d\delta w}{dz}\right)dz=0,$$

which shows that the work done by these stresses is zero. Such a system of stresses may, therefore, be applied or removed without interfering with the equilibrium or motion of the shell

Let us now suppose that the rectangle OADB, instead of being under the action of the remainder of the shell, is isolated, and that its state of strain is maintained by means of constraining stresses applied to its edges, then it follows that if, instead of the torsional couples H_1, H_2, due to the action of contiguous portions of the shell, we apply torsional couples $\mathfrak{H}_1$, $\mathfrak{H}_2$, where

$$\mathfrak{H}_1=\mathrm{H}_1+\mathrm{H}' \qquad . \qquad . \qquad (53),$$

$$\mathfrak{H}_2=\mathrm{H}_2-\mathrm{H}' \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (54),$$

* 'CRELLE,' vol 40, p. 51, 1850, and Collected Works, p 237

the state of strain will remain unchanged, provided we apply in addition the stresses

$$\left.\begin{aligned}\mathfrak{M}_2 &= \mathrm{M}_2 + \mathrm{H}'/a \\ \mathfrak{N}_2 &= \mathrm{N}_2 + \frac{1}{a}\frac{d\mathrm{H}'}{d\phi}\end{aligned}\right\} \tag{55}$$

and

$$\mathfrak{N}_1 = \mathrm{N}_1 + \frac{d\mathrm{H}'}{dz} \tag{56},$$

whence, eliminating H′ between (53), (55), and (54), (56) respectively, we obtain

$$\left.\begin{aligned}\mathfrak{M}_2 a - \mathfrak{H}_1 &= \mathrm{M}_2 a - \mathrm{H}_1 \\ \mathfrak{N}_2 - \frac{1}{a}\frac{d\mathfrak{H}_1}{d\phi} &= \mathrm{N}_2 - \frac{1}{a}\frac{d\mathrm{H}_1}{d\phi}\end{aligned}\right\} \tag{57},$$

and

$$\mathfrak{N}_1 + \frac{d\mathfrak{H}_2}{dz} = \mathrm{N}_1 + \frac{d\mathrm{H}_2}{dz} \tag{58}.$$

In these equations the Roman letters denote the stresses due to the action of contiguous portions of the shell, whose values are given by (43) and (44), whilst the Old English letters denote the values of the actual stresses applied to the boundary If, therefore, the shell consists of a portion of a cylinder which is bounded by four lines of curvature and whose edges are free, the boundary conditions along the circular edges are obtained by equating the right hand sides of the first and fourth of (43), and the right hand sides of (57) to zero, the first two of which express the condition that the tension perpendicular to, and the flexural couple about, a line element of the circular edge must vanish when the edge is free; and the boundary conditions along the straight edge are similarly obtained by equating the right hand sides of the first, second, and fourth of (44), and the right hand side of (58) to zero, the first three of which express the conditions that the tangential shearing stress, the tension and the flexural couple must vanish when the free edge is a generating line. We may also, if we do not wish to introduce the time and the bodily forces into these equations, substitute for $\ddot{u} - \mathrm{X}$, $\ddot{v} - \mathrm{Y}$, $\ddot{w} - \mathrm{Z}$ their approximate values from (32)

12. We have now obtained all the materials we require, for a perfectly accurate approximate solution of any problem relating to the vibrations of a thin cylindrical shell as far as the terms involving the cube of the thickness, but before proceeding to discuss any problems, it will be necessary to make some remarks respecting Mr. LOVE's paper. The first line of my expression for the potential energy which is given in (24), and which involves h and not h^3, agrees with the expression obtained by Mr. LOVE and other writers; also the approximate equations of motion (32) agree, as has been already pointed out, with the corresponding equations obtained by him, and by means of which he has discussed the extensional vibrations of a cylinder. It will

also hereafter appear, that observations of a precisely similar character apply to the corresponding equations which determine to a first approximation the extensional vibrations of a spherical shell This portion of his paper therefore appears to be perfectly satisfactory, but that portion which involves the terms depending upon the cube of the thickness is open to criticism.

In the first place, he appears to have employed a system of *rectangular* axes, consisting of the normal at a point on the middle surface, and the tangents to the two lines of curvature through that point Now, although it is immaterial, so long as we confine our attention to infinitesimals of the first order, whether a quantity is measured along the tangents to three orthogonal curves or along the curves themselves, yet when it is necessary to take into consideration infinitesimals of higher orders, which is always the case whenever an investigation involves changes of curvature, a method in which everything is referred to rectangular axes requires care, and on comparing the terms in h^3 in (24) with the corresponding terms in Mr LOVE's expression for the potential energy, it will be seen that he has omitted several terms which involve the extensions of the middle surface, which partly, although not entirely, arises from his having omitted the factor $1 + h'/a$ It is not improbable that these terms may be small, but at the same time we are not at liberty to neglect them altogether, for it is quite evident that a term such as $\delta(\mathfrak{B}\mu)$ in the variational equation, will give rise to terms in the equations of motion and the equations giving the values of the sectional stresses, which do not involve the extension of the middle surface

In the second place, on comparing Mr LOVE's variational equation of motion* with my equations (25), (26), (28), and (29), it will be seen that he has omitted several terms in the expressions for $\delta\mathfrak{T}$, δU, and $\delta\mathfrak{L}$

In the third place he states (p. 521) that the extensional quantities "σ_1, σ_2, ϖ may not, in general, be regarded as of a higher order of small quantities than κ_2, λ_1, κ_1," which are the quantities upon which the bending depends The argument of Lord RAYLEIGH† appears to me to show, that at points whose distance from the edge is large in comparison with the thickness, the extensional terms are usually small in comparison with the terms upon which the bending depends It must be obvious to every one, that a thin plate of metal or a steel spring can be bent with the greatest ease by means of the fingers, whereas the production of any extension of the middle surface which would be capable of measurement, would involve considerable muscular effort. These considerations indicate that when a thin shell is vibrating, the change of curvature is so greatly in excess of the extension of the middle surface, that notwithstanding the smallness of h^3 compared with h, the product $h^3(\delta\rho^{-1})^2$ is large‡ compared with

* 'Phil Trans,' A, 1888, p 514, equation (19).

† 'Roy Soc Proc,' vol. 45, p 105

‡ The problem discussed in § 14 shows that the product $h\sigma^2$ *may be* of the order $h^5(\delta\rho^{-1})^2$, except in the neighbourhood of a free edge, but in the equations of motion we have to deal with the quantities $h\sigma$ and $h^3\,\delta\rho^{-1}$.

the product $h\sigma^2$. At the same time, inasmuch as the production of change of curvature involves some extension or contraction of all but the central layers, and consequently of those portions of the shell which are near its external surface, it does not seem unreasonable to suppose that in the neighbourhood of a free edge, an extension or contraction of the middle surface may take place, which is comparable with the change of curvature

In the fourth place, Mr LOVE appears to have argued as if the equations of motion of a shell, whose middle surface undergoes no extension or contraction throughout the motion, might be obtained from his general equations (30), (31), (32), by putting $\sigma_1 = \sigma_2 = \varpi = 0$, but it has already been pointed out, that the correct equations for this kind of motion must be obtained by taking the variation subject to the conditions of inextensibility, and introducing indeterminate multipliers It will be shown in the next section, that in the case of the flexural vibrations of an indefinitely long complete cylindrical shell considered by HOPPE and Lord RAYLEIGH,* the differential equation for the tangential displacement v is of the sixth degree, and that when the cross section of the shell consists of a circular arc, this equation contains sufficient constants to enable all the conditions of the problem to be satisfied

13 The first problem which we shall consider will be that of the flexural vibrations of an indefinitely long cylinder, in which the displacement of every element lies in a plane perpendicular to the axis of the cylinder, and which has been discussed by HOPPE and Lord RAYLEIGH

In this problem the middle surface is supposed to undergo no extension or contraction throughout the motion, and the solution is most easily obtained by means of the general equations (11). In these equations we must omit all the terms on the right hand sides which involve h^3, for they would, if retained, give rise to a term involving h^4 in the period equation, which must be rejected, since we do not carry the approximation further than h^2 in determining the period

We evidently have† $M_1 = N_2 = H_2 = 0$; also none of the quantities are functions of z. The equations of motion are thus

$$\frac{dT_2}{d\phi} + N_1 = 2\rho h a \ddot{v},$$

$$\frac{dN_1}{d\phi} - T_2 = 2\rho h a \ddot{w},$$

$$\frac{dG_1}{d\phi} + N_1 a = 0,$$

* 'Theory of Sound,' vol 1, p 324, 'Roy. Soc. Proc,' vol 45, p 129 Equation (51)

† We shall presently see that these conditions imply a constraint at infinity

also the condition of inextensibility gives

$$\frac{dv}{d\phi} + w = 0$$

Eliminating N_1, T_2, and w, and substituting the value of G_1 from the first of (52), we obtain

$$\frac{4mnh^2}{3\rho a^4(m+n)}\left(\frac{d^3}{d\phi^3} + \frac{d}{d\phi}\right)^2 v + \frac{d^2\ddot{v}}{d\phi^2} - \ddot{v} = 0 \qquad (59),$$

whence, putting

$$v = A\epsilon^{\iota pt + \iota s\phi},$$

we obtain

$$p^2 = \frac{4mnh^2 s^2 (s^2 - 1)^2}{3\rho a^4 (m+n)(s^2+1)} \quad . \quad . \quad . \quad (60),$$

which is the required result.

If the cylinder is complete, s is any integer, unity excluded, but if the cross-section of the cylinder consists of a circular arc of length $2a\alpha$, s will not be an integer Its values in terms of p are the six roots of (60), but in order to obtain the frequency equation, the value of s in terms of the dimensions and elastic constants is required The additional equations are obtained from the boundary conditions, which have to be satisfied along the straight edges of the shell, and these require that the tension T_2, the normal shearing stress N_1, and the flexural couple G_1 should vanish at the edges where $\phi = \pm\alpha$.

Since

$$G_1 = -\frac{8mnh^3}{3(m+n)}\mu,$$

where

$$\mu = \frac{1}{a^2}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right),$$

the boundary conditions are obviously

$$\mu = 0,$$

$$\frac{d\mu}{d\phi} = 0,$$

$$\frac{4mnh^2}{3\rho a^2(m+n)}\frac{d^2\mu}{d\phi^2} + \frac{d^3v}{d\phi\, dt^2} = 0.$$

These conditions have to be satisfied at each of the edges of the shell where $\phi = \pm\alpha$, and there are, therefore, six equations of condition; hence the six constants

which appear in the solution of (59) can be eliminated and the resulting determinantal equation, combined with (60), will give the frequency.

If a complete cylinder of *finite length* were vibrating in this manner, it would be necessary to satisfy the conditions at the circular ends, and this would require that $T_1 = 0$, $G_2 = 0$ at the ends for all values of ϕ, and from the first and fourth of (43) we see that this requires that $\mu = 0$, or

$$\frac{d^2w}{d\phi^2} + w = 0.$$

whence

$$w = A\cos\phi + B\sin\phi$$

for all values of ϕ. Since it is impossible to satisfy this condition for the kind of motion considered, it follows that when the cylinder is of finite length it would be necessary to apply at every point of the circular boundary a tension T_1 and a couple G_2 of the requisite amount.

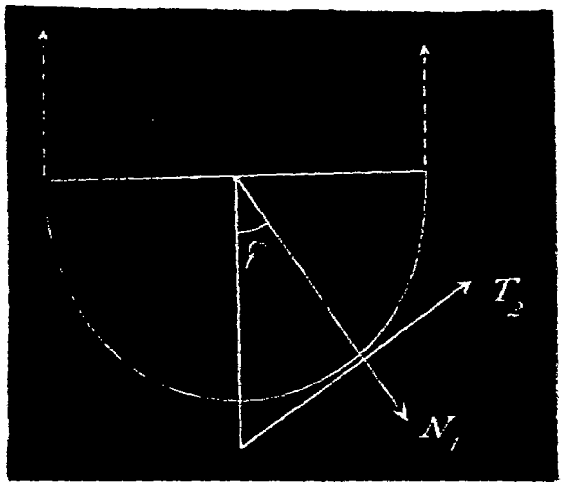
This is the question upon which Lord RAYLEIGH and Mr LOVE are at issue, and the preceding investigation shows that Mr LOVE is right in supposing that it is impossible to satisfy the boundary conditions along the curved edges of a cylindrical shell when these edges are free, although he does not appear to have noticed that it is possible to satisfy these conditions when the free edges are generating lines. In order to obtain a complete mathematical solution of this question, it would be necessary to work out the problem of the free vibrations of a complete cylindrical shell of given length $2l$, which is deformed in such a manner that $dv/d\phi + w = 0$, where v and w are functions of ϕ alone, and is then let go, without assuming that the middle surface remains unextended during the subsequent motion.

Owing unfortunately to the extremely complicated nature of the general equations, a rigorous solution of this problem would be exceedingly difficult. We shall, however, be able to throw some light upon this question, by solving and discussing the following much simpler statical problem.

14. Let us consider a heavy cylindrical shell, whose cross section is a semicircle, and which is suspended by means of vertical bands attached to its straight edges, so that its axis is horizontal; and let us investigate the state of strain produced by its own weight.

In order to simplify the problem as much as possible, we shall suppose that the displacement of every point of the middle surface lies in a plane perpendicular to the axis, and we shall afterwards investigate the stresses which must be applied to the circular edges,* in order to maintain this state of things.

* [This problem is of a similar character to that of a bar, whose natural form is circular, and which has been discussed by LAMB. 'London Math. Soc. Proc.,' vol. 19, p. 365.—June, 1890.]



We have

$$Y = -g \sin\phi, \quad Z = g\cos\phi,$$

whence, if $W = 2g\rho a h$, the equations of equilibrium are

$$\frac{dT_2}{d\phi} + N_1 = W\sin\phi,$$

$$\frac{dN_1}{d\phi} - T_2 = -W\cos\phi,$$

$$\frac{dG_1}{d\phi} + N_1 a = -\frac{h^2}{3a} W \sin\phi,$$

from which we obtain

$$\frac{d^2T_2}{d\phi^2} + T_2 = 2W\cos\phi,$$

the integral of which is

$$T_2 = A\cos\phi + B\sin\phi + W\phi\sin\phi,$$

and, therefore,

$$N_1 = A\sin\phi - B\cos\phi - W\phi\cos\phi$$

Since $N_1 = 0$ when $\phi = \frac{1}{2}\pi$, $A = 0$, also since $T_2 = \frac{1}{2}\pi W$ when $\phi = \frac{1}{2}\pi$, $B = 0$, whence

$$T_2 = W\phi\sin\phi, \qquad N_1 = -W\phi\cos\phi \quad \ldots \quad (61),$$

and, therefore,

$$\frac{dG_1}{d\phi} = Wa\left(\phi\cos\phi - \frac{h^2}{3a^2}\sin\phi\right),$$

whence

$$G_1 = Wa\left(\phi\sin\phi + \cos\phi + \frac{h^2}{3a^2}\cos\phi\right) + C.$$

Since $G_1 = 0$ when $\phi = \frac{1}{2}\pi$, $C = -\frac{1}{2}W\pi a$, accordingly

$$G_1 = Wa\left\{\phi\sin\phi + \left(1 + \frac{h^2}{3a^2}\right)\cos\phi - \tfrac{1}{2}\pi\right\} \quad . \quad . \quad (62)$$

But

$$G_1 = -\tfrac{4}{3} nh^3 \mathfrak{F} = -\tfrac{4}{3} nh^3 (1 + E) \mu \quad . \tag{63}$$

whence

$$\tfrac{4}{3} nh^3 \mu = -\frac{Wa}{1+E} \left\{ \phi \sin \phi + \left(1 + \frac{h^2}{3a^2}\right) \cos \phi - \tfrac{1}{2} \pi \right\} \tag{64}$$

Again, if R denote the change of curvature along a circular section, so that

$$R = -\frac{1}{a^2} \left(\frac{d^2 w}{d\phi^2} + w \right)$$

we have

$$\mu = R - E\sigma_2 / a \quad . \tag{65}$$

Also by (18)

$$\mu' = -(2 + E) \mu / a + \frac{E}{a^2} \frac{d^2 \sigma_2}{d\phi^2}$$

and, therefore,

$$\mathfrak{F}' = (1 + E) \mu' = -(1 + E)(2 + E) \mu / a + \frac{(1 + E) E}{a^2} \frac{d^2 \sigma_2}{d\phi^2},$$

whence, by the second of (44),

$$T_2 = 4nh (1 + E) \sigma_2 - \frac{2nh^3}{3a} (1 + E)(3 + 2E) \mu + \frac{2nh^3}{3a^2} (1 + E) E \frac{d^2 \sigma_2}{d\phi^2} + \frac{h^2}{3a^2} EW \cos \phi.$$

Substituting the values of T_2 and μ from (61) and (64), we obtain

$$2nh (1 + E) \left\{ 2\sigma_2 + \frac{h^2 E}{3a^2} \frac{d^2 \sigma_2}{d\phi^2} \right\} + \tfrac{1}{2} (2 + 3E) \left\{ \phi \sin \phi + \left(1 + \frac{h^2}{3a^2}\right) \cos \phi - \tfrac{1}{2} \pi \right\} + \frac{h^2 E}{3a^2} W \cos \phi - W\phi \sin \phi = 0.$$

This equation might, if necessary, be solved by successive approximation, but a first approximation will be sufficient. Omitting the terms in h^2, and recollecting that W involves h as a factor, we obtain

$$4nh (1 + E) \sigma_2 + W (\cos \phi - \tfrac{1}{2} \pi) + \tfrac{3}{2} EW (\phi \sin \phi + \cos \phi - \tfrac{1}{2} \pi) = 0 \tag{66}$$

whence from (64), (65), and (66), we obtain

$$\frac{R}{\sigma_2} = \frac{E}{a} + \frac{3a (\tfrac{1}{2} \pi - \phi \sin \phi - \cos \phi)}{h^2 \{ \tfrac{1}{2} \pi - \cos \phi + \tfrac{3}{2} E (\tfrac{1}{2} \pi - \phi \sin \phi - \cos \phi) \}} \quad . \quad . \tag{67}$$

Since the numerator of this fraction is an even function of ϕ, it does not change sign with ϕ, also the numerator is always positive between the limits $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, and its maximum value occurs when $\phi = 0$ and is equal to $\frac{1}{2}\pi - 1$, and its minimum value occurs when $\phi = \frac{1}{2}\pi$ and is equal to zero. We, therefore, see that when $\phi = 0$,

$$\frac{R}{\sigma_2} = \frac{E}{a} + \frac{3a}{h^2},$$

and when $\phi = \frac{1}{2}\pi$,

$$\frac{R}{\sigma_2} = \frac{E}{a}.$$

Since the thickness of the shell is supposed to be small compared with its radius, it follows that the change of curvature is large compared with the extension of the middle surface, except when c $(\frac{1}{2}\pi - \phi)$ is comparable with h, *i.e.*, in the neighbourhood of the straight edges of the shell, and therefore at all points of the shell whose distances from the edges are large in comparison with its thickness, the terms depending upon the product of the change of curvature and the cube of the thickness, *i.e.*, the terms upon which the bending depends, are of the same order as the terms depending upon the product of the extension of the middle surface and the thickness, but at points whose distances from the edge are comparable with the thickness of the shell, the extension of the middle surface is of the same order as the change of curvature, and therefore the terms depending upon the product of the change of curvature and the cube of the thickness are small in comparison with the terms depending upon the product of the extension and the thickness

We shall now calculate the stresses which must be applied to the circular edges in order to maintain this particular kind of strain. From (43) we have

$$G_2 = \tfrac{4}{3} nh^3 E(\mu + \sigma_2/a)$$

Substituting the values of μ and σ_2 from (62), (63), and (64), we see that the terms in σ_2 may be omitted, and we obtain

$$G_2 = \frac{EWa}{1+E}(\tfrac{1}{2}\pi - \phi\sin\phi - \cos\phi) \quad . \quad . \quad . \quad (68),$$

which shows that G_2 is positive.

Also

$$\begin{aligned} T_1 &= 4nh\,E\,\sigma_2 - \frac{2nh^3}{3a}E(2+3E)\mu \\ &= \frac{EW}{1+E}\phi\sin\phi \quad . \quad . \quad . \quad . \quad . \quad (69), \end{aligned}$$

which shows that T_1 is positive.

Comparing (68) and (69) with (61) and (62) we see that ratios of the tension T_1 and the couple G_2, to T_2 and G_1 are numerically equal to $E/(1+E)$; we further

see that G_1 is negative, and, therefore, the strain tends to increase the curvature of the circular sections. Now when a cylindrical shell is bent about a generating line in such a manner that its curvature is increased, all lines parallel to the axis which lie on the convex side of the middle surface will be contracted, whilst all such lines which lie on the concave side will be extended, and this contraction and extension will give rise to a couple about the circular sections which tends to produce anticlastic curvature of the generating lines In order to prevent this taking place it is necessary to apply at every point of the circular edges a couple G_2 tending to produce synclastic curvature, and a tension T_1 parallel to the axis, whose values are given by (68) and (69) If this couple and tension were removed, the middle surface would bend about its circular sections, and anticlastic curvature of the generating lines would be produced, and this would necessarily involve extension or contraction parallel to the axis, so that the problem could no longer be treated as one of two dimensions

It must, however, be within the experience of everyone that when a thin cylindrical shell of finite length, whose cross section is the arc of a circle, is bent about its generating lines, the shell does not assume a saddle-back form, and consequently the anticlastic curvature of the generating lines must be so small as to be inappreciable. This circumstance furnishes an additional argument in favour of the supposition that the extension of the middle surface is only sensible in the neighbourhood of the free edges

We therefore conclude that if the circular edges were free, some extension or contraction of the middle surface must necessarily take place, but that this extension or contraction is small compared with the change of curvature along a circular section, except just in the neighbourhood of the edges From these considerations the inference is, that if by means of proper constraints applied to the circular edges, a cylindrical shell were enabled to execute the non-extensional vibrations discussed in § 13, the vibrations would cease to be non-extensional if the constraints were removed; but that the amplitudes of those portions of the displacements upon which the extension depends, would be very small compared with the amplitudes of those portions upon which the change of curvature along a circular section depends, except just in the neighbourhood of the edges. Moreover, the theory of plane plates shows, that the frequency of the extensional vibrations is expressible* by means of a series of even powers of h, *commencing with a term independent of* h, whilst the frequency of the flexural vibrations is expressible by means of a similar series *commencing with* h^2. It therefore follows, that the pitch of the notes arising from the former class of vibrations, is high compared with the pitch of those arising from the latter class. And although, except under special circumstances, it is not possible in the case of curved shells whose edges are free, for these two classes of vibrations to coexist independently, as in the case of a plane plate; yet recent investigations show, that the pitch of

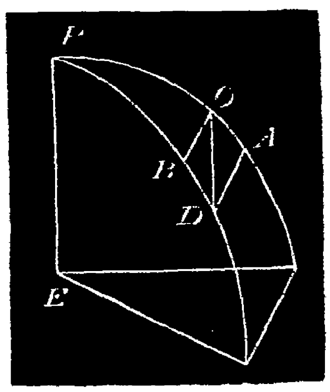
* Lord RAYLEIGH, 'London Math Soc. Proc.,' vol. 20, p. 225 See especially equations (38), (45), and (53).

notes which mainly depend upon the extension is usually high in comparison with the pitch of notes which mainly depend upon bending, and consequently the notes arising from the former cause, both on account of the smallness of their amplitudes and the highness of their pitch, would probably be so feeble in comparison with those which arise from the latter cause, as to be scarcely capable of producing any appreciable effect upon the ear. Judging from the usual course of such investigations, the probable form of the exact solution of the problems suggested at the end of § 13 would be that of an infinite series, the periods of the different components of which would satisfy a transcendental equation having an infinite number of roots; but the preceding considerations point to the conclusion that the frequency of the gravest* note given by (60), viz., $p^2 = 48mnh^2/5\rho a^4 (m + n)$, although perhaps not rigorously accurate, is a close approximation to the truth.

Spherical Shells

15. The fundamental equations for a spherical shell can be investigated in precisely the same manner as in the case of a cylindrical shell.

If u', v', w' be the component displacements at any point of the substance of the shell in the directions, θ, ϕ, r, the equations connecting the displacements and strains are



$$\left.\begin{aligned}
\sigma'_1 &= \frac{1}{r}\left(\frac{du'}{d\theta} + w'\right) \\
\sigma'_2 &= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{dv'}{d\phi} + u'\cot\theta + w'\right) \\
\sigma'_3 &= \frac{dw'}{dr} \\
\varpi'_1 &= \frac{1}{r\sin\theta}\frac{dw'}{d\phi} + \frac{dv'}{dr} - \frac{v'}{r} \\
\varpi'_2 &= \frac{du'}{dr} - \frac{u'}{r} + \frac{1}{r}\frac{dw'}{d\theta} \\
\varpi'_3 &= \frac{1}{r}\left(\frac{dv'}{d\theta} - v'\cot\theta + \frac{1}{\sin\theta}\frac{du'}{d\phi}\right)
\end{aligned}\right\} \quad (1),$$

* [The experiments of Lord RAYLEIGH, 'Phil. Mag.,' Jan., 1890, show that the effective pitch of a bell is usually not the same as that of its gravest tone; and, in the bells which he examined, the fifth tone in order was the one which agreed with the nominal pitch of the bell.—June, 1890.]

whence

$$\left.\begin{aligned}\left(\frac{du}{dr}\right) &= \varpi_2 + \frac{u}{a} - \frac{1}{a}\frac{dw}{d\theta}\\ \left(\frac{dv}{dr}\right) &= \varpi_1 + \frac{v}{a} - \frac{1}{a\sin\theta}\frac{dw}{d\phi}\\ \left(\frac{dw}{dr}\right) &= \frac{A}{m+n} - EK\end{aligned}\right\} \quad . \quad . \quad . \quad (2),$$

also

$$\left.\begin{aligned}\left(\frac{d^2u}{dr^2}\right) &= \left(\frac{d\varpi_2}{dr}\right) + \frac{\varpi_2}{a} - \frac{1}{a(m+n)}\frac{dA}{d\theta} + \frac{E}{a}\frac{dK}{d\theta}\\ \left(\frac{d^2v}{dr^2}\right) &= \left(\frac{d\varpi_1}{dr}\right) + \frac{\varpi_1}{a} - \frac{1}{a(m+n)\sin\phi}\frac{dA}{d\phi} + \frac{E}{a\sin\theta}\frac{dK}{d\phi}\\ \left(\frac{d^2w}{dr^2}\right) &= \frac{A_1}{m+n} - E(\lambda+\mu)\end{aligned}\right\} \quad . \quad (3).$$

16 We can now obtain the equations of motion in terms of the sectional stresses.

If dS be an element of the middle surface whose coordinates are (a, θ, ϕ), and dS' an element of a layer of the shell whose coordinates are $(a+h', \theta, \phi)$, then $dS' = (1+h'/a)^2\, dS$, whence, if in the figure OA, OB respectively coincide with the meridians and circular sections, we obtain by resolving parallel to OA,

$$\frac{d}{d\theta}(T_1 a\sin\theta\,\delta\phi)\,\delta\theta - T_2 a\cos\theta\,\delta\theta\,\delta\phi + \frac{d}{d\phi}(M_1 a\,\delta\theta)\,\delta\phi + N_2 a\sin\theta\,\delta\theta\,\delta\phi$$

$$= \rho\, dS\int_{-h}^{h}(\ddot{u}' - X)(1+h'/a)^2\, dh' \qquad (4)$$

But

$$u' = u + h'\left(\frac{du}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2u}{dr^2}\right),$$

accordingly if we substitute the values of (du/dr) and (d^2u/dr^2) from (2) and (3), and recollect that all quantities which vanish with h may be omitted when multiplied by h^3, the right hand side of (4) becomes

$$\rho dS\left\{2h\left(1+\frac{h^2}{a^2}\right)\ddot{u} + \frac{h^3E}{3a}\frac{d\ddot{K}}{d\theta} - \frac{4h^3}{3a^3}\frac{d\ddot{w}}{d\theta} - 2h\left(1+\frac{h^2}{3a^2}\right)X\right\}.$$

Resolving parallel to OB, OC, and then taking moments about OA, OB, OC we shall obtain in a similar way five other equations, which, together with (4), may be written,

$$\left.\begin{aligned}
&\frac{d}{d\theta}(\mathrm{T}_1 \sin\theta) - \mathrm{T}_2\cos\theta + \frac{d\mathrm{M}_1}{d\phi} + \mathrm{N}_2\sin\theta \\
&\qquad = \left\{2h\left(1+\frac{h^2}{a^2}\right)u + \frac{h^3\mathrm{E}}{3a}\frac{d\mathrm{K}}{d\theta} - \frac{4h^3}{3a^2}\frac{dw}{d\theta} - 2h\left(1+\frac{h^2}{3a^2}\right)\mathrm{X}\right\}\rho a\sin\theta, \\
&\frac{d\mathrm{T}_2}{d\phi} + \frac{d}{d\theta}(\mathrm{M}_2\sin\theta) + \mathrm{M}_1\cos\theta + \mathrm{N}_1\sin\theta \\
&\qquad = \left\{2h\left(1+\frac{h^2}{a^2}\right)v + \frac{h^3\mathrm{E}}{3a\sin\theta}\frac{d\mathrm{K}}{d\phi} - \frac{4h^3}{3a^2\sin\theta}\frac{dw}{d\phi} - 2h\left(1+\frac{h^2}{3a^2}\right)\mathrm{Y}\right\}\rho a\sin\theta, \\
&\frac{d}{d\theta}(\mathrm{N}_2\sin\theta) + \frac{d\mathrm{N}_1}{d\phi} - (\mathrm{T}_1+\mathrm{T}_2)\sin\theta \\
&\qquad = \left\{2h\left(1+\frac{h^2}{3a^2}\right)w - \tfrac{1}{3}h^3\mathrm{E}(\lambda+\mu) - \frac{4h^3}{3a}\mathrm{EK} - 2h\left(1+\frac{h^2}{3a^2}\right)\mathrm{Z}\right\}\rho a\sin\theta, \\
&\frac{d\mathrm{G}_1}{d\phi} + \mathrm{N}_1 a\sin\theta + \frac{d}{d\theta}(\mathrm{H}_1\sin\theta) - \mathrm{H}_2\cos\theta \\
&\qquad = \tfrac{2}{3}\rho h^3\left(\frac{1}{\sin\theta}\frac{du}{d\theta} - 3v + 2\mathrm{Y}\right)\sin\theta, \\
&\frac{d}{d\theta}(\mathrm{G}_2\sin\theta) + \mathrm{G}_1\cos\theta - \mathrm{N}_2 a\sin\theta + \frac{d\mathrm{H}_2}{d\theta} \\
&\qquad = -\tfrac{2}{3}\rho h^3\left(\frac{du}{d\theta} - 3u + 2\mathrm{X}\right)\sin\theta, \\
&\qquad\qquad (\mathrm{M}_1 - \mathrm{M}_2)a - \mathrm{H}_1 - \mathrm{H}_2 = 0
\end{aligned}\right\} \quad (5)$$

17 We shall now (as in the case of a cylindrical shell) proceed to obtain the values of the couples and the stresses M_1, M_2 by direct calculation

We have

$$\mathrm{T}_1\, a\sin\theta\,\delta\phi = \int_{-h}^{h}\mathrm{P}'(a+h')\sin\theta\,\delta\phi\,dh';$$

whence

$$\left.\begin{aligned}
\mathrm{T}_1 &= 2h\mathrm{P} + \tfrac{1}{3}h^3\left(\frac{d^2\mathrm{P}}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{d\mathrm{P}}{dr}\right) \\
\mathrm{T}_2 &= 2h\mathrm{Q} + \tfrac{1}{3}h^3\left(\frac{d^2\mathrm{Q}}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{d\mathrm{Q}}{dr}\right) \\
\mathrm{M}_1 = \mathrm{M}_2 &= 2nh\varpi + \tfrac{1}{3}nh^3\left(\frac{d^2\varpi_3}{dr^2}\right) + \frac{2nh_3}{3a}\left(\frac{d\varpi_3}{dr}\right) \\
\mathrm{G}_1 &= -\tfrac{2}{3}h^3\left\{\left(\frac{d\mathrm{Q}}{dr}\right) + \frac{\mathrm{Q}}{a}\right\} \\
\mathrm{G}_2 &= \tfrac{2}{3}h^3\left\{\left(\frac{d\mathrm{P}}{dr}\right) + \frac{\mathrm{P}}{a}\right\} \\
\mathrm{H}_1 = -\mathrm{H}_2 &= -\tfrac{2}{3}nh^3\left\{\left(\frac{d\varpi_3}{dr}\right) + \frac{\varpi_3}{a}\right\}
\end{aligned}\right\} \quad . \quad (6).$$

The third and sixth of these equations satisfy the last of (5) as ought to be the case. Also, employing our previous notation, we see that

$$\left.\begin{aligned} G_1 = -\tfrac{4}{3}nh^3\left(\mathfrak{F} + \frac{\mathfrak{B}}{a}\right),\quad G_2 = \tfrac{4}{3}nh^3\left(\mathfrak{E} + \frac{\mathfrak{A}}{a}\right) \\ H_1 = -H_2 = -\tfrac{2}{3}nh^3\left(p + \frac{\varpi_3}{a}\right) \end{aligned}\right\} \qquad (7)$$

Since the couples are proportional to the cube of the thickness, it follows from the fourth and fifth of (5) that the normal shearing stresses N_1, N_2 are also proportional to the cube of the thickness, and, therefore, that the shearing strains ϖ'_1, ϖ'_2 are quadratic functions of h and h'.

Employing our previous notation, the next thing is to calculate the quantities λ, μ, p, λ', μ', p'. We have

$$\left.\begin{aligned} \lambda &= \left(\frac{d\sigma_1}{dr}\right) = -\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2) \\ \mu &= \left(\frac{d\sigma_2}{dr}\right) = -\frac{1}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2w}{d\phi^2} + \cot\theta\frac{dw}{d\theta} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2) \\ p &= \left(\frac{d\varpi_3}{dr}\right) = \frac{2}{a^2\sin\theta}\left(\cot\theta\frac{dw}{d\phi} - \frac{d^2w}{d\theta\, d\phi}\right) \end{aligned}\right\} \qquad (8),$$

in which equations we have omitted all quantities which vanish with h, because λ, μ, p occur in expressions which are multiplied by h^3. Similarly

$$\left.\begin{aligned} \lambda' &= -\frac{2\lambda}{a} - \frac{E}{a}(\lambda + \mu) + \frac{E}{a^2}\frac{d^2K}{d\theta^2} \\ \mu' &= -\frac{2\mu}{a} - \frac{E}{a}(\lambda + \mu) + \frac{E}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2K}{d\phi^2} + \cot\theta\frac{dK}{d\theta}\right) \\ p' &= -\frac{2p}{a} - \frac{2E}{a^2\sin\theta}\left(\cot\theta\frac{dK}{d\phi} - \frac{d^2K}{d\theta\, d\phi}\right) \end{aligned}\right\} \qquad (9)$$

18. The variational equation may be written

$$\delta W + \delta\mathfrak{T} = \delta U + \delta\mathfrak{L} \qquad (10),$$

and we must now calculate the values of the four terms in it, and we shall begin with W.

Since we may omit ϖ'_1, ϖ'_2, and may, therefore, write ϖ for ϖ_3, the potential energy of any portion of the shell is

$$W = \tfrac{1}{2}\iiint_{-h}^{h}[(m+n)\Delta'^2 + n\{\varpi'^2 - 4(\sigma'_1\sigma'_2 + \sigma'_2\sigma'_3 + \sigma'_3\sigma'_1)\}](1 + h'/a)^2\, dh'\, dS \quad (11)$$

where the integration with respect to S extends over the middle surface of the portion considered. Since

$$\Delta' = \Delta + h'\left(\frac{d\Delta}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\Delta}{dr^2}\right) + \ldots$$

we obtain

$$\tfrac{1}{2}(m+n)\int_{-h}^{h}\Delta'^2(1+h'/a)^2\,dh'$$

$$= (m+n)\left\{h\left(1+\frac{h^2}{3a^2}\right)\Delta^2 + \tfrac{1}{3}h^3\left(\frac{d\Delta}{dr}\right)^2 + \tfrac{1}{3}h^3\Delta\left(\frac{d^2\Delta}{dr^2}\right) + \frac{4h^3}{3a}\Delta\left(\frac{d\Delta}{dr}\right)\right\}$$

$$= \frac{4n^2}{m+n}\left\{h\left(1+\frac{h^2}{3a^2}\right)(\sigma_1+\sigma_2+\mathrm{A}/2n)^2 + \tfrac{1}{3}h^3(\lambda+\mu)^2\right.$$

$$\left. + \tfrac{1}{3}h^3(\sigma_1+\sigma_2)(\lambda'+\mu') + \frac{4h^3}{3a}(\sigma_1+\sigma_2)(\lambda+\mu)\right\} \quad \ldots \quad (12),$$

also

$$2n\int_{-h}^{h}\sigma'_1\sigma'_2(1+h'/a)^2\,dh' = 4nh(1+\tfrac{1}{3}h^2/a^2)\,\sigma_1\sigma_2 + \tfrac{4}{3}nh^3\lambda\mu$$

$$+ \tfrac{2}{3}nh^3(\lambda'\sigma_2+\mu'\sigma_1) + \frac{8nh^3}{3a}(\lambda\sigma_2+\mu\sigma_1) \qquad (13),$$

and

$$2n\int_{-h}^{h}(\sigma'_1+\sigma'_2)\,\sigma'_3(1+h'/a)^2\,dh'$$

$$= 4nh(1+\tfrac{1}{3}h^2/a^2)\left\{\frac{\mathrm{A}}{m+n} - \mathrm{E}(\sigma_1+\sigma_2)\right\}(\sigma_1+\sigma_2) - \tfrac{4}{3}nh^3\mathrm{E}(\lambda+\mu)^2$$

$$- \tfrac{4}{3}nh^3\mathrm{E}(\sigma_1+\sigma_2)(\lambda'+\mu') - \frac{16nh^3}{3a}\mathrm{E}(\lambda+\mu)(\sigma_1+\sigma_2) \quad \ldots \quad (14),$$

lastly

$$\tfrac{1}{2}n\int_{-h}^{h}\varpi'^2(1+h'/a)^2\,dh' = nh(1+\tfrac{1}{3}h^2/a^2)\,\varpi^2 + \tfrac{1}{3}nh^3p^2 + \tfrac{1}{3}nh^3\varpi p' + \frac{4nh^3}{3a}\varpi p \qquad (15)$$

Substituting from (12), (13), (14) and (15) in (11), the value of W per unit of area of the middle surface is,

$$\mathrm{W} = 2nh\left(1+\frac{h^2}{3a^2}\right)\{\sigma_1^2+\sigma_2^2+\mathrm{E}(\sigma_1+\sigma_2)^2+\tfrac{1}{2}\varpi^2\}$$

$$+ \tfrac{2}{3}nh^3\{\lambda^2+\mu^2+\mathrm{E}(\lambda+\mu)^2+\tfrac{1}{2}p^2\}$$

$$+ \tfrac{2}{3}nh^3(\mathfrak{A}\lambda' + \mathfrak{B}\mu' + \tfrac{1}{2}\varpi p')$$

$$+ \frac{8nh^3}{3a}(\mathfrak{A}\lambda + \mathfrak{B}\mu + \tfrac{1}{2}\varpi p) \quad \ldots \quad (16)$$

We must now obtain $\delta\mathfrak{T}$. We have

$$\delta\mathfrak{T} = \rho \iint\int_{-h}^{h} (u'\,\delta u' + v'\,\delta v' + w'\,\delta w')\,(1 + h'/a)^2\,dh'\,dS\,;$$

also

$$\int_{-h}^{h} u'\,\delta u'\,(1 + h'/a)^2\,dh' = 2h\left(1 + \frac{h^2}{3a^2}\right) u\,\delta u + \tfrac{2}{3}\,h^3\,\frac{du}{dr}\,\frac{d\delta u}{dr} + \tfrac{1}{3}\,h^3\left(u\,\frac{d^2\delta u}{dr^2} + \frac{d^2u}{dr^2}\,\delta u\right)$$

$$+ \frac{4h^3}{3a}\left(u\,\frac{d\delta u}{dr} + \frac{du}{dr}\,\delta u\right)$$

$$= 2h\left(1 + \frac{h^2}{3a^2}\right) u\,\delta u + \frac{2h^3}{3a^2}\left(\frac{du}{d\theta} - u\right)\left(\frac{d\delta w}{d\theta} - \delta u\right)$$

$$+ \frac{h^3\mathrm{E}}{3a}\left(u\,\frac{d\delta\mathrm{K}}{d\theta} + \frac{d\mathrm{K}}{d\theta}\,\delta u\right) - \frac{4h^3}{3a^2}\left\{u\left(\frac{d\delta w}{d\theta} - \delta u\right) + \left(\frac{dw}{d\theta} - u\right)\delta u\right\}$$

by (2) and (3). Treating the other terms in a similar way, we obtain

$$\delta\mathfrak{T} = 2\rho h\,(1 + \tfrac{1}{3}h^2/a^2) \iint (\dot u\,\delta u + \dot v\,\delta v + \dot w\,\delta w)\,dS$$

$$+ \tfrac{2}{3}\,\rho h^3 \iint \left\{\frac{1}{a^2}\left(\frac{d\dot w}{d\theta} - 3\dot u\right)\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{1}{a^2}\left(\frac{1}{\sin\theta}\,\frac{d\dot w}{d\phi} - 3\dot v\right)\left(\frac{1}{\sin\theta}\,\frac{d\delta w}{d\phi} - \delta v\right)\right.$$

$$\left. + \mathrm{E}\left(\mathrm{E}\dot{\mathrm{K}} - \frac{2\dot u}{a}\right)\delta\mathrm{K}\right\} dS$$

$$+ \tfrac{1}{3}\,\rho h^3\mathrm{E} \iint \left\{\frac{\dot u}{a}\,\frac{d\delta\mathrm{K}}{d\theta} + \frac{\dot v}{a\sin\theta}\,\frac{d\delta\mathrm{K}}{d\phi} - \dot w\,(\delta\lambda + \delta\mu) + \frac{1}{a}\,\frac{d\dot{\mathrm{K}}}{d\theta}\,\delta u + \frac{1}{a\sin\theta}\,\frac{d\dot{\mathrm{K}}}{d\phi}\,\delta v\right.$$

$$\left. - (\dot\lambda + \dot\mu)\,\delta w\right\} dS$$

$$- \frac{4\rho h^3}{3a} \iint \left\{\frac{1}{a}\left(\frac{d\dot w}{d\theta} - \dot u\right)\delta u + \frac{1}{a}\left(\frac{1}{\sin\theta}\,\frac{d\dot w}{d\phi} - \dot v\right)\delta v + \mathrm{E}\dot{\mathrm{K}}\,\delta w\right\} dS \quad . \quad . \quad (17).$$

We must next find $\delta\mathfrak{L}$.

We have

$$\delta\mathfrak{L} = \iint_{-h}^{h} (\mathrm{P}'\,\delta u' + \mathrm{U}'\,\delta v')\,(a + h')\sin\theta\,dh'\,d\phi + \iint_{-h}^{h} (\mathrm{Q}'\,\delta v' + \mathrm{U}'\,\delta u')\,(a + h')\,dh'\,d\theta$$

$$+ \int \mathrm{N}_2 a\sin\theta\,\delta w\,d\phi + \int \mathrm{N}_1 a\,\delta w\,d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

whence

$$\delta\mathfrak{L} = \int\left\{T_1\,\delta u + M_2\,\delta v + N_2\,\delta w - \frac{G_2}{a}\left(\frac{d\delta u}{d\theta} - \delta u\right) + \frac{H_1}{a}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right)\right.$$

$$\left. + \frac{2nh^3E\mathfrak{A}}{3a}\frac{d\delta K}{d\theta} + \frac{nh^3E\varpi}{3a\sin\theta}\frac{d\delta K}{d\phi}\right\}a\sin\theta\,d\phi$$

$$+ \int\left\{M_1\,\delta u + T_2\,\delta v + N_1\,\delta w + \frac{G_1}{a}\left(\frac{1}{\sin\theta}\frac{d\delta u}{d\phi} - \delta v\right) - \frac{H_2}{a}\left(\frac{d\delta u}{d\theta} - \delta u\right)\right.$$

$$\left. + \frac{2nh^3E\mathfrak{B}}{3a\sin\theta}\frac{d\delta K}{d\phi} + \frac{nh^3E\varpi}{3a}\frac{d\delta K}{d\theta}\right\}a\,d\theta \quad . \qquad (19)$$

Lastly,

$$\delta U = \rho\iiint_{-h}^{h}(X\,\delta u' + Y\,\delta v' + Z\,\delta w')\,(1 + h'/a)^2\,dh'dS$$

$$= 2\rho h\left(1 + \frac{h^2}{3a^2}\right)\iint(X\,\delta u + Y\,\delta v + Z\,\delta w)\,dS$$

$$+ \tfrac{1}{3}\rho h^3E\iint\left\{\frac{X}{a}\frac{d\delta K}{d\theta} + \frac{Y}{a\sin\theta}\frac{d\delta K}{d\phi} - Z\,(\delta\lambda + \delta\mu)\right\}dS$$

$$- \frac{4\rho h^3}{3a}\iint\left\{\frac{X}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{Y}{a}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right) + ZE\,\delta K\right\}dS \qquad (20)$$

19 We shall, as in the case of a cylindrical shell, denote the four lines of W by W_1, W_2, W_3, W_4. Whence

$$\delta W_1 = 4nh\,(1 + \tfrac{1}{3}h^2/a^2)\iint(\mathfrak{A}\delta\sigma_1 + \mathfrak{B}\delta\sigma_2 + \tfrac{1}{2}\varpi\delta\varpi)\,dS$$

$$= 4nh\left(1 + \frac{h^2}{3a^2}\right)\left\{\int(\mathfrak{A}\delta u + \tfrac{1}{2}\varpi\,\delta v)\,a\sin\theta\,d\phi + \int(\mathfrak{B}\,\delta v + \tfrac{1}{2}\varpi\,\delta u)\,a\,d\theta\right\}$$

$$- 4nh\left(1 + \frac{h^2}{3a^2}\right)\iint\left[\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\}\delta u\right.$$

$$\left. + \left\{\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d}{d\theta}(\varpi\sin\theta) + \tfrac{1}{2}\varpi\cos\theta\right\}\delta v - (\mathfrak{A} + \mathfrak{B})\sin\theta\,\delta w\right]a\,d\theta\,d\phi \qquad (21),$$

from which we obtain the approximate equations

$$\left.\begin{array}{c} T_1 = 4nh\mathfrak{A}, \qquad T_2 = 4nh\mathfrak{B} \\ M_1 = M_2 = 2nh\varpi \end{array}\right\} \quad . \qquad (22).$$

$$\left.\begin{array}{l} \rho\ddot{u} = \dfrac{2n}{a\sin\theta}\left\{\dfrac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\dfrac{d\varpi}{d\phi}\right\} + \rho X \\ \rho\ddot{v} = \dfrac{2n}{a\sin\theta}\left\{\dfrac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\dfrac{d}{d\theta}(\varpi\sin\theta) + \tfrac{1}{2}\varpi\cos\theta\right\} + \rho Y \\ \rho\ddot{w} = -\dfrac{2n}{a}(\mathfrak{A} + \mathfrak{B}) + \rho Z \end{array}\right\} \quad . \qquad (23)$$

These are the equations which have been obtained by Mr LOVE,* and which have been employed by him in discussing the extensional vibrations of a spherical shell

Again

$$\delta W_2 = \tfrac{4}{3}\mu h^3 \iint (\mathfrak{E}\delta\lambda + \mathfrak{F}\delta\mu + \tfrac{1}{2}p\delta p)\, dS$$

Substituting the values of λ, μ, p from (8) we obtain

$$\begin{aligned}
\iint \mathfrak{E}\delta\lambda\, dS = &-\iint \mathfrak{E} \left\{\frac{d^2\delta w}{d\theta^2} + \delta w + E\left(\frac{d\delta u}{d\theta} + \frac{1}{\sin\theta}\frac{d\delta v}{d\phi} + \delta u \cot\theta + 2\delta w\right)\right\} \sin\theta\, d\theta\, d\phi \\
= &-\int\left\{E\mathfrak{E}\sin\theta\delta u - \frac{d}{d\theta}(\mathfrak{E}\sin\theta)\,\delta w + \mathfrak{E}\sin\theta\frac{d\delta w}{d\theta}\right\} d\phi - \int E\mathfrak{E}\delta v\, d\theta \\
&+ \iint\Bigg[E\sin\theta\frac{d\mathfrak{E}}{d\theta}\,\delta u + E\frac{d\mathfrak{E}}{d\phi}\,\delta v \\
&\qquad - \left\{\frac{d^2}{d\theta^2}(\mathfrak{E}\sin\theta) + (1+2E)\,\mathfrak{E}\sin\theta\right\}\delta w\Bigg]\, d\theta\, d\phi \quad (24),
\end{aligned}$$

also

$$\begin{aligned}
\iint \mathfrak{F}\delta\mu\, dS = &-\iint \mathfrak{F}\left\{\frac{1}{\sin\theta}\frac{d^2\delta w}{d\phi^2} + \cos\theta\frac{d\delta w}{d\theta} + \delta w\sin\theta \right. \\
&\qquad \left. + E\left(\frac{d\delta u}{d\theta}\sin\theta + \frac{d\delta v}{d\phi} + \delta u\cos\theta + 2\delta w\sin\theta\right)\right\} d\theta\, d\phi \\
= &-\int(E\mathfrak{F}\sin\theta\delta u + \mathfrak{F}\cos\theta\delta w)\, d\phi \\
&-\int\left(E\mathfrak{F}\delta v - \frac{1}{\sin\theta}\frac{d\mathfrak{F}}{d\phi}\,\delta w + \frac{\mathfrak{F}}{\sin\theta}\frac{d\delta w}{d\phi}\right) d\theta \\
&+\iint\Bigg[E\sin\theta\frac{d\mathfrak{F}}{d\theta}\,\delta u + E\frac{d\mathfrak{F}}{d\phi}\,\delta v \\
&\qquad - \left\{\frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2} - \frac{d}{d\theta}(\mathfrak{F}\cos\theta) + (1+2E)\,\mathfrak{F}\sin\theta\right\}\delta w\Bigg] d\theta\, d\phi \quad (25)
\end{aligned}$$

In the last term $p\,\delta p$, we must treat the integral which involves $d^2\delta w/d\theta\, d\phi$ exactly in the same way as in the corresponding case of a cylindrical shell, and we shall thus obtain

$$\begin{aligned}
\tfrac{1}{2}\iint p\,\delta p\, dS = &\iint p\left(\cot\theta\frac{d\delta w}{d\phi} - \frac{d^2\delta w}{d\theta d\phi}\right) d\theta\, d\phi \\
= &\ \tfrac{1}{2}\int\left(\frac{dp}{d\phi}\,\delta w - p\frac{d\delta w}{d\phi}\right) d\phi + \int\left\{\left(p\cot\theta + \tfrac{1}{2}\frac{dp}{d\theta}\right)\delta w - \tfrac{1}{2}p\frac{d\delta w}{d\theta}\right\} d\theta \\
&-\iint\left(\cot\theta\frac{dp}{d\phi} + \frac{d^2p}{d\theta d\phi}\right)\delta w\, d\theta\, d\phi \quad . \quad . \qquad (26).
\end{aligned}$$

* 'Phil Trans,' A, 1888, p 527. Equation (23) corresponds to LOVE's equations (46), (47), and (48) and (22) to (72).

Adding (24), (25), and (26), we finally obtain

$$\delta W_2 = \frac{4nh^3}{3a}\int\Big[-\{\mathfrak{E} + E(\mathfrak{E} + \mathfrak{F})\}\,\delta u - \tfrac{1}{2}p\,\delta v + \frac{1}{\sin\theta}\Big\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta$$

$$+ \tfrac{1}{2}\frac{dp}{d\phi}\Big\}\,\delta w - \mathfrak{E}\left(\frac{d\delta u}{d\theta} - \delta u\right) - \tfrac{1}{2}p\left(\frac{1}{\sin\theta}\frac{d\delta v}{d\phi} - \delta v\right)\Big]\,a\sin\theta\,d\phi$$

$$+ \frac{4nh^3}{3a}\int\Big[-\tfrac{1}{2}p\,\delta u - \{\mathfrak{F} + E(\mathfrak{E} + \mathfrak{F})\}\,\delta v + \frac{1}{\sin\theta}\left(\frac{d\mathfrak{F}}{d\phi} + p\cos\theta + \tfrac{1}{2}\sin\theta\frac{dp}{d\theta}\right)\delta w$$

$$- \mathfrak{F}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right) - \tfrac{1}{2}p\left(\frac{d\delta w}{d\theta} - \delta u\right)\Big]\,a\,d\theta$$

$$+ \tfrac{4}{3}nh^3\iint\Big[E\sin\theta\frac{d}{d\theta}(\mathfrak{E} + \mathfrak{F})\,\delta u + E\frac{d}{d\phi}(\mathfrak{E} + \mathfrak{F})\,\delta v$$

$$- \left\{\frac{d^2}{\delta\theta^2}(\mathfrak{E}\sin\theta) + (1 + 2E)(\mathfrak{E} + \mathfrak{F})\sin\theta + \frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2} - \frac{d}{d\theta}(\mathfrak{F}\cos\theta)\right.$$

$$\left. + 2(1 + E)\,\mathfrak{F}\sin\theta + \cot\theta\frac{dp}{d\phi} + \frac{d^2p}{d\theta\,d\phi}\right\}\delta w\Big]\,d\theta\,d\phi \qquad (27)$$

The expressions for W_3, W_4 may, as in the case of a cylindrical shell, be divided into two parts W_3', W_3'', W_4', W_4''. The values of $\delta W_3'$, $\delta W_4'$, may at once be written down from (21) by changing $\mathfrak{A}$, $\mathfrak{B}$, ϖ into $\mathfrak{E}'$, $\mathfrak{F}'$, p' and $\mathfrak{E}$, $\mathfrak{F}$, p respectively, and by altering the coefficient into $\frac{2}{3}nh^3$ and $8nh^3/3a$ respectively. With regard to W_3'' we have

$$\delta W_3'' = \tfrac{2}{3}nh^3\iint(\mathfrak{A}\,\delta\lambda' + \mathfrak{B}\,\delta\mu' + \tfrac{1}{2}\varpi\,\delta p')\,dS$$

Substituting the value of λ' from (9) and integrating once by parts, we shall obtain

$$\iint\mathfrak{A}\,\delta\lambda'\,dS = E\int\mathfrak{A}\frac{d\,\delta K}{d\theta}\sin\theta\,d\phi$$

$$- \iint\Big[E\frac{d}{d\theta}(\mathfrak{A}\sin\theta)\frac{d\,\delta K}{d\theta} + \mathfrak{A}a\sin\theta\,\{2\,\delta\lambda + E(\delta\lambda + \delta\mu)\}\Big]\,d\theta\,d\phi$$

Treating the other terms in a similar manner, we shall finally obtain

$$\delta W_3'' = \int\left\{\frac{2nh^3E\mathfrak{A}}{3a}\frac{d\,\delta K}{d\theta} + \frac{nh^3E\varpi}{3a\sin\theta}\frac{d\,\delta K}{d\phi}\right\}a\sin\theta\,d\phi$$

$$+ \int\left\{\frac{2nh^3E\mathfrak{B}}{3a\sin\theta}\frac{d\,\delta K}{d\phi} + \frac{nh^3E\varpi}{3a}\frac{d\,\delta K}{d\theta}\right\}a\,d\theta$$

$$- \frac{2nh^3}{3a^2}\iint\Big[\frac{E}{\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\}\frac{d\delta K}{d\theta}$$

$$+ \frac{E}{\sin^2\theta}\left(\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d\varpi}{d\theta}\sin\theta + \varpi\cos\theta\right)\frac{d\,\delta K}{d\phi} + Ea(\mathfrak{A} + \mathfrak{B})(\delta\lambda + \delta\mu)\Big]\,dS$$

$$- \frac{4nh^3}{3a}\iint(\mathfrak{A}\,\delta\lambda + \mathfrak{B}\,\delta\mu + \tfrac{1}{2}\varpi\,\delta p)\,dS.$$

If in the first surface integral in this equation we substitute the approximate values of the coefficients of $d\,\delta K/d\theta$ &c from (23), which we may do since this integral is multiplied by h^3, and then substitute the values of $\delta W_3''$, $\delta\mathfrak{T}$, $\delta\mathfrak{L}$, and δU in (10), it will be found that all the terms involving $d\,\delta K/d\theta$, $d\,\delta K/d\phi$, and $\delta\lambda + \delta\mu$ cut out, we are therefore no longer concerned with them, and the value of $\delta W_3''$ reduces to the last line, on this understanding we may write

$$\delta W_3'' + \delta W_4'' = \frac{4nh^3}{3a}\iint (\mathfrak{A}\,\delta\lambda + \mathfrak{B}\,\delta\mu + \tfrac{1}{2}\varpi\,\delta p)\,dS \qquad (28)$$

The variation of the right-hand side of (28) might at once be written down from (27) by substituting $\mathfrak{A}$, $\mathfrak{B}$, and ϖ for $\mathfrak{E}$, $\mathfrak{F}$, and p, but it will be more convenient to present the results in another form Taking the first term, and integrating the second differential coefficients *once* by parts, we obtain

$$\iint \mathfrak{A}\,\delta\lambda\,dS = -\int \mathfrak{A}\frac{d\,\delta u}{d\theta}\sin\theta\,d\phi + \iint\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta)\frac{d\,\delta w}{d\theta} - \mathfrak{A}\sin\theta\,\delta w \right.$$
$$\left. - E\mathfrak{A}a\sin\theta\,\delta K\right\} d\theta\,d\phi$$

Treating the other terms in a similar way, and adding to the result from (21) that portion of δW_1 which depends upon h^3, and finally replacing the coefficients of $d\,\delta w/d\theta - \delta u$, &c, by their approximate values from (23), the final result will be

$$- \frac{4nh^3}{3a^2}\int\left\{\mathfrak{A}\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + \tfrac{1}{2}\varpi\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right)\right\} a\sin\theta\,d\phi$$
$$- \frac{4nh^3}{3a^2}\int\left\{\mathfrak{B}\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right) + \tfrac{1}{2}\varpi\left(\frac{d\,\delta w}{d\theta} - \delta u\right)\right\} a\,d\theta$$
$$+ \frac{2\rho h^3}{3a^2}\iint\left\{(u - X)\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + (v - Y)\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right)\right.$$
$$\left. + Ea(w - Z)\,\delta K\right\} dS \qquad (29)$$

This result enables us to test the accuracy of a portion of our work, and the fundamental hypothesis on which the theory is based; for if we substitute in (10) the expression (29), and also the value of $\delta\mathfrak{L}$ from (19), it will be seen that we have reproduced the values of the couples which are given by (7), also comparing with $\delta\mathfrak{L}$, the line integral parts of δW_2, given by (27), the line integral parts of $\delta W_3'$ and $\delta W_4'$, which, as we have explained above, are obtained from (21) by changing certain letters, we see that we have also reproduced the values of M_1, M_2, given by the third of (6). We may, therefore, omit the couple terms, and also the terms in M; also, since we

have disposed of the terms in δW_1, which involve h^3, we shall write $\delta W_1'$ for the remaining portion which depends upon h, and the variational equation finally becomes

$$\delta W_1' + \delta W_2' + \delta W_3' + \delta W_4' + 2\rho h\,(1 + \tfrac{1}{3}h^2/a^2)\iint(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w)\,dS$$

$$+ \tfrac{2}{3}\rho h^3 \iint\left\{\frac{1}{a^2}\left(\frac{d\ddot{w}}{d\theta} - 2\ddot{u}\right)\left(\frac{d\,\delta u}{d\theta} - \delta u\right) + \frac{1}{a^2}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v}\right)\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right)\right.$$

$$\left. + \mathrm{E}\left(\mathrm{E\ddot{K}} - \frac{\ddot{u}}{a}\right)\delta\mathrm{K}\right\}dS$$

$$+ \tfrac{1}{3}\rho h^3\mathrm{E}\iint\left\{\frac{1}{a}\frac{d\ddot{\mathrm{K}}}{d\theta}\,\delta u + \frac{1}{a\sin\theta}\frac{d\ddot{\mathrm{K}}}{d\phi}\,\delta v - (\lambda + \mu)\,\delta w\right\}dS$$

$$- \frac{4\rho h^3}{3a}\iint\left\{\frac{1}{a}\left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\delta u + \frac{1}{a}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - \ddot{v}\right)\delta v + \mathrm{E\ddot{K}}\,\delta w\right\}dS$$

$$= 2\rho h\left(1 + \frac{h^2}{3a^2}\right)\iint\left(\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w\right)dS$$

$$- \frac{2\rho h^3}{3a^2}\iint\left\{\mathrm{X}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \mathrm{Y}\left(\frac{1}{\sin\theta}\frac{d\delta v}{d\phi} - \delta v\right) + \mathrm{ZE}a\,\delta\mathrm{K}\right\}dS$$

$$+ \int(\mathrm{T}_1\,\delta u + \mathrm{N}_2\,\delta w)\,a\sin\theta\,d\phi + \int(\mathrm{T}_2\,\delta v + \mathrm{N}_1\,\delta w)\,a\,d\theta \qquad (30).$$

We have now got rid of all the terms involving the second differential coefficients of δu, δv, δw; and all that remains to be done is to integrate by parts the terms which involve the first differential coefficients. Putting

$$\alpha = \frac{d\ddot{w}}{d\theta} - 2\ddot{u} + \mathrm{X}, \qquad \beta = \frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}, \qquad \gamma = \mathrm{E}\,(a\mathrm{E\ddot{K}} - \ddot{w} + \mathrm{Z}) \quad (31),$$

we have

$$\frac{2\rho h^3}{3a^2}\iint\left\{\alpha\,\frac{d\delta w}{d\theta} + \frac{\beta}{\sin\theta}\frac{d\delta w}{d\phi} + a\gamma\delta\mathrm{K}\right\}dS$$

$$= \frac{2\rho h^3}{3a}\int(\gamma\delta u + \alpha\delta w)\,a\sin\theta d\phi + \frac{2\rho h^3}{3a}\int(\gamma\delta v + \beta\delta w)\,ad\theta$$

$$- \frac{2\rho h^3}{3a^2}\iint\left[\left\{\frac{d}{d\theta}(\gamma\sin\theta) - \gamma\cos\theta\right\}\delta u + \frac{d\gamma}{d\phi}\,\delta v + \left\{\frac{d}{d\theta}(\alpha\sin\theta) + \frac{d\beta}{d\phi} - 2\gamma\sin\theta\right\}\delta w\right]\frac{dS}{\sin\theta} \quad (32).$$

Substituting the values of $\delta W_1'$, $\delta W_2'$, $\delta W_3'$, $\delta W_4'$, and the right hand side of (32) in (30), and picking out the line integral terms, we obtain the following equations for the sectional stresses, viz.,

$$\left.\begin{aligned}
T_1 &= 4nh\mathfrak{A} + \frac{4nh^3}{3a}\{\mathfrak{E} - E(\mathfrak{E} + \mathfrak{F})\} + \tfrac{2}{3}nh^3\mathfrak{E}' + \frac{2\rho h^3}{3a}E(aEK - w + z) \\
M_2 &= 2nh\varpi + \frac{2nh^3}{3a}p + \tfrac{1}{3}nh^3p' \\
N_2 &= \frac{4nh^3}{3a\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta + \tfrac{1}{2}\frac{dp}{d\phi}\right\} + \frac{2\rho h^3}{3a}\left(\frac{dw}{d\theta} - 2u + X\right) \\
G_2 &= \tfrac{4}{3}nh^3(\mathfrak{E} + \mathfrak{A}/a) \\
H_1 &= -\tfrac{2}{3}nh^3(p + \varpi/a)
\end{aligned}\right\} \quad (33)$$

which give the values of the sectional stresses across a parallel of latitude, and

$$\left.\begin{aligned}
M_1 &= 2nh\varpi + \frac{2nh^3}{3a}p + \tfrac{1}{3}nh^3p' \\
T_2 &= 4nh\mathfrak{B} + \frac{4nh^3}{3a}\{\mathfrak{F} - E(\mathfrak{E} + \mathfrak{F})\} + \tfrac{2}{3}nh^3\mathfrak{F}' + \frac{2\rho h^3}{3a}E(aEK - w + Z) \\
N_1 &= \frac{4nh^3}{3a\sin\theta}\left(\frac{d\mathfrak{F}}{d\phi} + p\cos\theta + \tfrac{1}{2}\sin\theta\frac{dp}{d\theta}\right) + \frac{2\rho h^3}{3a}\left(\frac{1}{\sin\theta}\frac{dw}{d\phi} - 2\dot{v} + Y\right) \\
G_1 &= -\tfrac{4}{3}nh^3(\mathfrak{F} + \mathfrak{B}/a) \\
H_2 &= \tfrac{2}{3}nh^3(p + \varpi/a)
\end{aligned}\right\} \quad (34)$$

which give the values of the sectional stresses across a meridian.

In these equations we may, if we please, substitute the approximate values of $\ddot{u}$, $\dot{v}$, $\dot{w}$ from (23), and by means of these values it can be shown that the values of N_1, N_2 agree with the values which are obtained by substituting the values of the couples in the fourth and fifth of (5)

Picking out the coefficients of δu, δv, δw, in the surface integrals, we obtain the equations of motion, which are

$$\begin{aligned}
&\left\{2\left(1 + \frac{h^3}{a^3}\right)\ddot{u} + \frac{h^2E}{3a}\frac{d\dot{K}}{d\theta} - \frac{4h^2}{3a^2}\frac{d\dot{w}}{d\theta} - 2\left(1 + \frac{h^2}{3a^2}\right)X\right\}\rho a\sin\theta \\
&\quad = 4n\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\} \\
&\qquad - \frac{4nh^2}{3a^2}E\sin\theta\frac{d}{d\theta}(\mathfrak{E} + \mathfrak{F}) + \tfrac{2}{3}nh^2\left\{\frac{d}{d\theta}(\mathfrak{E}'\sin\theta) - \mathfrak{F}'\cos\theta + \tfrac{1}{2}\frac{dp'}{d\phi}\right\} \\
&\qquad + \frac{8nh^2}{3a}\left\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta + \tfrac{1}{2}\frac{dp}{d\phi}\right\} + \frac{2\rho h^2}{3a}\left\{\alpha\sin\theta + \frac{d\gamma}{d\theta}\sin\theta - \gamma\cos\theta\right\} \quad (35),
\end{aligned}$$

$$\left\{2\left(1+\frac{h^2}{a^2}\right)v+\frac{h^2\mathrm{E}}{3a\sin\theta}\frac{d\mathrm{K}}{d\phi}-\frac{4h^2}{3a^2\sin\theta}\frac{dw}{d\phi}-2\left(1+\frac{h^2}{3a^2}\right)\mathrm{Y}\right\}\rho a\sin\theta$$
$$=4n\left\{\frac{d\mathfrak{B}}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}(\varpi\sin\theta)+\tfrac{1}{2}\varpi\cos\theta\right\}$$
$$-\frac{4nh^2}{3a}\mathrm{E}\frac{d}{d\phi}(\mathfrak{E}+\mathfrak{F})+\tfrac{2}{3}nh^2\left\{\frac{d\mathfrak{F}'}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}(p'\sin\theta)+\tfrac{1}{2}p'\cos\theta\right\}$$
$$+\frac{8nh^2}{3a}\left\{\frac{d\mathfrak{F}}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}(p\sin\theta)+\tfrac{1}{2}p\cos\theta\right\}+\frac{2\rho h^2}{3a}\left(\beta\sin\theta+\frac{d\gamma}{d\phi}\right) \qquad (36)$$

$$\left\{2\left(1+\frac{h^2}{3a^2}\right)(w-\mathrm{Z})-\tfrac{1}{3}h^2\mathrm{E}(\dot\lambda+\dot\mu)-\frac{4h^2}{3a}\mathrm{E}\dot{\mathrm{K}}\right\}\rho a\sin\theta$$
$$=-4n(\mathfrak{A}+\mathfrak{B})\sin\theta+\frac{4nh^2}{3a}\left\{\frac{d^2}{d\theta^2}(\mathfrak{E}\sin\theta)+(1+2\mathrm{E})(\mathfrak{E}+\mathfrak{F})\sin\theta\right.$$
$$\left.+\frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2}-\frac{d}{d\theta}(\mathfrak{F}\cos\theta)+\cot\theta\frac{dp}{d\phi}+\frac{d^2p}{d\theta\,d\phi}\right\}$$
$$-\tfrac{2}{3}nh^2(\mathfrak{E}'+\mathfrak{F}')\sin\theta-\frac{8nh^2}{3a}(\mathfrak{E}+\mathfrak{F})\sin\theta$$
$$+\frac{2\rho h^2}{3a}\left\{\frac{d}{d\theta}(\alpha\sin\theta)+\frac{d\beta}{d\phi}-2\gamma\sin\theta\right\}. \qquad (37)$$

The correctness of these equations may be tested by substituting the values of the sectional stresses from (33) and (34) in the first three of (5), when it will be found that we shall reproduce (35), (36), and (37)

20 The boundary conditions for a spherical shell may be investigated in exactly the same manner as in the case of a cylindrical shell, by means of STOKES' theorem , for in the present case the theorem may be written

$$\int\left(\frac{d\mathrm{H}'}{d\phi}\delta w+\mathrm{H}'\frac{d\delta w}{d\phi}\right)d\phi+\int\left(\frac{d\mathrm{H}'}{d\theta}\delta w+\mathrm{H}'\frac{d\delta w}{d\theta}\right)d\theta=0,$$

the integration extending round any curvilinear rectangle bounded by two meridians and two parallels of latitude If, therefore, in the figure we apply to the side AD the stresses,

$$\mathrm{M}_2'=\mathrm{H}'/a,\qquad \mathrm{N}_2'=\frac{1}{a\sin\theta}\frac{d\mathrm{H}'}{d\phi},\qquad \mathrm{H}_1'=\mathrm{H}',$$

to the side BD the stresses

$$\mathrm{M}_1'=\mathrm{H}'/a,\qquad \mathrm{N}_1'=\frac{1}{a}\frac{d\mathrm{H}'}{d\theta},\qquad \mathrm{H}_2'=-\mathrm{H}';$$

and to the sides OB, OA, corresponding and opposite stresses respectively, the preceding integral becomes

$$\int\left\{\mathrm{M}_2'\,\delta v + \mathrm{N}_2'\,\delta w + \frac{\mathrm{H}_1'}{a}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right)\right\} a\sin\theta\, d\phi$$

$$+\int\left\{\mathrm{M}_1'\,\delta u + \mathrm{N}_1'\,\delta w - \frac{\mathrm{H}_2'}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right)\right\} a\, d\theta = 0\,,$$

which shows that the work done by this system of stresses is zero

If, therefore, we suppose that the rectangle OADB, instead of being under the action of the remainder of the shell, is isolated, and that its state of strain is maintained by stresses applied to its edges, then it follows that if instead of the torsional couples H_1, H_2, due to the action of contiguous portions of the shell, we apply torsional couples $\mathfrak{H}_1$, $\mathfrak{H}_2$, where

$$\mathfrak{H}_1 = \mathrm{H}_1 + \mathrm{H}' \qquad (38),$$

$$\mathfrak{H}_2 = \mathrm{H}_2 - \mathrm{H}' \qquad (39),$$

the state of strain will remain unchanged, provided we apply in addition the stresses

$$\left.\begin{aligned} \mathfrak{M}_2 &= \mathrm{M}_2 + \mathrm{H}'/a \\ \mathfrak{N}_2 &= \mathrm{N}_2 + \frac{1}{a\sin\theta}\frac{d\mathrm{H}'}{d\phi} \end{aligned}\right\} \qquad (40)$$

and

$$\left.\begin{aligned} \mathfrak{M}_1 &= \mathrm{M}_1 + \mathrm{H}'/a \\ \mathfrak{N}_1 &= \mathrm{N}_1 + \frac{1}{a}\frac{d\mathrm{H}'}{d\theta} \end{aligned}\right\} \qquad (41),$$

whence eliminating H′ between (38) and (40), and between (39) and (41) respectively, we obtain

$$\left.\begin{aligned} \mathfrak{M}_2 a - \mathfrak{H}_1 &= \mathrm{M}_2 a - \mathrm{H}_1 \\ \mathfrak{N}_2 a\sin\theta - \frac{d\mathfrak{H}_1}{d\phi} &= \mathrm{N}_2 a\sin\theta - \frac{d\mathrm{H}_1}{d\phi} \end{aligned}\right\} \qquad (42)$$

and

$$\left.\begin{aligned} \mathfrak{M}_1 a + \mathfrak{H}_2 &= \mathrm{M}_1 a + \mathrm{H}_2 \\ \mathfrak{N}_1 a + \frac{d\mathfrak{H}_2}{d\theta} &= \mathrm{N}_2 a + \frac{d\mathrm{H}_2}{d\theta} \end{aligned}\right\} \qquad (43)$$

in which we are to remember that $\mathfrak{H}_1 = -\mathfrak{H}_2$, and $\mathrm{H}_1 = -\mathrm{H}_2$.

In these equations the Roman letters denote the stresses due to the action of contiguous portions of the shell, whilst the Old English letters denote the values of the actual stresses applied to the boundary. If, therefore, the shell consists of a

portion of a sphere bounded by two meridians and two parallels of latitude, and whose edges are free, the boundary conditions along a parallel of latitude are obtained by equating the right hand sides of the first and fourth of (33) and of (42) to zero, whilst the boundary conditions along a meridian are similarly obtained by equating the right hand sides of the first and fourth of (34) and of (43) to zero

21 If the shell is supposed to vibrate in such a manner, that its middle surface does not experience any extension or contraction throughout the motion, the equations of motion can be obtained by taking the variation subject to the conditions of inextensibility, and introducing indeterminate multipliers

22 It will now be convenient to make a short digression for the purpose of considering some of the quantities involved

Let P be any point on the deformed middle surface whose undisplaced coordinates are (a, θ, ϕ) The coordinates of P after deformation are

$$\mathrm{R} = a + w, \qquad \Theta = \theta + u/a, \qquad \Phi = \phi + v/a \sin\theta \qquad . \quad (44),$$

and since u, v, w are functions of θ and ϕ, the elimination of the latter quantities from (44) will give a relation between R, Θ, Φ, which is the equation of the deformed middle surface

If ρ_1 be the radius of curvature at any point of a meridian section after deformation, and P the perpendicular from the centre on to the tangent at that point to the deformed section,

$$\frac{1}{\rho_1} = \frac{1}{\mathrm{R}}\frac{d\mathrm{P}}{d\mathrm{R}}.$$

Now

$$\frac{1}{\mathrm{P}^2} = \frac{1}{\mathrm{R}^2}\left\{1 + \left(\frac{d\mathrm{R}}{\mathrm{R}d\Theta}\right)^2\right\}$$

$$= \frac{1}{\mathrm{R}^2}\left\{1 + \frac{(dw/d\theta)^2}{\mathrm{R}^2(1 + du/ad\theta)^2}\right\},$$

and therefore, neglecting cubes of displacements,

$$\mathrm{P} = a + w - \frac{1}{2a}\left(\frac{dw}{d\theta}\right)^2$$

Also

$$d\mathrm{R} = \frac{dw}{d\theta}\,d\theta,$$

whence

$$\frac{1}{\rho_1} - \frac{1}{a} = -\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2} + w\right) \qquad . \quad . \quad . \quad . \quad . \quad (45),$$

which gives the change of curvature along a meridian.

We shall now find an expression for the change of curvature along any great circle which makes an angle γ with a meridian.

In the figure on p 463 join OD, and let the angle OED $= \chi$, and the angle DOA $= \gamma$, then by (45) the change of curvature along OD is

$$-\frac{1}{a^2}\left(\frac{d^2w}{d\chi^2}+w\right)$$

If $w+\delta w$ be the normal displacement at D, it follows by equating the two values of δw that

$$\frac{dw}{d\chi}\delta\chi+\tfrac{1}{2}\frac{d^2w}{d\chi^2}\delta\chi^2=\frac{dw}{d\theta}\delta\theta+\frac{dw}{d\phi}\delta\phi+\tfrac{1}{2}\left(\frac{d^2w}{d\theta^2}\delta\theta^2+2\frac{d^2w}{d\theta d\phi}\delta\theta\,\delta\phi+\frac{d^2w}{d\phi^2}\delta\phi^2\right) \quad (46)$$

From the spherical triangle ODP we have

$$-\cos\gamma=\frac{\cos \mathrm{PD}-\cos\theta\cos\delta\chi}{\sin\theta\sin\delta\chi},$$

whence

$$\begin{aligned}\sin\theta\,\delta\theta&=\cos\gamma\sin\theta\,\delta\chi+\tfrac{1}{2}\cos\theta\,\delta\chi^2-\tfrac{1}{2}\cos\theta\,\delta\theta^2\\&=\cos\gamma\sin\theta\,\delta\chi+\tfrac{1}{2}\cos\theta\sin^2\gamma\,\delta\chi^2\end{aligned}$$

Again

$$\frac{\sin\delta\phi}{\sin\delta\chi}=\frac{\sin\gamma}{\sin(\theta+\delta\theta)},$$

whence

$$\delta\phi=\frac{\sin\gamma}{\sin\theta}(\delta\chi-\cot\theta\cos\gamma\,\delta\chi^2)$$

Substituting these values of $\delta\theta$, $\delta\phi$ in (46) and equating coefficients of $\delta\chi^2$, we obtain

$$\frac{d^2w}{d\chi^2}=\frac{\sin^2\gamma}{\sin^2\theta}\frac{d^2w}{d\phi^2}+\frac{\sin 2\gamma}{\sin\theta}\frac{d^2w}{d\theta d\phi}+\cos^2\gamma\frac{d^2w}{d\theta^2}-\frac{\sin 2\gamma\cos\theta}{\sin^2\theta}\frac{dw}{d\phi}+\sin^2\gamma\cot\theta\frac{dw}{d\theta} \quad (47).$$

Whence it follows, that if ρ_1, ρ_2 are the principal radii of curvature along and perpendicular to a meridian

$$\left.\begin{aligned}\frac{1}{\rho_1}-\frac{1}{a}&=-\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2}+w\right)\\\frac{1}{\rho_2}-\frac{1}{a}&=-\frac{1}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2w}{d\phi^2}+\cot\theta\frac{dw}{d\theta}+w\right)\end{aligned}\right\}\quad . \;.\; .\; .\; . \quad (48).$$

23. When the middle surface is inextensible, it has been shown by Lord RAYLEIGH* that the displacements are given by the equations

$$\left.\begin{aligned} u &= -\Sigma A_s \epsilon^{\iota s\phi} \sin\theta \tan^s \tfrac{1}{2}\theta \\ v &= \Sigma \iota A_s \epsilon^{\iota s\phi} \sin\theta \tan^s \tfrac{1}{2}\theta \\ w &= \Sigma A_s \epsilon^{\iota s\phi} (s + \cos\theta) \tan^s \tfrac{1}{2}\theta \end{aligned}\right\} \qquad (49)$$

where $s = 2, 3, 4$ and A_s is a complex function of the time From these equations it can easily be shown by means of (48) that

$$\frac{1}{\rho_1} - \frac{1}{a} = -\Sigma \frac{A_s (s^2 - s) \epsilon^{\iota s\phi} \tan^s \frac{1}{2}\theta}{a \sin^2\theta} = -\left(\frac{1}{\rho_2} - \frac{1}{a}\right) \qquad (50)$$

The value of the potential energy is given by the second line of (16), also by the first two of (8) and by (48)

$$\lambda = -\mu = \frac{1}{\rho_1} - \frac{1}{a} \qquad (51),$$

and by the last of (8)

$$p = -2\iota \Sigma \frac{A_s (s^2 - s) \epsilon^{\iota s\phi} \tan^s \frac{1}{2}\theta}{a \sin^2\theta},$$

whence

$$W = \frac{4nh^3}{3a^2 \sin^4\theta} [\{\Sigma A_s (s^3 - s) \cos s\phi \tan^s \tfrac{1}{2}\theta\}^2 + \{\Sigma A_s (s^3 - s) \sin s\phi \tan^s \tfrac{1}{2}\theta\}^2] \qquad (52),$$

which agrees with Lord RAYLEIGH's result.

Let us now suppose that a bell which consists of a spherical shell, bounded by a small circle whose latitude is $\frac{1}{2}\pi - \alpha$, is vibrating in such a manner that its middle surface does not undergo any extension or contraction throughout the motion. One of the boundary conditions requires that the flexural couple G_2 should vanish along the circle of latitude which constitutes the free edge of the bell By (7) and (51)

$$\begin{aligned} G_2 &= \tfrac{4}{3} nh^3 \mathfrak{E} = \tfrac{4}{3} nh^3 \{\lambda + E(\lambda + \mu)\} \\ &= \tfrac{4}{3} nh^3 \left(\frac{1}{\rho_1} - \frac{1}{a}\right). \end{aligned}$$

From (50) we see that G_2 cannot vanish for any value of θ except $\theta = 0$, that is, at the pole, provided $s > 2$. It, therefore, follows that a spherical bell whose edge is free cannot vibrate in this manner if the middle surface is supposed to remain absolutely inextensible throughout the motion. If, however, extension or contraction

* 'London Math. Soc. Proc.,' vol 13, p 4 (1881)

were to take place in the neighbourhood of the edge, it would be possible for G_2 to vanish there and also to satisfy the other boundary conditions

There seems no reason to doubt that the argument which has been employed in the case of a cylindrical shell, would apply equally to the case of a spherical shell, and probably also to a shell of any shape, in which case, the portions of the displacements upon which extension principally depends, would be small compared with the portions upon which bending principally depends, except at points whose distances from a free edge are comparable with the thickness. At the same time it would be very desirable to obtain the solution of some problem relating to the vibrations of a shell whose edges are free, in which no supposition is made as to the relative magnitudes of the extensional and flexural terms

VII. *Memoir on Symmetric Functions of the Roots of Systems of Equations.*

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§ 1. *Preliminary Ideas.*

1. The theory of the symmetrical functions of a single system of quantities has been investigated in a large number of memoirs, but so far, only a few attempts have been made to develop an analogous theory with regard to several systems of quantities. The chief authors are Schlafli* and Cayley,† both of whom have however, restricted themselves to the outlines of the commencement of such a theory. In the theory of the single system it is found convenient to regard the quantities as the roots of an equation, since the coefficients of such an equation are themselves those particular symmetric functions of the quantities which have been variously termed fundamental, elementary, and unitary; they are fundamental because all other rational integral functions are expressible by their products of the same or lower degree; elementary because they are those which, first of all, naturally arise; unitary because their partitions are composed wholly of units. The left hand side of the equation referred to is a product of binomial linear functions of a single variable x, so that, $\alpha_1, \alpha_2, \ldots \alpha_n$ being the quantities which compose the system, the fundamental relation may be written

$$
\begin{aligned}
(1+\alpha_1 x)(1+\alpha_2 x)\ldots(1+\alpha_n x) &= 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \\
&= 1 + (1)x + (1^2)x^2 + \ldots + (1^n)x^n,
\end{aligned}
$$

in the ordinary partition notation.

In a general discussion it is convenient and advantageous to suppose the number of quantities infinite, so that the relation becomes

$$
(1+\alpha_1 x)(1+\alpha_2 x)\ldots = (1 + a_1 x + a_2 x^2 + \ldots) = 1 + (1)x + (1^2)x^2 + \ldots
$$

* "Ueber die Resultante eines Systemes mehrerer algebraischen Gleichungen." 'Vienna Academy Denkschriften,' vol. 4, 1852.

† "On the Symmetric Functions of the Roots of certain Systems of Two Equations." 'Phil. Trans.,' vol. 147 (1857).

2 Instead of taking a product of binomial linear functions of one variable, as above, we can, for m systems of quantities, take a product of non-homogeneous linear functions of m variables, and each such linear function may be taken of the form

$$1+\alpha_{s1}x_1+\alpha_{s2}x_2+\qquad+\alpha_{sm}x_m$$

As indicative of this general case it is sufficient to consider merely the case of two systems of quantities Complexity of formulas is thereby avoided, but it must be distinctly borne in mind that all the succeeding theorems can be at once extended to the general case of m systems by an easy enlargement of the nomenclature and notation.

I consider, then, two systems of quantities

$$\begin{array}{llll}\alpha_1, & \alpha_2, & \qquad & \alpha_n\,, \\ \beta_1, & \beta_2, & & \beta_n\,,\end{array}$$

as connected with two non-homogeneous equations, in two variables, in such wise that the values α_s, β_s, of the variables respectively constitute one solution of the two simultaneous equations In order to avoid identical relations between fundamental forms, as well as for other reasons which will appear, I take the number of quantities n in each system to be infinite.

By analogy the fundamental relation is written

$$\begin{aligned}(1+\alpha_1x+\beta_1y)(1+\alpha_2x+\beta_2y)\qquad(1+\alpha_sx+\beta_sy)\\ =1+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+\ \ ..+a_{pq}x^py^q+\ \ ..\end{aligned}$$

As shown by SCHLAFLI this equation may be directly formed and exhibited as the resultant of the two given equations, and an arbitrary, linear, non-homogeneous equation in two variables Beyond the preliminary idea this investigation has little to do with the original equations or with the theory of resultants It starts with the fundamental equation just written, the right-hand side of which may be put into the form

$$1+\Sigma\alpha_1x+\Sigma\beta_1y+\Sigma\alpha_1\alpha_2\,x^2+\Sigma\alpha_1\beta_2\,xy+\Sigma\beta_1\beta_2\,y^2+\quad .$$

The most general symmetric function to be considered is

$$\Sigma\alpha_1^{p_1}\beta_1^{q_1}\alpha_2^{p_2}\beta_2^{q_2}\alpha_3^{p_3}\beta_3^{q_3}\quad .$$

which I represent symbolically by

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\qquad).$$

Observe that the summation is in regard to the expressions obtained by permuting the n suffices

$$1, 2, 3, \ldots n.$$

The weight of the function must be considered as bipartite, it consists of the two numbers

$$p_1 + p_2 + p_3 + \ldots = \Sigma p,$$
$$q_1 + q_2 + q_3 + \ldots = \Sigma q,$$

and I speak of the biweight Σp, Σq

The sum $\Sigma p + \Sigma q$ may be called the whole weight, or simply the weight Associated with any number w there will be a weight w and a biweight corresponding to every composition of w by means of two numbers, including zero as a number By composition is meant partition, in which regard is paid to the order of the parts, for instance, 21 and 12 are different binary compositions of 3, and 30, 21, 12, 03 constitute the system

3 It is necessary to introduce the notion of the partition of the bipartite number which denotes the biweight

Thus of the biweight Σp, Σq the expression

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\ldots)$$

may be termed a partition

The dual symbols $\overline{p_1q_1}$, $\overline{p_2q_2}$, $\overline{p_3q_3}$, ... are the parts of this partition; the parts are themselves bipartite and may be termed biparts

We have thus a biweight denoted by a bipartite number partitioned into a number of bipartite numbers termed biparts.

It is convenient to arrange the biparts so that the sums of the symbols which compose them are in descending order of magnitude from left to right

According to usual practice repetitions of biparts are denoted by power symbols, thus

$$(\overline{p_1q_1}^2) \equiv (\overline{p_1q_1}\,\overline{p_1q_1})$$

4 In the notation just explained the fundamental relation is written

$$\begin{aligned}(1 + \alpha_1 x + \beta_1 y)\,(1 + \alpha_2 x + \beta_2 y)\ldots\\ = 1 + (\overline{10})\,x + (\overline{01})\,y + (\overline{10}^2)\,x^2 + (\overline{10}\,\overline{01})\,xy + (\overline{01}^2)\,y^2\\ + (\overline{10}^3)\,x^3 + (\overline{10}^2\,\overline{01})\,x^2y + (\overline{10}\,\overline{01}^2)\,xy^2 + (\overline{01}^3)\,y^3 + \ldots\end{aligned}$$

where $(\overline{10}\,\overline{01}^2)$ denotes $\Sigma\,\alpha_1\,\beta_2\,\beta_3$ and in general $a_{pq} = (\overline{10}^p\,\overline{01}^q)$

Observe that here the number of quantities in each system is considered to be infinite, and that the right hand side of the equation is taken with unit and not

multinomial coefficients (*cf.* CAYLEY, *loc. cit.*) This is done because it is the universal practice in the theory of the single system, and because otherwise it appears to possess undoubted advantages

The symmetric functions which appear in the relation are fundamental since, as will appear, they serve to express all other rational integral symmetric functions, and they may be further termed single-unitary, in that, not only is each composed entirely of units, but also each bipart comprises but a single unit

It is obvious that the number of biweights connected with the weight w is $w+1$

5 It may be asked in how many ways it is possible to partition a biweight into biparts

In the ordinary theory of partitions the number of partitions of a number w is the coefficient of x^w in the ascending expansion of

$$\frac{1}{1-x\;1-x^2\;1-x^3\;1-x^4} .$$

In the present case, the number of partitions of the biweight pq into biparts is the coefficient of x^py^q in the ascending expansion of

$$\frac{1}{1-x\;1-y\;1-x^2\;1-xy\;1-y^2\;1-x^3\;1-x^2y\;1-xy^2\;1-y^3} .$$

or, putting y equal to x, we see that the whole number of partitions of the weight $p+q$ into biparts is the coefficient of x^{p+q} in the ascending expansion of

$$\frac{1}{(1-x)^2(1-x^2)^3(1-x^3)^4}$$

Further, it is clear that the number of partitions of the biweight pq into exactly μ biparts is the coefficient of $a^\mu x^py^q$ in the expansion of

$$\frac{1}{1-ax\;1-ay\;1-ax^2.1-axy.1-ay^2.1-ax^3\;1-ax^2y\;1-axy^2\;1-ay^3} .$$

6. It is convenient now to have before us a list of the symmetric functions up to weight 4 inclusive.

The expanded generating function is

$$1+x+y+2x^2+2xy+2y^2+3x^3+4x^2y+4xy^2+3y^3$$
$$+5x^4+7x^3y+9x^2y^2+7xy^3+5y^4+\ldots ,$$

and we have

WEIGHT 1,

Biweight 10,	Biweight 01,
$(\overline{10})$;	$(\overline{01})$,

WEIGHT 2,

Biweight 20,	Biweight 11,	Biweight 02,
$(\overline{20})$	$(\overline{11})$	$(\overline{02})$
$(\overline{10}^2)$;	$(\overline{10}\,\overline{01})$,	$(\overline{01}^2)$.

WEIGHT 3,

Biweight 30,	Biweight 21,	Biweight 12,	Biweight 03,
$(\overline{30})$	$(\overline{21})$	$(\overline{12})$	$(\overline{03})$
$(\overline{20}\,\overline{10})$	$(\overline{20}\,\overline{01})$	$(\overline{10}\,\overline{02})$	$(\overline{01}\,\overline{02})$
$(\overline{10}^3)$,	$(\overline{11}\,\overline{10})$	$(\overline{01}\,\overline{11})$	$(\overline{01}^3)$.
	$(\overline{10}^2\,\overline{01})$;	$(\overline{10}\,\overline{01}^2)$,	

WEIGHT 4,

Biweight 40,	Biweight 31,	Biweight 22,	Biweight 13,	Biweight 04,
$(\overline{40})$	$(\overline{31})$	$(\overline{22})$	$(\overline{13})$	$(\overline{04})$
$(\overline{30}\,\overline{10})$	$(\overline{21}\,\overline{10})$	$(\overline{21}\,\overline{01})$	$(\overline{12}\,\overline{01})$	$(\overline{03}\,\overline{01})$
$(\overline{20}^2)$	$(\overline{30}\,\overline{01})$	$(\overline{12}\,\overline{10})$	$(\overline{03}\,\overline{10})$	$(\overline{02}^2)$
$(\overline{20}\,\overline{10}^2)$	$(\overline{20}\,\overline{11})$	$(\overline{20}\,\overline{02})$	$(\overline{02}\,\overline{11})$	$(\overline{02}\,\overline{01}^2)$
$(\overline{10}^4)$,	$(\overline{20}\,\overline{10}\,\overline{01})$	$(\overline{11}^2)$	$(\overline{02}\,\overline{10}\,\overline{01})$	$(\overline{01}^4)$
	$(\overline{11}\,\overline{10}^2)$	$(\overline{20}\,\overline{01}^2)$	$(\overline{11}\,\overline{01}^2)$	
	$(\overline{10}^3\,\overline{01})$,	$(\overline{02}\,\overline{10}^2)$	$(\overline{10}\,\overline{01}^3)$,	
		$(\overline{11}\,\overline{10}\,\overline{01})$		
		$(\overline{10}^2\,\overline{01}^2)$;		

§ 2. *Preliminary Algebraic Theory.*

7. The partitions with one bipart correspond to the sums of the powers in the single system, or unipartite theory. They are easily expressed in terms of the fundamental symmetric functions.

The right hand side of the relation

$$(1+\alpha_1 x+\beta_1 y)(1+\alpha_2 x+\beta_2 y) \quad = 1 + a_{10}x + a_{01}y + \quad + a_{pq}x^py^q + \;.\;,$$

may be written

$$\overline{\exp}\,(a_{10}x + a_{01}y),$$

or, since it is convenient to write the symmetric function $(\overline{pq})$ in the form s_{pq}, this is

$$\overline{\exp}\,(s_{10}x + s_{01}y),$$

where the bar over exp indicates a symbolism by which $\frac{s_{10}{}^p s_{01}{}^q}{p!\,q!}$ denotes $(\overline{10^p}\,\overline{01^q}) \equiv a_{pq}$

Hence the relation

$$1 + a_{10}x + a_{01}y + . \quad + a_{pq}x^py^q + \quad = \overline{\exp}\,(s_{10}x + s_{01}y),$$

which is important in connection with the collateral theory of operations to be presently brought into view.

8 Taking logarithms of both sides of the relation

$$(1+\alpha_1 x+\beta_1 y)(1+\alpha_2 x+\beta_2 y) \quad = 1 + a_{10}x + a_{01}y + \quad + a_{pq}x^py^q + \quad .\,,$$

there results

$$s_{10}x + s_{01}y - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2) + \tfrac{1}{3}(s_{30}x^3 + 3s_{21}x^2y + 3s_{12}xy^2 + s_{03}y^3) - \quad *$$
$$= \log(1 + a_{10}x + a_{01}y + . \quad + a_{pq}x^py^q + . \quad .)$$

Hence

$$1 + a_{10}x + a_{01}y + .\,. \quad a_{pq}x^py^q + . \quad = \overline{\exp}\,(s_{10}x + s_{01}y)$$
$$= \exp\{s_{10}x + s_{01}y - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2)$$
$$+ \tfrac{1}{3}(s_{30}x^3 + 3s_{21}x^2y + 3s_{12}xy^2 + s_{03}y^3) - . \quad \}$$

Also we have the series of relations —

$$\begin{cases} s_{10} = a_{10}, \\ s_{01} = a_{01}, \end{cases}$$
$$\begin{cases} s_{20} = a_{10}{}^2 - 2a_{20}, \\ s_{11} = a_{10}a_{01} - a_{11}, \\ s_{02} = a_{01}{}^2 - 2a_{02}, \end{cases}$$
$$\begin{cases} s_{30} = a_{10}{}^3 - 3a_{20}a_{10} + 3a_{30}, \\ s_{21} = a_{10}{}^2 a_{01} - a_{20}a_{01} - a_{11}a_{10} + a_{21}, \\ s_{12} = a_{01}{}^2 a_{10} - a_{02}a_{10} - a_{11}a_{01} + a_{12}, \\ s_{03} = a_{01}{}^3 - 3a_{02}a_{01} + 3a_{03}, \end{cases}$$
$$\vdots$$

* Viz :- $s_{10}x + s_{01}y = \Sigma(\alpha_1 x + \beta_1 y)$; $s_{20}x^2 + 2s_{11}xy + s_{02}y^2 = \Sigma(\alpha_1 x + \beta_1 y)^2$, &c

and in general

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}s_{pq}=\Sigma_\pi(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}a_{p_1q_1}^{\pi_1}a_{p_2q_2}^{\pi_2}\ldots$$

9 Moreover, the fundamental symmetric functions are expressed in the terms of the forms s_{pq} by the formula

$$(-)^{p+q-1}a_{pq}=\Sigma\left\{\frac{(p_1+q_1-1)!}{p_1!\,q_1!}\right\}^{\pi_1}\left\{\frac{(p_2+q_2-1)!}{p_2!\,q_2!}\right\}^{\pi_2}\ldots\frac{(-)^{\Sigma\pi-1}}{\pi_1!\,\pi_2!\ldots}s_{p_1q_1}^{\pi_1}s_{p_2q_2}^{\pi_2}\ldots,$$

as will be evident by simply applying the multinomial theorem to one of the above written general identities of Art 8

10 The single-bipart functions having been actually expressed in terms of the fundamental symmetric functions, it remains to show that all other rational algebraic symmetric functions are also so expressible SCHLAFLI (*loc cit*) has established this by induction, and it is not necessary to further discuss the theorem here. In Art 43 of the present memoir, will be found the actual expression of a given symmetric function by means of single-bipart forms, a formula which, combined with one given above, Art 8 serves to establish the theorem conclusively

The Symmetric Function h_{pq}

11 Write

$$(1+\alpha_1x+\beta_1y)(1+\alpha_2x+\beta_2y)\ldots=1+a_{10}x+a_{01}y+\ldots+a_{pq}x^py^q+\ldots$$
$$=\frac{1}{1-h_{10}x-h_{01}y+\ldots+(-)^{p+q}h_{pq}x^py^q+\ldots},$$

as the definition of the function h_{pq}

Writing $-x$, $-y$ for x, y, we have

$$1+h_{10}x+h_{01}y+\ldots+h_{pq}x^py^q=\frac{1}{(1-\alpha_1x-\beta_1y)(1-\alpha_2x-\beta_2y)\ldots},$$

and expanding the right hand in ascending powers of x and y

$$h_{pq}=\Sigma\frac{(p_1+q_1)!}{p_1!\,q_1!}\frac{(p_2+q_2)!}{p_2!\,q_2!}\ldots(\overline{p_1q_1}\,\overline{p_2q_2}\ldots),$$

the summation being for all partitions of the biweight. Changing the signs of x and y in the relation first written down, we obtain

$$1+h_{10}x+h_{01}y+\ldots+h_{pq}x^py^q+\ldots=\frac{1}{1-a_{10}x-a_{01}y+\ldots+(-)^{p+q}a_{pq}x^py^q+\ldots},$$

an identity which arises from the former by interchanging the letter h with the letter a

Hence, if f and ϕ be any two functions, such that

$$f(a_{10}, a_{01}, \ldots a_{pq}, \ldots) = \phi(h_{10}, h_{01}, \ldots h_{pq}, \ldots),$$

then also

$$\phi(a_{10}, a_{01}, \ldots a_{pq}, \ldots) = f(h_{10}, h_{01}, \ldots h_{pq}, \ldots),$$

and, in general, in any relation connecting the functions a with the functions h, an identity will still remain if the letters a and h be transposed.

By the multinomial theorem

$$(-)^{p+q-1} h_{pq} = \Sigma_\pi (-)^{\Sigma\pi-1} \frac{(\Sigma\pi)!}{\pi_1!\,\pi_2!} a_{p_1q_1}^{\pi_1} a_{p_2q_2}^{\pi_2}$$

From a previous result in this article, by taking logarithms and expanding

$$\frac{(p+q-1)!}{p!\,q!} s_{pq} = \Sigma_\pi (-)^{\Sigma\pi-1} \frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!} h_{p_1q_1}^{\pi_1} h_{p_2q_2}^{\pi_2} \ldots,$$

which is to be compared with the formula

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} s_{pq} = \Sigma_\pi (-)^{\Sigma\pi-1} \frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!} a_{p_1q_1}^{\pi_1} a_{p_2q_2}^{\pi_2}$$

and it will be noticed that s_{pq} remains unchanged when h is written for a, except for a change of sign, when the weight $p+q$ is even.

§ 3. *The Differential Operations*

12 The beautiful properties of these symmetric functions are most easily established by means of the differential operations whose theory I proceed to establish.

Consider the identity

$$\begin{aligned}(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_n x + \beta_n y) \\ = 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots,\end{aligned}$$

where n may be as large as we please.

Multiply each side by $(1 + \mu x + \nu y)$.

The right hand side becomes

$$\begin{aligned}1 + (a_{10} + \mu)x + (a_{01} + \nu)y + (a_{20} + \mu a_{10})x^2 + (a_{11} + \mu a_{01} + \nu a_{10})xy \\ + (a_{02} + \nu a_{01})y^2 + \ldots,\end{aligned}$$

and, in general, a_{pq} becomes converted into

$$a_{pq} + \mu a_{p-1,q} + \nu a_{p,q-1}$$

Hence any rational integral function of the coefficients

$$a_{10}, a_{01}, a_{20}, a_{11}, a_{02} \ldots$$

viz.,

$$f(a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \ldots) \equiv f,$$

is converted into

$$f + (\mu g_{10} + \nu g_{01}) f + \frac{1}{2!}(\mu g_{10} + \nu g_{01})^2 f + \frac{1}{3!}(\mu g_{10} + \nu g_{01})^3 f + \ldots,$$

where

$$g_{10} = \Sigma a_{p-1,q}\,\partial_{a_{pq}}, \quad g_{01} = \Sigma a_{p,q-1}\,\partial_{a_{pq}},$$

and the multiplication of operators is symbolic.*

The new value of f is

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01}) f,$$

where the bar is placed over exp to denote that the multiplication of operators is symbolic (*vide* Art. 7).

13. Write

$$G_{pq} = \frac{1}{p!\,q!}\overline{g_{10}{}^p\,g_{01}{}^q},$$

the bar denoting symbolic multiplication, then

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01}) f$$
$$= (1 + \mu G_{10} + \nu G_{01} + \mu^2 G_{20} + \mu\nu G_{11} + \nu^2 G_{02} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f$$

(Compare Art. 7.)

Now suppose the symmetric function f expressed in terms of

$$\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \ldots \alpha_n, \beta_n$$

to be

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3} \ldots).$$

The introduction of the new quantities μ, ν results in the addition to

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3} \ldots)$$

of the terms

$$\mu^{p_1}\nu^{q_1}(\overline{p_2q_2}\,\overline{p_3q_3} \ldots) + \mu^{p_2}\nu^{q_2}(\overline{p_1q_1}\,\overline{p_3q_3} \ldots) + \mu^{p_3}\nu^{q_3}(\overline{p_1q_1}\,\overline{p_2q_2} \ldots) + \ldots;$$

* By "symbolic" is to be understood "non-operational," as in what is commonly known as the "symbolic" form of TAYLOR's Theorem.

and hence

$$f + \mu^{p_1}\nu^{q_1}(\overline{p_2q_2}\,\overline{p_3q_3}\;\ldots) + \mu^{p_2}\nu^{q_2}(\overline{p_1q_1}\,\overline{p_3q_3}\;\ldots) + \mu^{p_3}\nu^{q_3}(\overline{p_1q_1}\,\overline{p_2q_2}\;\ldots) + \ldots$$
$$= (1 + \mu G_{10} + \nu G_{01} + \mu^2 G_{20} + \mu\nu G_{11} + \nu^2 G_{02} + \ldots) f,$$

and equating coefficients of like products $\mu^p\nu^q$, we find

$$G_{p_1q_1}(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\;\ldots) = (\overline{p_2q_2}\,\overline{p_3q_3}\ldots),$$
$$G_{p_2q_2}(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\;\ldots) = (\overline{p_1q_1}\,\overline{p_3q_3}\ldots),$$
$$G_{p_3q_3}(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\;\ldots) = (\overline{p_1q_1}\,\overline{p_2q_2}\ldots),$$
$$G_{p_1q_1}(\overline{p_1q_1}) = 1,$$
$$G_{p_1q_1}G_{p_2q_2}\ldots G_{p_nq_n}(\overline{p_1q_1}\,\overline{p_2q_2}\ldots\overline{p_nq_n}) = 1,$$

and $G_{rs}f = 0$, unless the bipart $\overline{rs}$ is involved in the expression of f

From the above we gather the very important fact that the effect of the operation G_{pq} upon a partition is to obliterate one bipart $\overline{pq}$ when such bipart is present, and to annihilate the partition if it contains no bipart $\overline{pq}$.

14. I return to the result

$$1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \ldots = \overline{\exp}\,(\mu g_{10} + \nu g_{01}),$$

wherein be it remembered the multiplication of operators in the right hand expression is symbolic. I seek to replace $\overline{\exp}\,(\mu g_{10} + \nu g_{01})$ by an expression containing products of linear partial differential operations in which the multiplication is not symbolic

We have by definition

$$g_{10} = \partial_{a_{10}} + a_{10}\,\partial_{a_{20}} + a_{01}\,\partial_{a_{11}} + \ldots,$$
$$g_{01} = \partial_{a_{01}} + a_{01}\,\partial_{a_{02}} + a_{10}\,\partial_{a_{11}} + \ldots,$$

let further

$$g_{pq} = \partial_{a_{pq}} + a_{10}\,\partial_{a_{p+1,q}} + a_{01}\,\partial_{a_{p\,q+1}} + \ldots + a_{rs}\,\partial_{a_{p+r\,q+s}} + \ldots,$$

a definition which includes the former.

15 I will establish the relation

$$\overline{\exp}\,(m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots)$$
$$= \exp\,(M_{10}g_{10} + M_{01}g_{01} + \ldots + M_{pq}g_{pq} + \ldots),$$

where on the left and right hand sides the multiplications of operators are respectively symbolic and not symbolic, and

$$\exp(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots)$$
$$= 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots,$$

where ξ, η are the undetermined algebraic quantities

For the multiplication of two operators $g_{p_1q_1}$, $g_{p_2q_2}$, we have the formula

$$g_{p_1q_1} g_{p_2q_2} = \overline{g_{p_1q_1} g_{p_2q_2}} + g_{p_1q_1} \dagger g_{p_2q_2},$$

wherein the symbol $\dagger$ denotes explicit operation upon the operand, regarding the latter as a function of symbols of quantity only, and not of the differential inverses

Also

$$g_{p_1q_1} \dagger g_{p_2q_2} = g_{p_1+p_2, q_1+q_2}$$

Put

$$u_1 = m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots,$$

which may be written

$$u_1 = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots) g,$$

in which $m_{pq}g_{pq}$ is symbolically written $m_{pq}g$

But

$$u_2 = u_1 \dagger u_1 = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)^2 g,$$

where, after expansion of the right hand side, $m_{p_1q_1}m_{p_2q_2}g$ is to be written

$$m_{p_1q_1}m_{p_2q_2}g_{p_1+p_2, q_1+q_2}.$$

Then, with a similar convention,

$$u_s = u_1 \dagger u_{s-1} = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)^s g$$

Further, it is easy to prove the relation

$$u_s \dagger u_t = u_{s+t}$$

But for a series of linear partial differential operators enjoying this property, it is a well known and easily established theorem of SYLVESTER'S that

$$\overline{\exp} \, u_1 = \exp(u_1 - \tfrac{1}{2} u_2 + \tfrac{1}{3} u_3 - \ldots).$$

Hence, substituting, we easily reach the relation

$$\exp(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots) = 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots,$$

wherein ξ and η are undetermined algebraic quantities.

This establishes the theorem.

16. To apply it to the case in hand, put

$$m_{10} = \mu, \qquad m_{01} = \nu, \qquad m_{pq} = 0 \text{ in other cases}$$

Then

$$M_{10}\xi + M_{01}\eta + \quad + M_{pq}\xi^p\eta^q + \quad . \quad = \log(1 + \mu\xi + \nu\eta)$$

$$\therefore \quad M_{pq} = (-)^{p+q+1}\frac{(p+q-1)!}{p!\,q!}\mu^p\nu^q,$$

and the result is

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01}) = \exp\{\mu g_{10} + \nu g_{01} - \tfrac{1}{2}(\mu^2 g_{20} + 2\mu\nu g_{11} + \nu^2 g_{02}) + \tfrac{1}{3}(\mu^3 g_{30} + 3\mu^2\nu g_{21} + 3\mu\nu^2 g_{12} + \nu^3 g_{03}) - \quad .\}.$$

Combining with a former result

$$\mu g_{10} + \nu g_{01} - \tfrac{1}{2}(\mu^2 g_{20} + 2\mu\nu g_{11} + \nu^2 g_{02}) + \tfrac{1}{3}(\mu^3 g_{30} + 3\mu^2\nu g_{21} + 3\mu\nu^2 g_{12}\,\nu^3 g_{03}) - \,.$$
$$= \log(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \,.\quad.).$$

(Compare Art. 8.)

17 By expanding the right hand side we can express each linear operator of the form g_{pq} in terms of products of the obliterating operators which have the form G_{pq}

The law is identical with that which expresses the functions containing one bipart in terms of the fundamental symmetric functions We find

$$\begin{cases} g_{10} = G_{10} \\ g_{01} = G_{01} \end{cases}$$

$$\begin{cases} g_{20} = G_{10}^2 - 2G_{20} \\ g_{11} = G_{10}G_{01} - G_{11} \\ g_{02} = G_{01}^2 - 2G_{02} \end{cases}$$

$$\begin{cases} g_{30} = G_{10}^3 - 3G_{20}G_{10} + 3G_{30} \\ g_{21} = G_{10}^2G_{01} - G_{20}G_{01} - G_{11}G_{10} + G_{21} \\ g_{12} = G_{01}^2G_{10} - G_{02}G_{10} - G_{11}G_{01} + G_{12} \\ g_{03} = G_{01}^3 - 3G_{02}G_{01} + 3G_{03} \end{cases}$$

$$\vdots$$

and in general

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq} = \Sigma_\pi(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\,..}G_{p_1q_1}^{\pi_1}\;G_{p_2q_2}^{\pi_2}\quad .\quad .$$

while

$$(-)^{p+q-1}G_{pq} = \Sigma_\pi\left\{\frac{(p_1+q_1-1)!}{p_1!\,q_1!}\right\}^{\pi_1}\left\{\frac{(p_2+q_2-1)!}{p_2!\,q_2!}\right\}^{\pi_2}.\quad .\ \frac{(-)^{\Sigma\pi-1}}{\pi_1!\,\pi_2!\ .}g_{p_1q_1}^{\pi_1}\;g_{p_2q_2}^{\pi_2}\quad .$$

(Compare Art. 8.)

18 By comparison of these relations with the corresponding algebraic ones to which reference has been made, it is manifest that g_{pq} and G_{pq} are respectively in co-relation with s_{pq} and a_{pq} In other words these operations respectively correspond to the partitions $(\overline{pq})$ and $(\overline{10^p}\ \overline{01^q})$ It is necessary to find the operations which correspond to the remaining partitions which symbolize symmetric functions

We have the easily derivable results in operations

$$
\begin{aligned}
g_{p_1q_1} g_{p_2q_2} &= \overline{g_{p_1q_1} g_{p_2q_2}} + g_{p_1+p_2,\, q_1+q_2}, \\
g_{p_1q_2}{}^2 &= \overline{g_{p_1q_2}{}^2} + g_{2p_1,\, 2q_1}, \\
g_{p_1q_1} g_{p_2q_2} g_{p_3q_3} &= \overline{g_{p_1q_1} g_{p_2q_2} g_{p_3q_3}} + \overline{g_{p_1q_1} g_{p_2+p_3,\, q_2+q_3}} + \overline{g_{p_2q_2} g_{p_3+p_1,\, q_3+q_1}} \\
&\quad + \overline{g_{p_3q_3} g_{p_1+p_2,\, q_1+q_2}} + g_{p_1+p_2+p_3,\, q_1+q_2+q_3}, \\
g_{p_1q_1}{}^2 g_{p_2q_2} &= \overline{g_{p_1q_1}{}^2 g_{p_2q_2}} + \overline{2g_{p_1q_1} g_{p_1+p_2,\, q_1+q_2}} + \overline{g_{2p_1,\, 2q_1} g_{p_2q_2}} + g_{2p_1+p_2,\, 2q_1+q_2}, \\
g_{p_1q_1}{}^3 &= \overline{g_{p_1q_1}{}^3} + \overline{3g_{2p_1,\, 2q_1} g_{p_1q_1}} + g_{3p_1,\, 3q_1};
\end{aligned}
$$

where as usual the bar denotes symbolic multiplication, and comparing these with the algebraic formulæ

$$
\begin{aligned}
(\overline{p_1q_1})(\overline{p_2q_2}) &= (\overline{p_1q_1}\ \overline{p_2q_2}) + (\overline{p_1+p_2,\, q_1+q_2}) \\
(\overline{p_1q_1})^2 &= 2(\overline{p_1q_1}^2) + (\overline{2p_1,\, 2q_1}) \\
(\overline{p_1q_1})(\overline{p_2q_2})(\overline{p_3q_3}) &= (\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3}) + (\overline{p_1q_1}\ \overline{p_2+p_3,\, q_2+q_3}) + (\overline{p_2q_2}\ \overline{p_3+p_1,\, q_3+q_1}) \\
&\quad + (\overline{p_3q_3}\ \overline{p_1+p_2,\, q_1+q_2}) + (\overline{p_1+p_2+p_3,\, q_1+q_2+q_3}) \\
(\overline{p_1q_1})^2(\overline{p_2q_2}) &= 2(\overline{p_1q_1}^2\ \overline{p_2q_2}) + 2(\overline{p_1q_1}\ \overline{p_1+p_2,\, q_1+q_2}) + (\overline{2p_1,\, 2q_1}\ \overline{p_2q_2}) \\
&\quad + (\overline{2p_1+p_2,\, 2q_1+q_2}) \\
(\overline{p_1q_1})^3 &= 6(\overline{p_1q_1}^3) + 3(\overline{2p_1,\, 2q_1}\ \overline{p_1q_1}) + (\overline{3p_1,\, 3q_1})
\end{aligned}
$$

it is evident that the operations

$$
\frac{\overline{g_{p_1q_1}{}^2}}{2!}, \quad \frac{\overline{g_{p_1q_1}{}^2 g_{p_2q_2}}}{2!}, \quad \frac{\overline{g_{p_1q_1}{}^3}}{3!},
$$

are produced according to the same *law* as the symmetric functions

$$
(\overline{p_1q_1}^2), \quad (\overline{p_1q_1}^2\ \overline{p_2q_2}), \quad (\overline{p_1q_1}^3);
$$

further the *law* is perfectly general and indicates that the operation

$$
\frac{1}{\pi_1!}\frac{1}{\pi_2!}\cdots\overline{g_{p_1q_1}{}^{\pi_1} g_{p_2q_2}{}^{\pi_2}\cdots},
$$

is in co-relation with the symmetric function

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\quad .)$$

19 There is thus complete correspondence between quantity and operation, and any formula of quantity may be at once translated into a formula of operation.

Observe that a product of symmetric functions

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\qquad)\,(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\qquad)$$

is in correspondence with the operation

$$\frac{1}{\pi_1!}\,\frac{1}{\pi_2!}\qquad\overline{g_{p_1q_1}{}^{\pi_1}\,g_{p_2q_2}{}^{\pi_2}\,\ldots}\;\frac{1}{\rho_1!}\,\frac{1}{\rho_2!}\qquad\overline{g_{r_1s_1}{}^{\rho_1}\,g_{r_2s_2}{}^{\rho_2}\quad .}$$

the notation indicating that the two operations

$$\frac{1}{\rho_1!}\,\frac{1}{\rho_2!}\;\cdot\;\overline{g_{r_1s_1}{}^{\rho_1}\,g_{r_2s_2}{}^{\rho_2}\quad\ldots}\quad\text{and}\quad\frac{1}{\pi_1!}\,\frac{1}{\pi_2!}\;\cdot\;\overline{g_{p_1q_1}{}^{\pi_1}\,g_{p_2q_2}{}^{\pi_2}\,.}$$

are to be successively performed.

For an example take the algebraic formula

$$(\overline{31}\,\overline{01})=-\tfrac{1}{2}(\overline{21}\,\overline{10})(\overline{01})+\tfrac{1}{2}(\overline{21}\,\overline{01})(\overline{10})+\tfrac{1}{2}(\overline{10}\,\overline{01})(\overline{21})-\tfrac{1}{2}(\overline{21}\,\overline{10}\,\overline{01}),$$

which is translated into the operator formula

$$\overline{g_{31}g_{01}}=-\tfrac{1}{2}\,\overline{g_{21}g_{10}}\cdot g_{01}+\tfrac{1}{2}\,\overline{g_{21}g_{01}}\quad g_{10}+\tfrac{1}{2}\,\overline{g_{10}g_{01}}\cdot g_{21}-\tfrac{1}{2}\,\overline{g_{21}g_{10}g_{01}}.$$

20. It is now necessary to enquire into the laws which appertain to the performance of these operations upon symmetric functions We have seen. *ante* Art 13, the law by which the obliterator G_{pq} is performed upon a monomial symmetric function.

Since

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq}=\Sigma_\pi(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!}\,G_{p_1q_1}^{\pi_1}\;G_{p_2q_2}^{\pi_2}\quad .\quad ,$$

we can operate with g_{pq} upon a monomial form by operating independently with the successive G products on the right, and adding the results together. As a particular result, observe that a term on the right is G_{pq}, and hence

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq}s_{pq}=G_{pq}s_{pq}=G_{pq}(\overline{pq})=1,$$

or

$$g_{pq}s_{pq}=(-)^{p+q-1}\frac{p!\,q!}{(p+q-1)!};$$

and g_{pq} causes every other single bipart function to vanish, it must, indeed, cause any monomial function to vanish which does not comprise one of the partitions of the biweight pq amongst its biparts

21. The relation just obtained yields the equivalence

$$g_{pq} = (-)^{p+q-1} \frac{p!\,q!}{(p+q-1)!} \partial_{s_{pq}},$$

and further results of the nature

$$\overline{g_{p_1q_1}g_{p_2q_2}} = (-)^{p_1+p_2+q_1+q_2} \left\{ \frac{p_1!\,q_1!\,p_2!\,q_2!}{(p_1+q_1-1)!\,(p_2+q_2-1)!} \partial_{s_{p_1q_1}} \partial_{s_{p_2q_2}} \right.$$
$$\left. + \frac{(p_1+p_2)!\,(q_1+q_2)!}{(p_1+p_2+q_1+q_2-1)!} \partial_{s_{p_1+p_2,q_1+q_2}} \right\}$$

which are of use in connexion with the theory of function with single biparts

Since every symmetric function is expressible in terms of the fundamental symmetric functions, every operation $\overline{g_{p_1q_1}^{\pi_1} g_{p_2q_2}^{\pi_2}}$ is necessarily expressible as a sum of G products and can be performed upon a monomial symmetric function

22. The solutions of the partial differential equation

$$g_{pq} = 0,$$

are the single bipart forms omitting s_{pq} (Art 21), while the solution of the partial differential equation

$$G_{pq} = 0,$$

are those monomial symmetric functions in which the bipart $\overline{pq}$ is absent (Art. 13).

23 The operation $\partial_{a_{pq}}$ is expressible by means of the operations g_{pq}.

Reversing the formula

$$g_{pq} = \partial_{a_{pq}} + a_{10}\,\partial_{a_{p+1,q}} + a_{01}\,\partial_{a_{p,q-1}} + \ldots + a_{rs}\,\partial_{a_{p+r,q-s}} + \ldots,$$

we obtain

$$\partial_{a_{pq}} = g_{pq} - h_{10}g_{p+1,q} - h_{01}g_{p,q+1} + \ldots + (-)^{r+s}h_{rs}g_{p+r,q+s} + \ldots,$$

where as before (Art 11),

$$(-)^{r+s-1}h_{rs} = \Sigma_\rho (-)^{\Sigma\rho-1} \frac{(\Sigma\rho)!}{\rho_1!\,\rho_2!\ldots} a_{r_1s_1}^{\rho_1} a_{r_2s_2}^{\rho_2} \ldots$$

§ 4 *The Theory of Three Identities.*

24 The course of the investigation at this point necessitates the introduction of two identities similar to, and in addition to, the fundamental identity

Let

$$1+a_{10}x+a_{01}y+\ldots+a_{pq}x^py^q+\ldots=(1+\alpha_1x+\beta_1y)(1+\alpha_2x+\beta_2y)\ldots \quad (\text{I})$$

$$1+b_{10}x+b_{01}y+\ldots+b_{pq}x^py^q+\ldots=(1+\alpha_1^{(1)}x+\beta_1^{(1)}y)(1+\alpha_2^{(1)}x+\beta_2^{(1)}y)\ldots \quad (\text{II})$$

$$1+c_{10}x+c_{01}y+\ldots+c_{pq}x^py^q+\ldots=(1+\alpha_1^{(2)}x+\beta_1^{(2)}y)(1+\alpha_2^{(2)}x+\beta_2^{(2)}y)\ldots \quad (\text{III})$$

wherein x and y may be regarded as undetermined quantities and the identities as merely expressing the relations between the coefficients on the left and quantities α, β on the right.

Assume the coefficients and quantities in the first two identities to be given and the coefficients in the third identity to be then determined by the relations —

$$\begin{aligned}1+c_{01}\xi+c_{01}\eta+\ldots+c_{pq}\xi^p\eta^q+\ldots\\ =\Pi_s(1+\alpha_sb_{10}\xi+\beta_sb_{01}\eta+\ldots+\alpha_s^p\beta_s^qb_{pq}\xi^p\eta^q+\ldots),\end{aligned}$$

ξ and η being undetermined quantities

Multiplying out the right-hand side of this relation, it is found to be equivalent to the series of relations .—

$$\begin{aligned}
c_{10}&=(\overline{10})\,b_{10}\,,\\
c_{01}&=(\overline{01})\,b_{01}\,,\\
c_{20}&=(\overline{20})\,b_{20}+(\overline{10^2})\,b_{10}^2\,,\\
c_{11}&=(\overline{11})\,b_{11}+(\overline{10}\,\overline{01})\,b_{10}b_{01}\,,\\
c_{02}&=(\overline{02})\,b_{02}+(\overline{01^2})\,b_{01}^2\,,\\
c_{30}&=(\overline{30})\,b_{30}+(\overline{20}\,\overline{10})\,b_{20}b_{10}+(\overline{10^3})\,b_{10}^3\,,\\
c_{21}&=(\overline{21})\,b_{21}+(\overline{20}\,\overline{01})\,b_{20}b_{01}+(\overline{11}\,\overline{10})\,b_{11}b_{10}+(\overline{10^2}\,\overline{01})\,b_{10}^2b_{01}\,,\\
c_{12}&=(\overline{12})\,b_{12}+(\overline{02}\,\overline{10})\,b_{02}b_{10}+(\overline{11}\,\overline{01})\,b_{11}b_{01}+(\overline{10}\,\overline{01^2})\,b_{10}b_{01}^2\,,\\
c_{03}&=(\overline{03})\,b_{03}+(\overline{02}\,\overline{01})\,b_{02}b_{01}+(\overline{01^3})\,b_{01}^3\,,\\
&\vdots\qquad\cdots\qquad\cdots
\end{aligned}$$

and generally in the expression of c_{p}, every symmetric function of biweight pq of the quantities in the first identity occurs, each attached to the corresponding product of coefficients from the second identity.

25. Represent the symmetric functions of the quantities occurring in the second and third identities by partitions in brackets $(\;)_1$, $(\;)_2$ respectively.

Now

$$\Pi_s (1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s^p\beta_s^q b_{pq}\xi^p\eta^q + \ldots)$$

is from the identity II. equal to

$$\Pi_s [(1 + \alpha_s\alpha_1^{(1)}\xi + \beta_s\beta_1^{(1)}\eta)(1 + \alpha_s\alpha_2^{(1)}\xi + \beta_s\beta_2^{(1)}\eta)(\ldots)\ldots]$$

which is

$$\Pi_s\Pi_t (1 + \alpha_s\alpha_t^{(1)}\xi + \beta_s\beta_t^{(1)}\eta).$$

26. Hence the assumed relation becomes on taking logarithms

$$\Sigma_s \log (1 + \alpha_s^{(2)}\xi + \beta_s^{(2)}\eta) = \Sigma_s\Sigma_t \log (1 + \alpha_s\alpha_t^{(1)}\xi + \beta_s\beta_t^{(1)}\eta),$$

and expanding and equating coefficients of $\xi^p\eta^q$

$$(\overline{pq})_2 = (\overline{pq})(\overline{pq})_1;$$

an important relation which shows that the assumed relation is unaltered when the set of quantities α is interchanged with the set $\alpha^{(1)}$, in such wise that α_s and $\alpha_s^{(1)}$ are transposed. It is indeed of fundamental importance, and will be brought prominently forward in the sequel. Its consideration must be postponed until a further step has been taken in the theory of the operators.

27. Let the operators

$$\begin{array}{ll} g_{pq}, & G_{pq}, \\ g_{pq}', & G_{pq}', \\ g_{pq}'', & G_{pq}'', \end{array}$$

refer to identities I., II., III., respectively.

Writing the relation

$$\begin{aligned} 1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots \\ = \Pi_s (1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s^p\beta_s^q b_{pq}\xi^p\eta^q + \ldots), \end{aligned}$$

in the abbreviated form

$$U = u_{\alpha_1\beta_1}u_{\alpha_2\beta_2}u_{\alpha_3\beta_3}\ldots,$$

and performing the operation

$$g_{pq}' = \partial_{b_{pq}} + b_{10}\,\partial_{b_{p+1,q}} + b_{01}\,\partial_{b_{p,q-1}} + \dots + b_{rs}\,\partial_{b_{p+r\,q+s}} + \dots$$

we have

$$g_{pq}'\mathrm{U} = (g_{pq}'u_{\alpha_1\beta_1})\,u_{\alpha_2\beta_2}u_{\alpha_3\beta_3} \dots + u_{\alpha_1\beta_1}(g_{pq}'u_{\alpha_2\beta_2})\,u_{\alpha_3\beta_3} + \dots$$

Moreover,

$$g_{pq}'u_{\alpha_s\beta_s} = \alpha_s^p\beta_s^q\xi^p\eta^q u_{\alpha_s\beta_s},$$

hence

$$g_{pq}'\mathrm{U} = (\overline{pq})\xi^p\eta^q\mathrm{U},$$

and replacing U by its value, we have

$$g_{pq}'c_{pq} = (\overline{pq}),$$

while in general

$$g_{pq}'c_{rs} = (\overline{pq})\,c_{r-p,s-q}$$

Now regarding the coefficients b_{pq} as functions of the coefficients c_{pq} only, we have

$$\begin{aligned} g_{pq}' &= (g_{pq}'c_{pq})\,\partial_{c_{pq}} + \dots + (g_{pq}'c_{rs})\,\partial_{c_{rs}} + \dots \\ &= (\overline{pq})\,(\partial_{c_{pq}} + c_{10}\,\partial_{c_{p+1,q}} + c_{01}\,\partial_{c_{p\,q+1}} + \dots + c_{r-p,s-q}\,\partial_{c_{rs}} + \dots) \end{aligned}$$

Thus

$$g_{pq}' = (\overline{pq})\,g_{pq}''.$$

But the assumed relation is symmetrical as regards the quantities in the first two identities; hence also

$$g_{pq} = (\overline{pq})_1\,g_{pq}'',$$

and thence, since $(\overline{pq})_2 = (\overline{pq})\,(\overline{pq})_1$, we have

$$(\overline{pq})_2\,g_{pq}'' = (\overline{pq})_1\,g_{pq}' = (\overline{pq})\,g_{pq}$$

If we then regard the assumed relation as defining a transformation of the quantities occurring in the identity III into either of the sets of quantities associated with I. or II., the operation

$$(\overline{pq})_2\,g_{pq}''$$

is an invariant

28 Since $g_{pq}' = (\overline{pq})\,g_{pq}''$

$$\begin{aligned} &\xi g_{10}' + \eta g_{01}' - \tfrac{1}{2}(\xi^2 g_{20}' + 2\xi\eta g_{11}' + \eta^2 g_{02}') + \tfrac{1}{3}(\xi^3 g_{30}' + 3\xi^2\eta g_{21}' + 3\xi\eta^2 g_{12}' + \eta^3 g_{03}') - \dots \\ &\quad = \xi(\overline{10})\,g_{10}'' + \eta(\overline{01})\,g_{01}'' - \tfrac{1}{2}\{\xi^2(\overline{20})\,g_{20}'' + 2\xi\eta(\overline{11})\,g_{11}'' + \eta^2(\overline{02})\,g_{02}''\} \\ &\qquad + \tfrac{1}{3}\{\xi^3(\overline{30})\,g_{30}'' + 3\xi^2\eta(\overline{21})\,g_{21}' + 3\xi\eta^2(\overline{12})\,g_{12}'' + \eta^3(\overline{03})\,g_{03}''\} - \dots \end{aligned}$$

The left hand side of this equation is

$$\log\left(1 + \xi G_{10}' + \eta G_{01}' + \dots + \xi^p \eta^q G_{pq}' + \dots\right) \text{ (vide Art. 16)},$$

while if the operators g'' be replaced by their expressions in terms of the operators G'', it is easily seen that the right side is

$$\Sigma_s \log\left(1 + \alpha_s G_{10}''\xi + \beta_s G_{01}''\eta + \dots + \alpha_s^p \beta_s^q G_{pq}''\xi^p\eta^q + \dots\right)$$

29. Hence the operator relation

$$\begin{aligned} &1 + G_{10}'\xi + G_{01}'\eta + \dots + G_{pq}'\xi^p\eta^q + \dots \\ &\quad = \Pi_s\left(1 + \alpha_s G_{10}''\xi + \beta_s G_{01}''\eta + \dots + \alpha_s^p \beta_s^q G_{pq}''\xi^p\eta^q + \dots\right) \end{aligned}$$

This result must be compared with the relation (Art. 24)

$$\begin{aligned} &1 + c_{10}\xi + c_{01}\eta + \dots + c_{pq}\xi^p\eta^q + \dots \\ &\quad = \Pi_s\left(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \dots + \alpha_s^p \beta_s^q b_{pq}\xi^p\eta^q + \dots\right) \end{aligned}$$

30. I say that such a comparison yields the following theorem —

"In any relation connecting the quantities c_{pq} with the quantities b_{pq}, we are at liberty to substitute

$$G_{pq}' \text{ for } c_{pq}, \quad \text{and} \quad G_{pq}'' \text{ for } b_{pq},$$

and we in this manner obtain a relation between operators in correspondence."

To explain this further, observe that ξ, η being undetermined quantities in the assumed relation which connects the quantities of the three identities I., II., III., we are able to express any product whatever of the coefficients $c_{10}, c_{01}, \dots c_{pq}, \dots$ in terms of products of coefficients $b_{10}, b_{01}, \dots b_{pq}, \dots$ and of symmetrical functions of the quantities $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots$. The substitution in question can be made in any equation thus formed.

31. With regard to the relation of Art. 24, viz. —

$$\begin{aligned} &1 + c_{10}\xi + c_{01}\eta + \dots + c_{pq}\xi^p\eta^q + \dots \\ &\quad = \Pi_s\left(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \dots + \alpha_s^p \beta_s^q b_{pq}\xi^p\eta^q + \dots\right), \end{aligned}$$

two important facts have been established—

(i.) That the relation is unaltered when the quantities occurring in the first identity

$$1 + a_{10}x + a_{01}y + \dots + a_{pq}x^py^q + \dots = (1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \dots$$

are exchanged with those occurring in the second identity

$$1 + b_{10}x + b_{01}y + \ldots + b_{pq}x^py^q + \ldots = (1 + \alpha_1^{(1)}x + \beta_1^{(1)}y)(1 + \alpha_2^{(1)}x + \beta_2^{(1)}y)\ldots$$

each with each (Art 26)

(ii) That we can always proceed to a relation between the operations by writing

$$G'_{pq} \text{ for } c_{pq} \quad \text{and} \quad G''_{pq} \text{ for } b_{pq} \quad (\text{Art } 30)$$

I will refer to these facts as the first and second properties of the relation respectively

§ 5 *The First Law of Symmetry*

32 By means of the equality

$$(\overline{pq})_2 = (\overline{pq})(\overline{pq})_1$$

which has been established *ante* (Art. 26), it is clear that any symmetric function expressed in a bracket $(\quad)_2$ can be expressed as a linear function of products of symmetric functions of the form $(\quad)(\quad)_1$, it is also clear from the first property above defined, that such expression will remain unaltered when the brackets $(\quad)$ and $(\quad)_1$ are interchanged, it must, therefore, be a symmetric function in regard to these brackets.

We may, therefore, suppose an equation

$$\begin{aligned}(\overline{r_1s_1}^{\rho_1}\overline{r_2s_2}^{\rho_2}\ldots)_2 = \ldots &+ J(\overline{a_1b_1}^{\alpha_1}\overline{a_2b_2}^{\alpha_2}\ldots)(\overline{p_1q_1}^{\pi_1}\overline{p_2q_2}^{\pi_2}\ldots)_1 \\ &+ J(\overline{a_1b_1}^{\alpha_1}\overline{a_2b_2}^{\alpha_2}\ldots)_1(\overline{p_1q_1}^{\pi_1}\overline{p_2q_2}^{\pi_2}\ldots) + \ldots \qquad (A)\end{aligned}$$

Moreover, we can express any product of the coefficients $c_{10}, c_{01}, \ldots c_{pq},$ as a linear function of expressions each of which contains a monomial symmetric function of the quantities $\alpha_1, \beta_1; \alpha_2, \beta_2; \ldots$ and a product of coefficients $b_{10}, b_{01}, \ldots b_{pq}, \ldots$

Assume then

$$c_{p_1q_1}^{\pi_1}c_{p_2q_2}^{\pi_2}\ldots = \ldots + L(\overline{a_1b_1}^{\alpha_1}\overline{a_2b_2}^{\alpha_2}\ldots)\, b_{r_1s_1}^{\rho_1}b_{r_2s_2}^{\rho_2}\ldots + \ldots \qquad (B)$$

$$c_{a_1b_1}^{\alpha_1}c_{a_2b_2}^{\alpha_2}\ldots = \ldots + M(\overline{p_1q_1}^{\pi_1}\overline{p_2q_2}^{\pi_2}\ldots)\, b_{r_1s_1}^{\rho_1}b_{r_2s_2}^{\rho_2}\ldots + \ldots \qquad (C)$$

From equation (B) is derived by the second property the operator relation

$$G'^{\pi_1}_{p_1q_1}G'^{\pi_2}_{p_2q_2}\ldots = \ldots + L(\overline{a_1b_1}^{\alpha_1}\overline{a_2b_2}^{\alpha_2}\ldots)\, G''^{\rho_1}_{r_1s_1}G''^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

and performing each side of this equation upon the opposite side of the equation (A) we obtain, after cancelling of $(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)$,

$$LG''^{\rho_1}_{r_1s_1}G''^{\rho_2}_{r_2s_2}\ldots(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)_2 = JG'^{\pi_1}_{p_1q_1}G'^{\pi_2}_{p_2q_2}\ldots(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)_1,$$

no other terms surviving the operations, or

$$L = J,$$

since the symmetric function on either side is reduced to unity by the operation

Similarly the equation (C) yields the equation of operators

$$G'^{\alpha_1}_{a_1b_1}G'^{\alpha_2}_{a_2b_2}\ldots = \ldots + M(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)G''^{\rho_1}_{r_1s_1}G''^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

and this when performed on opposite sides of equation (A) gives

$$M = J$$

Hence

$$L = M,$$

and we have the law of symmetry expressed by the two relations

$$c^{\pi_1}_{p_1q_1}c^{\pi_2}_{p_2q_2}\ldots = \ldots + L(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)b^{\rho_1}_{r_1s_1}b^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

$$c^{\alpha_1}_{a_1b_1}c^{\alpha_2}_{a_2b_2}\ldots = \ldots + L(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)b^{\rho_1}_{r_1s_1}b^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

viz, if in the first of these relations the partitions $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, $(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)$ be interchanged the numerical coefficient L remains unaltered

The theorem is a consequence of the two properties that have been established in regard to the three identities and the relation assumed to exist between the quantities involved in them

It appears to be the most important theorem in symmetrical algebra

33 I now pass to certain consequences which flow straight from the theorem.

It is necessary to make a few definitions.

Definition.

"A partition is separated into separates by writing down a set of partitions, each separate partition in its own brackets, so that when all the parts of these partitions are assembled in a single bracket, the partition which is separated is reproduced."

For the purpose of this portion of the memoir alone it would have been expedient to use the word bipart in lieu of the word part in the foregoing definition, but I have retained the word part for the reason that the definition remains valid whatever be the order of multiplicity of the parts

Of a partition $(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3})$ the product $(\overline{p_1q_1}\ \overline{p_3q_3})\ (\overline{p_2q_2})$ is a separation composed of the separates $(\overline{p_1q_1}\ \overline{p_3q_3})$ and $(\overline{p_2q_2})$

Definition.

"A partition *quâ* its separations is termed a separable partition"

Definition.

"If the successive biweights of the separates of a separation be

$$w_1^{(1)}w_2^{(1)},\quad w_1^{(2)}w_2^{(2)},\quad w_1^{(3)}w_2^{(3)}\quad \ldots,$$

the separation is said to have the *specification*

$$(\overline{w_1^{(1)}w_2^{(1)}}\ \overline{w_1^{(2)}w_2^{(2)}}\ \overline{w_1^{(3)}w_2^{(3)}}\ \ldots)\text{"}$$

Observe that the biweights of the separation and of its specification are necessarily the same, and identical with the biweight of the separable partition.

Observation.

The separable partition is counted as one amongst its own separations

34 To take a concrete example of these definitions, consider a separable partition $(\overline{20}\ \overline{10}\ \overline{01})$

We have

Separations	Specifications
$(\overline{20}\ \overline{10}\ \overline{01})$	$(\overline{31})$,
$(\overline{20}\ \overline{10})\ (\overline{01})$	$(\overline{30}\ \overline{01})$,
$(\overline{20}\ \overline{01})\ (\overline{10})$	$(\overline{21}\ \overline{10})$,
$(\overline{10}\ \overline{01})\ (\overline{20})$	$(\overline{11}\ \overline{20})$,
$(\overline{20})\ (\overline{10})\ (\overline{01})$	$(\overline{20}\ \overline{10}\ \overline{01})$.

35. I will discuss the law of symmetry that has been established in the light of these definitions.

I recall the relations

$$c_{10} = (\overline{10})\, b_{10},$$
$$c_{01} = (\overline{01})\, b_{01},$$
$$c_{20} = (\overline{20})\, b_{20} + (\overline{10}^2)\, b_{10}^2,$$
$$c_{11} = (\overline{11})\, b_{11} + (\overline{10}\;\overline{01})\, b_{10} b_{01},$$
$$c_{02} = (\overline{02})\, b_{02} + (\overline{01}^2)\, b_{01}^2.$$

In the expression of c_{pq} each partition in brackets ( ) has the biweight pq, and each partition is attached to a product of quantities b_{rs}, such that each factor corresponds to a single bipart in the partition.

Hence on proceeding to form a relation

$$c_{p_1q_1}^{\pi_1} c_{p_2q_2}^{\pi_2} \ldots = \ldots + \mathrm{P}\, b_{r_1s_1}^{\rho_1} b_{r_2s_2}^{\rho_2} \ldots + \ldots,$$

wherein P represents the complete symmetric function cofactor of $b_{r_1s_1}^{\rho_1} b_{r_2s_2}^{\rho_2} \ldots$, it is clear that P is a linear function of symmetric function products, each of which has a specification

$$(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2} \ldots),$$

and is also a separation of the separable partition

$$(\overline{r_1s_1}^{\rho_1}\, \overline{r_2s_2}^{\rho_2} \ldots).$$

The partitions $(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2} \ldots)$, $(\overline{r_1s_1}^{\rho_1}\, \overline{r_2s_2}^{\rho_2} \ldots)$, as well as each of the separations which present themselves in the linear function P are of the same biweight.

When the separations in the function P are all expanded into a sum of monomial symmetric functions, each of the latter has the same biweight.

Taking the separable partition $(\overline{r_1s_1}^{\rho_1}\, \overline{r_2s_2}^{\rho_2} \ldots)$ as fixed, a definite number of specifications appertain to the separations. Forming then c products in correspondence with each of these specifications the law of symmetry indicates that the *same number* of different monomial symmetric functions will appear in the developments of the several linear functions P. Further, the partitions of these monomial symmetric functions will be, in some order, identical with the several specifications of the separations of the fixed separable partition.

Assume the specifications to be, in any order

$$\theta_1, \theta_2, \ldots \theta_k,$$

and write the identity

$$c_{p_1q_1}^{\pi_1} c_{p_2q_2}^{\pi_2} \ldots = \ldots + \mathrm{P}\, b_{r_1s_1}^{\rho_1} b_{r_2s_2}^{\rho_2} \ldots + \ldots$$

in the abbreviated notation

$$c_\theta = \quad + P_\theta b_r +$$

We have then the relations —

$$c_{\theta_k} = . \quad + P_{\theta_r} b_r + \qquad (k = 1, 2, 3, \quad . k),$$

and

$$P_{\theta_1} = m_{11}\theta_1 + m_{12}\theta_2 + \quad . + m_{1l}\theta_k,$$
$$P_{\theta_2} = m_{21}\theta_1 + m_{22}\theta_2 + \quad + m_{2k}\theta_k,$$
$$P_{\theta_k} = m_{k1}\theta_1 + m_{k2}\theta_2 + \quad + m_{kk}\theta_k,$$

the quantities m being numerical

The determinant of the numbers m is symmetrical, for by the law of symmetry

$$m_{rs} = m_{sr}.$$

Hence, the coefficient of symmetric function θ_s in the development of the assemblage of separations P_{θ_r} is identical with the coefficient of symmetric function θ_r in the development of the assemblage of separations P_{θ_s}.

We may now proceed to express the symmetric functions θ as linear functions of the assemblages of separations P_θ and by elementary theory of determinants, the determinant of the system of results is symmetrical Hence

$$\theta_1 = \mu_{11}P_{\theta_1} + \mu_{12}P_{\theta_2} + \quad + \mu_{1k}P_{\theta_k},$$
$$\theta_2 = \mu_{21}P_{\theta_1} + \mu_{22}P_{\theta_2} + \quad + \mu_{2k}P_{\theta_k},$$
$$. \quad . \quad . \quad . \quad .$$
$$\theta_k = \mu_{k1}P_{\theta_1} + \mu_{k2}P_{\theta_2} + \quad + \mu_{kk}P_{\theta_k},$$

wherein

$$\mu_{rs} = \mu_{sr}.$$

36. From this are deduced two important theorems, the one a theorem of expressibility, and the other a theorem of symmetry Any one of the monomial symmetric functions θ is expressed by a partition which is a specification of a separation of the partition $(\overline{r_1s_1}^{\rho_1} \overline{r_2s_2}^{\rho_2} \ldots)$. This implies that the biparts occurring in the partition of θ can be so partitioned into biparts that when assembled together they will be identical with the biparts of the partition $(\overline{r_1s_1}^{\rho_1} \overline{r_2s_2}^{\rho_2} . \quad)$ Hence the theorem of expressibility :—

37. *Theorem.*

"The biparts of the partition of a monomial symmetric function θ are partitioned in

any manner into biparts, which, when all assembled together in a single bracket, are represented by

$$(\overline{p_1 s_1}^{\rho_1}\, \overline{p_2 s_2}^{\rho_2} \ldots)$$

The symmetric function θ is expressible as a linear function of assemblages of separations of the symmetric function $(\overline{p_1 s_1}^{\rho_1}\, \overline{p_2 s_2}^{\rho_2} \ldots)$."

38. The theorem of symmetry is as follows:—

Theorem

"When the monomial symmetric function θ_r is expressed as a linear function of the assemblages of separations $P_{\theta_1}, P_{\theta_2}, \ldots P_{\theta_k}$, the coefficient of the assemblage P_{θ_s} is the same as the coefficient of the assemblage P_{θ_r} when θ_s is so expressed."

39. This theorem enables us to form a pair of symmetrical tables in regard to every partition of every biweight. The number of tables is therefore twice the number of partitions, the generating function for which has been already given.

§ 6. *The Functions composed of One Part.*

40. I will now establish a law by means of which any symmetric function expressed by a partition with a single bipart may be at once expressed in terms of separations of any partition of its biweight. It is merely necessary to interpret a result already obtained.

I recall the formula of Art. 26,

$$(\overline{pq})_2 = (\overline{pq})\,(\overline{pq})_1,$$

which may also be written by Art. 8,

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} c_{p_1q_1}^{\pi_1} c_{p_2q_2}^{\pi_2}\ldots = (\overline{pq})\,\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2}\ldots$$

Let us compare the cofactor of $b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2}\ldots$ in the development of the left hand side with its cofactor on the right hand side.

When the left hand side is multiplied out each symmetric function product which multiplies the term $b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2}\ldots$ is necessarily a separation of the symmetric function $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$. The result of the comparison will therefore be the expression of the function $(\overline{pq})$ in terms of such separations.

41. Let S_{pq} be the value assumed by c_{pq} when b_{pq} and other quantities b are put equal to unity.

Further, let $s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)}$ denote the expression of $s_{pq} = (\overline{pq})$ by means of separations of the symmetric function $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$.

Then we may write

$$\Sigma_{\pi}(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_{\pi}(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}S_{p_1q_1}^{\pi_1}S_{p_2q_2}^{\pi_2}\ldots$$

where S_{pq} denotes the sum of all the symmetric functions of biweight pq.

Represent the different separates of the partition $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$ by $(J_1), (J_2), \ldots$ and any separation by $(J_1)^{j_1}(J_2)^{j_2}\ldots$, substitute for the quantities S_{pq} their values in terms of symmetric functions; apply the multinomial theorem and equate corresponding portions of the two sides and there results the formula

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_j(-)^{\Sigma_j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots}(J_1)^{j_1}(J_2)^{j_2}\ldots$$

where the summation is taken for every separation of the given partition.

42. This important result is a generalisation of the VANDERMONDE-WARING law for the expression of the sums of the powers of the roots of an equation in terms of the coefficients.

43. The formula may be reversed so as to exhibit any symmetric function whatever in terms of single bipart functions. The result easily reached is

$$(-)^{\Sigma\pi-1}(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$$

$$= \Sigma_j(-)^{\Sigma j-1}\frac{(\Sigma\pi_1-1)!\,(\Sigma\pi_2-1)!}{j_1!\,j_2!\ldots\;\pi_{11}!\,\pi_{12}!\ldots\;\pi_{21}!\,\pi_{22}!\ldots}\,s^{j_1}_{(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)}\,s^{j_2}_{(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)}\ldots,$$

the summation being for every separation

$$(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)^{j_1}(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)^{j_2}\ldots$$

of the symmetric function

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$$

§ 7. *Second Law of Symmetry.*

44. The operation

$$g_{pq} = \partial_{a_{pq}} + a_{10}\,\partial_{a_{p+1,q}} + a_{01}\,\partial_{a_{p,q+1}} + \ldots + a_{rs}\,\partial_{a_{p+r,q+s}} + \ldots$$

may be said to be of biweight pq, since it lowers the weight of a symmetric function by the biweight pq. Further, its degree is zero, since it does not in general lower the degree of a symmetric function. If, however, g_{pq} operates upon a symmetric function of its own biweight, it is equivalent to the simple differential operation $\partial_{a_{pq}}$, and is of degree unity.

Similarly, the operation

$$\frac{\overline{g_{p_1q_1}^{\pi_1}\, g_{p_2q_2}^{\pi_2}}}{\pi_1!\, \pi_2!}$$

will be regarded as being of weight (biweight) pq, where $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$ is a partition of the biweight pq, and if, as a particular case, the operand be of the same biweight pq, it will be equivalent to the operation

$$\frac{\partial^{\pi_1}_{a_{p_1q_1}}\,\partial^{\pi_2}_{a_{p_2q_2}}}{\pi_1!\,\pi_2!},$$

and will be of degree equal to

$$\pi_1+\pi_2+\ldots=\Sigma\pi$$

Since therefore

$$\frac{\partial^{\pi_1}_{a_{p_1q_1}}\,\partial^{\pi_2}_{a_{p_2q_2}}}{\pi_1!\,\pi_2!}\ldots a^{\pi_1}_{p_1q_1}\,a^{\pi_2}_{p_2q_2}\ldots=1,$$

we have the result

$$\frac{\overline{g_{p_1q_1}^{\pi_1}\, g_{p_2q_2}^{\pi_2}\ldots}}{\pi_1!\,\pi_2!}\ldots a^{\pi_1}_{p_1q_1}\,a^{\pi_2}_{p_2q_2}\ldots=1,$$

assuming then a result

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)_2=\ldots+\mathrm{P}b^{\rho_1}_{r_1s_1}\,b^{\rho_2}_{r_2s_2}\ldots+\ldots$$

derived from the three initial identities of Art. 24 and the relation assumed to exist between the quantities involved, we are at once led to the operator relation

$$\frac{\overline{g'^{\pi_1}_{p_1q_1}\,g'^{\pi_2}_{p_2q_2}\ldots}}{\pi_1!\,\pi_2!\ldots}=\ldots+\mathrm{PG}''^{\rho_1}_{r_1s_1}\,\mathrm{G}''^{\rho_2}_{r_2s_2}\ldots+\ldots$$

where P consists entirely of symmetric functions of quantities which occur in the first identity. Further suppose a second result

$$(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)_2=\ldots+\mathrm{Q}b^{\pi_1}_{p_1q_1}\,b^{\pi_2}_{p_2q_2}\ldots+\ldots$$

Hence, operating on the left and right of this result with right and left sides of the foregoing operator relation, we obtain

$$(\ldots+\mathrm{PG}''^{\rho_1}_{r_1s_1}\,\mathrm{G}''^{\rho_2}_{r_2s_2}\ldots+\ldots)\,(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$$
$$=\frac{\overline{g'^{\pi_1}_{p_1q_1}\,g'^{\pi_2}_{p_2q_2}}}{\pi_1!\,\pi_2!\ldots}(\ldots+\mathrm{Q}b^{\pi_1}_{p_1q_1}\,b^{\pi_2}_{p_2q_2}\ldots+\ldots),$$

or from theorems established above (Arts. 13, 44)

$$\mathrm{P}=\mathrm{Q},$$

no other terms surviving the operation.

45. Hence a theorem of symmetry :—

Theorem.

"If

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\quad)_2 = \qquad + \mathrm{P}b_{r_1s_1}^{\rho_1}\,b_{r_2s_2}^{\rho_2}\qquad + \qquad,$$

the cofactor symmetric function P is unaltered when the partitions $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\quad)$, $(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\quad)$, are interchanged."

The function P presents itself in the first place as a linear function of separations of the partition of the b product to which it is attached The theorem supplies linear functions of separations of any two partitions $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\quad)$, $(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\quad)$ respectively, of the same biweight, which are equal to one another

46 To make the matter clear, form a table of biweight 21 as follows —

	b_{21}	$b_{20}b_{01}$	$b_{11}b_{10}$	$b_{10}^2b_{01}$
$(\overline{21})_2$	$(\overline{21})$	$(\overline{20}\,\overline{01})$ $-(\overline{20})(\overline{01})$	$(\overline{11}\,\overline{10})$ $-(\overline{11})(\overline{10})$	$(\overline{10}^2\,\overline{01})$ $-(\overline{10}^2)(\overline{01})$ $-(\overline{10}\,\overline{01})(\overline{10})$ $+(\overline{10})^2(\overline{01})$
$(\overline{20}\,\overline{01})_2$	$-(\overline{21})$	$-(\overline{20}\,\overline{01})$ $-(\overline{20})(\overline{01})$	$-(\overline{11}\,\overline{10})$ $+(\overline{11})(\overline{10})$	$-(\overline{10}^2\,\overline{01})$ $-(\overline{10}^2)(\overline{01})$ $+(\overline{10}\,\overline{01})(\overline{10})$
$(\overline{11}\,\overline{10})_2$	$-(\overline{21})$	$-(\overline{20}\,\overline{01})$ $+(\overline{20})(\overline{01})$	$-(\overline{11}\,\overline{10})$	$-(\overline{10}^2\,\overline{01})$ $+(\overline{10}^2)(\overline{01})$
$(\overline{10}^2\,\overline{01})_2$	$(\overline{21})$	$(\overline{20}\,\overline{01})$	$(\overline{11}\,\overline{10})$	$(\overline{10}^2\,\overline{01})$

which is to be read by rows *

Each term in a column is a separation of the partition of the b product at the head of the column

The separations in each line of terms as written possess the same specifications, and also the same numerical coefficients In the right hand column the partition separated is a fundamental symmetric function, and hence each separate therein appearing is so also. Each block of separations in the right hand column is the expression by means of fundamental symmetric functions of the monomial symmetric function of the same elements whose partition appears to the left of the same line The terms of the first three columns may be regarded as being formed according to the same *law* as the right hand column, and therefore according to a law defined by

* Each term in the left hand column is equal to the aggregate of terms in *any* block in the same row

the monomial function at the left of the same line For example, the terms in the second column and third line are separations of $(\overline{20}\,\overline{01})$ formed according to the law of the function $(\overline{11}\,\overline{10})$ Also the terms in the third column and second line are separations of $(\overline{11}\,\overline{10})$ formed according to the law of the function $(\overline{20}\,\overline{01})$ Now observe that the law of symmetry establishes that the table enjoys row and column symmetry Hence the assemblage of separations of $(\overline{20}\,\overline{01})$ formed according to the law of $(\overline{11}\,\overline{10})$ is equal to the assemblage of separations of $(\overline{11}\,\overline{10})$ formed according to the law of $(\overline{20}\,\overline{01})$

47 Hence in general the theorem.—

"The assemblage of separations of $(\overline{r_1 s_1}^{\rho_1}\,\overline{r_2 s_2}^{\rho_2} \quad)$, formed according to the law of $(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2} \quad)$, is equal to the assemblage of separations of $(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2} \quad)$, formed according to the law of $(\overline{r_1 s_1}^{\rho_1}\,\overline{r_2 s_2}^{\rho_2} \quad)$"

In the particular case considered the equality is

$$-(\overline{20}\,\overline{01}) + (\overline{20})(\overline{01}) = -(\overline{11}\,\overline{10}) + (\overline{11})(\overline{10})$$

To actually form separations of $(\overline{r_1 s_1}^{\rho_1}\,\overline{r_2 s_2}^{\rho_2} \quad)$, according to the law of $(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2} . \quad)$, the separations of the former must be written down, and also the expression of the latter, by means of fundamental symmetric functions. The separations are then given the same coefficients as the products of fundamental symmetric functions which possess the same specifications

§ 8. *Third Law of Symmetry*

48. From the relation

$$c_{p_1 q_1}^{\pi_1}\, c_{p_2 q_2}^{\pi_2} \quad . = . . \; + \mathrm{L} b_{r_1 s_1}^{\rho_1}\, b_{r_2 s_2}^{\rho_2} . \; . +$$

is derived the operator relation

$$\mathrm{G}'^{\pi_1}_{p_1 q_1}\, \mathrm{G}'^{\pi_2}_{p_2 q_2} \quad = \; . \; + \mathrm{L}\mathrm{G}''^{\rho_1}_{r_1 s_1}\, \mathrm{G}''^{\rho_2}_{r_2 s_2} \quad + .$$

and, thence, by the method already employed,

$$(\overline{r_1 s_1}^{\rho_1}\,\overline{r_2 s_2}^{\rho_2} . \quad)_2 = . \quad + \mathrm{L}\,(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2} \; . .)_1 + \quad .$$

This law of symmetry is of considerable importance and interest, but I do not stop to further discuss it.*

* *Vide* 'American Journal of Mathematics' "Third Memoir on a New Theory of Symmetric Functions," now in progress in vols 11, 12, and succeeding volumes.

§ 9. *The linear Partial Differential Operations of the Theory of Separations.*

49 For purposes of calculation it is necessary to adapt the operations

$$g_{10},\ g_{01},\ \ldots\ g_{pq}\ \ldots,$$

so that they may be performed on a symmetric function when the latter is expressed in terms of the separations of any given partition

Of any partition $(\overline{p_1q_1}^{\pi_1}\overline{p_2q_2}^{\pi_2}\ \ldots)$ separates (*vide* Definitions) are formed by taking all possible combinations of the parts These are precisely

$$(\pi_1 + 1)(\pi_2 + 1)\ \ldots\ -1,$$

distinct separates which must be regarded as independent variables

Put

$$(\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ \ldots\ \overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ \ldots)$$

for any separate of a given separable partition P

Then by a known theorem

$$g_{pq} = \Sigma g_{pq}(\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ \ldots\ \overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ \ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ \ldots\ \overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ \ldots)},$$

the summation being in regard to all the separates

Moreover (Art 17)

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq} = \Sigma\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ \ldots\ \pi_{p_1q_1}!\ \ldots}\mathrm{G}_{10}^{\pi_{10}}\mathrm{G}_{01}^{\pi_{01}}\ \ldots\ \mathrm{G}_{p_1q_1}^{\pi_{p_1q_1}}\ \ldots,$$

the summation being in regard to all the partitions $(\overline{10}^{\pi_{10}}\overline{01}^{\pi_{01}}\ \ldots\ \overline{p_1q_1}^{\pi_{p_1q_1}}\ \ldots)$ of the biweight pq; and also (Art 13)

$$\mathrm{G}_{10}^{\pi_{10}}\mathrm{G}_{01}^{\pi_{01}}\ldots\ \mathrm{G}_{p_1q_1}^{\pi_{p_1q_1}}\ldots\ (\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ \ldots\ \overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ \ldots) = (\overline{10}^{\rho_{10}}\overline{01}^{\rho_{01}}\ldots\ \overline{p_1q_1}^{\rho_{p_1q_1}}\ldots).$$

50. Hence

$$\begin{aligned}(-)^{p+q-1}&\frac{(p+q-1)!}{p!\,q!\ldots}g_{pq}\\ &= \Sigma_\pi\Sigma_\rho\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\ \pi_{p_1q_1}!\ldots}(\overline{10}^{\rho_{10}}\overline{01}^{\rho_{01}}\ \ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{01}}\overline{01}^{\pi_{01}+\rho_{01}}\ \ldots\ \overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ \ldots)}\end{aligned}$$

the summation being in regard to

(1) Every separate of the given separable partition;

(2) Every partition of the biweight pq

The right hand side of this relation may be broken up into fragments in each of which all the numbers $\pi_{10}, \pi_{01}, \ldots \pi_{p_1q_1}$ are constant

In fact we may write

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq}$$

$$=\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!\ldots}\Sigma_\rho\,(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}-\rho_{10}}\,\overline{01}^{\pi_{01}-\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}-\rho_{p_1q_1}}\ldots)},$$

wherein, following the summation sign Σ_ρ, the numbers $\pi_{10}, \pi_{01}, \ldots \pi_{p_1q_1}$ are constant, and the operator

$$\Sigma_\rho\,(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)}$$

is one of the fragments above mentioned

This operation has a biweight pq, and may be defined also in regard to the partition $(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)$ of the biweight pq

51 Write then for brevity and convenience

$$\Sigma_\rho\,(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)}=g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)},$$

so that we may write

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq}=\Sigma_\pi\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!}g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)},$$

where the summation is in regard to every partition $(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)$ of the biweight pq.

52 In general not every partition of the biweight pq will occur in the given separable partition, but it is convenient to consider the general result just written down as including every such partition It will be seen later that this result is of great importance in the theory

I remark that on the left-hand side we have a linear partial differential operation g_{pq} whose expression by means of the fundamental symmetric functions and their differential inverses is well known by what has preceded. Such expression is all that is needed so long as we are concerned only with the fundamental forms which, as they appear in the expression of a monomial symmetric function of biweight pq, present themselves in products which are separations of the symmetric function $(\overline{10}^p\,\overline{01}^q)$. In the present broader theory in which the leading idea is the consideration of any partition at pleasure of the biweight as the separable partition, we bring into view the exhibition of the operation g_{pq} as a linear function of operations, each of which is in correspondence with a partition of the biweight

We have, in fact, a biweight operator g_{pq} decomposed into a full number of bipart partition operators

$$g_{(\overline{10}^{\pi_{10}}\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}$$

Observe that the whole theory of separations is a generalisation from a weight to a partition of a weight. Here we have generalised from a weight operation to a partition operation, and I henceforward regard the partition operator as the essential linear partial differential operator of the theory. The biweight operator g_{pq} has been expressed *ante* (Art. 17) in terms of the obliterating operators of the form G_{pq}. These operations G_{pq} are equally available in the theory of the separable partition in general. The mode of their operation upon a symmetric function product will be subsequently explained (in § 11). So far I have merely considered their operation upon monomial forms.*

53. I observe that the biweight operator g_{pq} is expressed as a linear function of the partition operators of the same biweight, according to the same laws as—

(1) The operator g_{pq} is expressed in terms of the operations G_{pq} (Art. 17).

(2) The symmetric functions, containing one part only, are expressed in terms of the fundamental or single-unitary forms (Art. 8).

E.g., compare the three results (the first slightly modified) —

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}g_{(\overline{p_1q_1}^{\pi_1}\overline{p_2q_2}^{\pi_2}\ldots)},$$

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}G_{p_1q_1}^{\pi_1}G_{p_2q_2}^{\pi_2}\ldots,$$

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}(\overline{pq}) = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}a_{p_1q_1}^{\pi_1}a_{p_2q_2}^{\pi_2}\ldots$$

54. For convenience of reference, I write down the particular simplest cases of the decomposition.

$$\begin{aligned}
g_{10} &= g_{(\overline{10})},\\
g_{01} &= g_{(\overline{01})},\\
g_{20} &= g_{(\overline{10}^2)} - 2g_{(\overline{20})},\\
g_{11} &= g_{(\overline{10}\,\overline{01})} - g_{(\overline{11})},\\
g_{02} &= g_{(\overline{01}^2)} - 2g_{(\overline{02})},\\
g_{30} &= g_{(\overline{10}^3)} - 3g_{(\overline{20}\,\overline{10})} + 3g_{(\overline{30})},\\
g_{21} &= g_{(\overline{10}^2\,\overline{01})} - g_{(\overline{20}\,\overline{10})} - g_{(\overline{11}\,\overline{10})} + g_{(\overline{21})},\\
g_{12} &= g_{(\overline{01}^2\,\overline{10})} - g_{(\overline{02}\,\overline{10})} - g_{(\overline{11}\,\overline{01})} + g_{(\overline{12})},\\
g_{03} &= g_{(\overline{01}^3)} - 3g_{(\overline{02}\,\overline{01})} + 3g_{(\overline{03})}.
\end{aligned}$$

* The decomposition of the obliterating operation G_{pq} into partition obliterating operations is given *post* § 10.

55 I also give the developed expressions of a few of the partition operators. Thus

$$g_{(\overline{10})} = \partial_{(\overline{10})} + (\overline{10})\,\partial_{(\overline{10}^2)} + (\overline{01})\,\partial_{(\overline{10}\,\overline{01})} + (\overline{20})\,\partial_{(\overline{20}\,\overline{10})} + (\overline{10}^2)\,\partial_{(\overline{10}^3)}$$
$$+ (\overline{11})\,\partial_{(\overline{11}\,\overline{10})} + (\overline{10}\,\overline{01})\,\partial_{(\overline{10}^2\,\overline{01})} + (\overline{02})\,\partial_{(\overline{10}\,\overline{02})} + (\overline{01}^2)\,\partial_{(\overline{10}\,\overline{01}^2)}$$
$$+ \ldots ,$$

$$g_{(\overline{20}\,\overline{01})} = \partial_{(\overline{20}\,\overline{01})} + (\overline{10})\,\partial_{(\overline{20}\,\overline{10}\,\overline{01})} + (\overline{01})\,\partial_{(\overline{20}\,\overline{01}^2)} + (\overline{20})\,\partial_{(\overline{20}^2\,\overline{01})}$$
$$+ (\overline{10}^2)\,\partial_{(\overline{20}\,\overline{10}^2\,\overline{01})} + (\overline{11})\,\partial_{(\overline{20}\,\overline{11}\,\overline{01})} + (\overline{10}\,\overline{01})\,\partial_{(\overline{20}\,\overline{10}\,\overline{01}^2)}$$
$$+ (\overline{02})\,\partial_{(\overline{20}\,\overline{01}\,\overline{02})} + (\overline{01}^2)\,\partial_{(\overline{20}\,\overline{01}^3)} + \ldots$$

The mode of operation of the biweight operators in the separation theory is now manifest

56 Let

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)},\quad g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)},$$

be any two partition operators of the same or different biweights. Representing them for brevity by

$$g_{(\pi)},\quad g_{(\rho)},$$

we have

$$g_{(\pi)}g_{(\rho)} = \overline{g_{(\pi)}g_{(\rho)}} + g_{(\pi)} \dagger g_{(\rho)},$$

wherein the multiplication on the left denotes successive operation, the bar on the right denotes symbolical multiplication, and the symbol $\dagger$ denotes explicit differentiation on the operand regarded as a function of symbols of quantity only

It is easy to establish the result

$$g_{(\pi)} \dagger g_{(\rho)} = g_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)}$$
$$= g_{(\pi+\rho)} \text{ for brevity.}$$

Hence

$$g_{(\pi)} \dagger g_{(\rho)} = g_{(\rho)} \dagger g_{(\pi)} = g_{(\pi+\rho)},$$

and

$$g_{(\pi)}g_{(\rho)} = \overline{g_{(\pi)}g_{(\rho)}} + g_{(\pi+\rho)},$$

or at full length

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}\,g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)} = \overline{g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}\,g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)}}$$
$$+ g_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)},$$

the fundamental law of multiplication (compare Art. 15).

Also

$$g_{(\pi)} g_{(\rho)} - g_{(\rho)} g_{(\pi)} = 0,$$

or

$$g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1q_1}^{\pi_{p_1q_1}} \ldots)} g_{(\overline{10}^{\rho_{10}} \overline{01}^{\rho_{01}} \ldots \overline{p_1q_1}^{\rho_{p_1q_1}} \ldots)} - g_{(\overline{10}^{\rho_{10}} \overline{01}^{\rho_{01}} \ldots \overline{p_1q_1}^{\rho_{p_1q_1}} \ldots)} g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1q_1}^{\pi_{p_1q_1}} \ldots)} = 0,$$

shewing that any two partition operators are commutative

57 The left hand side of the result just reached is called by SOPHUS LIE* the "Zusammensetzung" or "Combination" of the operators which appear SYLVESTER† has also called it the 'Alternant' of the two operators

The whole system of partition operators forms an infinite group in co-relation with an infinite group of transformations—

We can state the theorem —

Theorem

"The Combination or Alternant of any two partition operators vanishes"

Considering the partial differential equation

$$g_{(\pi)} = 0,$$

and ϕ any function which is a solution, then must $g_{(\rho)}\phi$ be also a solution, since

$$g_{(\pi)} g_{(\rho)} \phi - g_{(\rho)} g_{(\pi)} \phi = 0$$

Theorem

"If $g_{(\pi)}$ and $g_{(\rho)}$ be any two partition operators, and ϕ a solution of the equation $g_{(\pi)} = 0$; then will $g_{(\rho)}\phi$ be also a solution of the same equation"

58. Consider now the partial differential equation of Art 51,

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} g_{pq} = \Sigma_{-} \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!\ldots} g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1q_1}^{\pi_{p_1q_1}} \ldots)} = 0$$

Assume the separable partition to be

$$(\overline{10}^{\sigma_{10}} \overline{01}^{\sigma_{01}} \ldots \overline{p_1q_1}^{\sigma_{p_1q_1}} \ldots),$$

so that the operand is a linear function of separations of this partition.

The effect of the partition operator

$$g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1q_1}^{\pi_{p_1q_1}} \ldots)},$$

is the production of terms each of which is a separation of the partition

$$(\overline{10}^{\sigma_{10}-\pi_{10}} \overline{01}^{\sigma_{01}-\pi_{01}} \ldots \overline{p_1q_1}^{\sigma_{p_1q_1}-\pi_{p_1q_1}} \ldots).$$

* 'Theorie der Transformationsgruppen,' Leipzig, 1888.

† 'Lectures on the Theory of Reciprocants.' ('American Journal of Mathematics,' and elsewhere)

Observe that separations of this partition cannot be produced by any *other* of the partition operators which present themselves on the left hand side of the differential equation Hence if the operand satisfies the differential equation

$$g_{pq} = 0,$$

it must also satisfy the differential equation

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\cdots\overline{p_s q_s}^{\pi_{p_s q_s}}\cdots)} = 0$$

59 This important theorem may be enunciated as follows —

Theorem

"If a function, expressed in terms of separations of a given monomial symmetric function, be annihilated by a biweight operator it must also be annihilated by every partition operator of that biweight"

As regards the calculation of Tables of Separations of Symmetric Functions, this is the cardinal theorem

As an example of its application I propose to utilise it for the purpose of exhibiting the function $(\overline{31}\,\overline{01})$ as a linear function of separations of $(\overline{21}\,\overline{10}\,\overline{01})$ The law of expressibility shows this to be possible, for $(\overline{21}\,\overline{10})$ is a partition of the biweight 31 Remarking that the separation $(\overline{21})\,(\overline{10})\,(\overline{01})$ cannot occur in the expression, since it is the only separation which produces the monomial $(\overline{32})$ when multiplied out, I put

$$(\overline{31}\,\overline{01}) = \mathrm{A}\,(\overline{21}\,\overline{10})\,(\overline{01}) + \mathrm{B}\,(\overline{21}\,\overline{01})\,(\overline{10}) + \mathrm{C}\,(\overline{10}\,\overline{01})\,(\overline{21}) + \mathrm{D}\,(\overline{21}\,\overline{10}\,\overline{01}).$$

A monomial symmetric function is caused to vanish by means of the operation of the biweight operator g_{pq} if no partition of the biweight pq is comprised amongst its parts In consequence of this, the only biweight operators which do not cause it to vanish are g_{31}, g_{01}, and g_{32} Hence all the partition operators of every other biweight operator annihilate the function $(\overline{31}\,\overline{01})$ It suffices to employ as annihilators the two partition operators $g_{(\overline{01})}$ and $g_{(\overline{21})}$

Hence, retaining only significant terms,

$$\{\partial_{(\overline{10})} + (\overline{01})\,\partial_{(\overline{10}\,\overline{01})} + (\overline{21})\,\partial_{(\overline{21}\,\overline{10})} + (\overline{21}\,\overline{01})\,\partial_{(\overline{21}\,\overline{10}\,\overline{01})}\}\,(\overline{31}\,\overline{01}) = 0,$$

$$\{\partial_{(\overline{21})} + (\overline{10})\,\partial_{(\overline{21}\,\overline{10})} + (\overline{01})\,\partial_{(\overline{21}\,\overline{01})} + (\overline{10}\,\overline{01})\,\partial_{(\overline{21}\,\overline{10}\,\overline{01})}\}\,(\overline{31}\,\overline{01}) = 0,$$

leading to

$$\mathrm{A} + \mathrm{C} = 0, \quad \mathrm{B} + \mathrm{D} = 0,$$
$$\mathrm{C} + \mathrm{D} = 0, \quad \mathrm{A} + \mathrm{B} = 0,$$

or

$$\mathrm{D} = -\,\mathrm{C} = -\,\mathrm{B} = \mathrm{A}.$$

Hence

$$(\overline{31}\,\overline{01}) = \mathrm{A}\,\{(\overline{21}\,\overline{10})\,(\overline{01}) - (\overline{21}\,\overline{01})\,(\overline{10}) - (\overline{10}\,\overline{01})\,(\overline{21}) + (\overline{21}\,\overline{10}\,\overline{01})\}\,,$$

and it is easy to see that $\mathrm{A} = -\frac{1}{2}$, for each of the products $(\overline{21}\,\overline{01})\,(\overline{10})$, $(\overline{10}\,\overline{01})\,(\overline{21})$ on multiplication produces a term $+(\overline{31}\,\overline{01})$, and this monomial is not produced by the development of either of the other two products The value of A may, however, be instructively obtained by means of the operator g_{01}

For

$$g_{01}\,(\overline{31}\,\overline{01}) = (\overline{31}),$$

and

$$g_{01} = g_{(\overline{01})} = \partial_{(\overline{01})} + (\overline{10})\,\partial_{(\overline{10}\,\overline{01})} + (\overline{21})\,\partial_{(\overline{21}\,\overline{01})} + (\overline{21}\,\overline{10})\,\partial_{(\overline{21}\,\overline{10}\,\overline{01})} \quad \text{(Art 53)}$$

Hence

$$(\overline{31}) = 2\mathrm{A}\,\{-(\overline{21})\,(\overline{10}) + (\overline{21}\,\overline{10})\},$$

and now A is obviously equal to $-\frac{1}{2}$

But we may further employ the operator g_{31}

For

$$g_{31} = -\mathrm{G}_{31} + \quad = + g_{(\overline{21}\,\overline{10})} + \quad . \quad \text{(Arts. 17, 53)},$$

significant terms only being retained, hence $-\mathrm{G}_{31}$ and $g_{(\overline{21}\,\overline{10})} \equiv \partial_{(\overline{21}\,\overline{10})}$ are equivalent operations in the present case, and performing them on their own sides $-1 = 2\mathrm{A}$ or $\mathrm{A} = -\frac{1}{2}$

Thus

$$(\overline{31}\,\overline{01}) = -\tfrac{1}{2}\,(\overline{21}\,\overline{10})(\overline{01}) + \tfrac{1}{2}\,(\overline{21}\,\overline{01})(\overline{10}) + \tfrac{1}{2}\,(\overline{10}\,\overline{01})(\overline{21}) - \tfrac{1}{2}\,(\overline{21}\,\overline{10}\,\overline{01})$$

§ 10. *The Partition obliterating Operators.*

60. In the foregoing a generalisation has been made from a number to the partition of a number in the case of the operations $g_{10}, g_{01}, \ldots g_{pq}, \quad .$ The possibility of the like generalisation in respect of the obliterating operators $\mathrm{G}_{10}, \mathrm{G}_{01}, . \quad \mathrm{G}_{pq}, \ldots$ is naturally presented as a subject for enquiry.

Consider a symmetric function

$$f\,(\alpha_{10}, \alpha_{01}, \ldots \alpha_{pq}, \ldots) = f$$

to be the product of m monomial functions, and write

$$f = f_1 f_2 \ldots f_m$$

Supposing a_{pq} changed into $a_{pq} + \mu a_{p-1,q} + \nu a_{p,q-1}$, we have from previous work

$$
\begin{aligned}
&(1 + \mu G_{10} + \nu G_{01} + \dots + \mu^p \nu^q G_{pq} + \dots) f \\
&= (1 + \mu G_{10} + \nu G_{01} + \dots + \mu^p \nu^q G_{pq} + \dots) f_1 \\
&\quad \times (1 + \mu G_{10} + \nu G_{01} + \dots + \mu^p \nu^q G_{pq} + \dots) f_2 \\
&\quad \times \dots \\
&\quad \times (1 + \mu G_{10} + \nu G_{01} + \dots + \mu^p \nu^q G_{pq} + \dots) f_m
\end{aligned}
$$

Expanding the right hand side and equating coefficients of like products of powers μ and ν, we get

$$
\begin{aligned}
G_{10} f &= \Sigma (G_{10} f_1) f_2 f_3 \dots f_m, \\
G_{20} f &= \Sigma (G_{10} f_1)(G_{10} f_2) f_3 \dots f_m + \Sigma (G_{20} f_1) f_2 f_3 \dots f_m, \\
G_{11} f &= \Sigma (G_{10} f_1)(G_{01} f_2) f_3 \dots f_m + \Sigma (G_{11} f_1) f_2 f_3 \dots f_m, \\
G_{30} f &= \Sigma (G_{10} f_1)(G_{10} f_2)(G_{10} f_3) f_4 \dots f_m + \Sigma (G_{20} f_1)(G_{10} f_2) f_3 \dots f_m + \Sigma (G_{30} f_1) f_2 \dots f_m, \\
G_{21} f &= \Sigma (G_{10} f_1)(G_{10} f_2)(G_{01} f_3) f_4 \dots f_m + \Sigma (G_{11} f_1)(G_{10} f_2) f_3 \dots f_m \\
&\quad + \Sigma (G_{20} f_1)(G_{01} f_2) f_3 \dots f_m + \Sigma (G_{21} f_1) f_2 f_3 \dots f_m,
\end{aligned}
$$

and so forth, where the summations are, in regard to the different terms obtained by permutation of the m suffixes of the functions $f_1, f_2, \dots f_m$.

In general in the expression for $G_{pq} f$ there will occur a summation corresponding to each partition of the biweight pq. If a partition be $(\overline{p_1 q_1}\ \overline{p_2 q_2} \dots \overline{p_s q_s})$ the summation is

$$\Sigma (G_{p_1 q_1} f_1)(G_{p_2 q_2} f_2) \dots (G_{p_s q_s} f_s) f_{s+1} \dots f_m$$

61. Thus, when performed upon a product of functions, the operator G_{pq} breaks up into as many distinct operations as the biweight pq possesses partitions. It is convenient to denote the operation indicated by the summation

$$\Sigma (G_{p_1 q_1} f_1)(G_{p_2 q_2} f_2) \dots (G_{p_s q_s}) f_s f_{s+1} \dots f_m,$$

by

$$G_{(\overline{p_1 q_1}\ \overline{p_2 q_2} \dots \overline{p_s q_s})},$$

and to speak of it as a partition operator.

62. We may now write down an equivalence

$$G_{pq} = \Sigma G_{(\overline{p_1 q_1}\ \overline{p_2 q_2} \dots \overline{p_s q_s})},$$

where the summation is in regard to every partition of the biweight pq. This is, in fact, a theorem for operating with G_{pq} upon a product of symmetric functions, and it is consistent with the more simple law previously established.

In particular

$$
\begin{aligned}
G_{10} &= G_{(\overline{10})}, \\
G_{01} &= G_{(\overline{01})}, \\
G_{20} &= G_{(\overline{10}^2)} + G_{(\overline{20})}, \\
G_{11} &= G_{(\overline{10}\,\overline{01})} + G_{(\overline{11})}, \\
G_{02} &= G_{(\overline{01}^2)} + G_{(\overline{02})}
\end{aligned}
$$

63. The relations between the partition g operators and the partition G operators are of great interest

Recalling the equivalence (Arts 53 and 41)

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, G_{p_1q_1}^{\pi_1} G_{p_2q_2}^{\pi_2}\ldots,$$

which should be compared with the algebraical result

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, S_{p_1q_1}^{\pi_1} S_{p_2q_2}^{\pi_2}\ldots,$$

there arise the relations :—

$$
\begin{aligned}
& g_{(\overline{10})} = G_{10} = G_{(\overline{10})}, \\
& g_{(\overline{01})} = G_{01} = G_{(\overline{01})}, \\
& g_{(\overline{10}^2)} - 2g_{(\overline{20})} = G_{10}^2 - 2G_{20} = G_{(\overline{10})}^2 - 2G_{(\overline{10}^2)} - 2G_{(\overline{20})}, \\
& g_{(\overline{10}\,\overline{01})} - g_{(\overline{11})} = G_{10}G_{01} - G_{11} = G_{(\overline{10})}G_{(\overline{01})} - G_{(\overline{10}\,\overline{01})} - G_{(\overline{11})}, \\
& g_{(\overline{10}^3)} - 3g_{(\overline{20}\,\overline{10})} + 3g_{(\overline{30})} = G_{10}^3 - 3G_{20}G_{10} + 3G_{30}, \\
& \quad = G_{(\overline{10})}^3 - 3G_{(\overline{10}^2)}G_{(\overline{10})} + 3G_{(\overline{10}^3)} - 3\left\{G_{(\overline{20})}G_{(\overline{10})} - G_{(\overline{20}\,\overline{10})}\right\} + 3G_{(\overline{30})},
\end{aligned}
$$

and so forth.

64 Now consider the relation last written.

I say that it may be broken up into three relations, viz.:—

$$
\begin{aligned}
g_{(\overline{10}^3)} &= G_{(\overline{10})}^3 - 3G_{(\overline{10}^2)}G_{(\overline{10})} + 3G_{(\overline{10}^3)}, \\
g_{(\overline{20}\,\overline{10})} &= G_{(\overline{20})}G_{(\overline{10})} - G_{(\overline{20}\,\overline{10})}, \\
g_{(\overline{30})} &= G_{(\overline{30})}\,;
\end{aligned}
$$

for suppose an operand to be composed of separations of a separable partition $(\overline{10}^{\pi_{10}}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}}\ldots)$, the performance of the operations on the two sides of the relation

produces the same result identically This result is composed of three portions containing separations of the partitions

$$(\overline{10}^{\pi_{10}-3}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}}\ \ldots)\qquad(\overline{10}^{\pi_{10}-1}\,\overline{20}^{\pi_{20}-1}\,\overline{30}^{\pi_{30}}\ \ldots)\qquad(\overline{10}^{\pi_{10}}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}-1}\ \ldots)$$

respectively Hence the operations which produce the three identically equal portions of the result must be equivalent, and the three relations between the operators therefore follow

In general, we may say that of the biweight pq there are as many relations between partition operations as there are partitions of the biweight pq.

65 The general law of the coefficients will be now investigated

In the result of Art 53 viz,

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, G_{p_1q_1}^{\pi_1}\, G_{p_2q_2}^{\pi_2}\ldots$$

we have to substitute for $G_{p_1q_1}$, $G_{p_2q_2}\ldots$, the sums of the partition G operators of weights p_1q_1, $p_2q_2\ldots$, respectively, we have then to collect on the right all the G products which are associated with separations of one and the same partition and to equate them to the corresponding g operator on the left It is evident that this process does not alter the *law* of the coefficients, and that representing the different separates of the given partition by (J_1), $(J_2)\ldots$, and any separation by $(J_1)^{j_1}(J_2)^{j_2}\ldots$, we may write

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_j (-)^{\Sigma j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots}\, G_{(J_1)}^{j_1}\, G_{(J_2)}^{j_2}\ldots$$

Observe that this is precisely the law which gives the expression of a single bipart function in terms of separations of the partition $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$ I recall the result of Art 41, viz,

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\, s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_j (-)^{\Sigma j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots}\,(J_1)^{j_1}(J_2)^{j_2}\ldots,$$

which renders the correspondence between the algebraic and differential theories very striking.

66 Reversing the formula as in the algebraic theory we get the important formula —

$$(-)^{\Sigma\pi-1}G_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)}$$
$$= \Sigma_j (-)^{\Sigma j-1}\frac{(\Sigma\pi_1-1)!\,(\Sigma\pi_2-1)!\ldots}{j_1!\,j_2!\ldots\,\pi_{11}!\,\pi_{12}!\ldots\,\pi_{21}!\,\pi_{22}!\ldots}\, g^{j_1}_{(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)}\, g^{j_2}_{(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)}\ldots;$$

the summation being for every separation

$$(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)^{j_1}(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)^{j_2}\ldots$$

of the partition $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$ (Compare Art 43)

Observe that in these formulæ the multiplications of operations are non-symbolic and denote successive operations

67 Remark the results of operations —

$$G_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)}\, s_{p_1q_1}^{\pi_1} s_{p_2q_2}^{\pi_2}\ldots = 1,$$

$$\frac{1}{j_1!}\frac{1}{j_2!}\ldots\; g^{j_1}_{(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)}\, g^{j_2}_{(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)}\ldots$$
$$\cdot\,(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)^{j_1}(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)^{j_2}\ldots = 1$$

§ 11 *The Multiplication of Symmetric Functions*

68. The partition G operators are of great service in multiplication An example will make this clear It is required to find the coefficient of $(\overline{11}^2)$ in the product $(\overline{10}^2)(\overline{01})^2$.

Put

$$(\overline{10}^2)(\overline{01})^2 = \ldots + A\,(\overline{11}^2) + \ldots$$

On operating with G_{11}^2 on the right the result is A, since every other term is annihilated , and since

$$G_{11} = G_{(\overline{11})} + G_{(\overline{10}\,\overline{01})},$$

we have

$$G_{11}^2\,(\overline{10}^2)(\overline{01})^2 = \{G_{(\overline{11})} + G_{(\overline{10}\,\overline{01})}\}^2(\overline{10}^2)(\overline{01})^2 = A,$$

therefore

$$\{G_{(\overline{11})} + G_{(\overline{10}\,\overline{01})}\}\,.\,2\,(\overline{10})(\overline{01}) = A,$$

hence

$$A = 2.$$

Similarly putting

$$(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)(\ldots) = \ldots + A\,(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots) + \ldots$$

we have merely to operate on the left-hand side with the partition operators equivalent to $G_{r_1s_1}^{\rho_1} G_{r_2s_2}^{\rho_2}\ldots$ in order to find A.

§ 12 *Symmetric Functions of Differences*

69 In the unipartite theory there is a transformation which connects the Symmetric Functions of the Differences of the roots of the equation

$$x^n - na_1x^{n-1} + n(n-1)a_2x^{n-2} - \quad + (-)^n n!\, a_n = 0$$

with the non-unitary symmetric functions of the roots of the equation

$$x^n - a_1x^{n-1} + a_2x^{n-2} - \quad + (-)^n a_n = 0. \qquad (n = \infty)$$

In fact, the annihilating operator is in each case found to be

$$g_1 = \partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \quad ..$$

The theory of the Invariants and Covariants of a Binary quantic may be thus brought to depend upon non-unitary symmetric functions (*Vide* 'American Journal of Mathematics,' vol 6, p 131)

In the present case, there is also a transformation. For the purpose in hand, write the fundamental identity in the form

$$(1 + \alpha_1x + \beta_1y)(1 + \alpha_2x + \beta_2y) \quad (1 + \alpha_nx + \beta_ny)$$
$$= 1 + na_{10}x + na_{01}y + \quad + \frac{n!}{(n-p-q)!} a_{pq}x^py^q +$$

Any function of the differences of the quantities on the left remains unaltered, when we write for the quantities α_s, β_s, respectively $\alpha_s + h$ and $\beta_s + h$. The coefficient of x^py^q on the right then becomes

$$\Sigma (\alpha_1 + h)(\alpha_2 + h) \ldots (\alpha_p + h)(\beta_{p+1} + h)(\beta_{p+2} + h) \quad . (\beta_{p+q} + h),$$

which is

$$(\overline{10}^p\, \overline{01}^q) + (n-p-q+1)\{(\overline{10}^{p-1}\, \overline{01}^q) + (\overline{10}^p\, \overline{01}^{q-1})\}h$$
$$+ \frac{(n-p-q+1)(n-p-q+2)}{2!}\{(\overline{10}^{p-2}\, \overline{01}^q) + 2(\overline{10}^{p-1}\, \overline{01}^{q-1}) + (\overline{10}^p\, \overline{01}^{q-2})\}h^2$$
$$+$$

But

$$(\overline{10}^p\, \overline{01}^q) = \frac{n!}{(n-p-q)!} a_{pq}.$$

Hence a_{pq} is transformed into

$$a_{pq} + (a_{p-1,q} + a_{p,q-1})h + (a_{p-2,q} + 2a_{p-1,q-1} + a_{p,q-2})\frac{h^2}{2!} + \ldots,$$

the general term being

$$\frac{a_{p-s\, q-t}\, h^{s+t}}{s!\, t!}.$$

Hence any symmetrical function

$$f(a_{10}, a_{01}, \ldots a_{pq} \ldots) \equiv f,$$

is transformed into

$$f\{a_{10} + h,\ a_{01} + h, \ \ldots a_{pq} + (a_{p-1,q} + a_{p,q-1})h + \ldots\},$$

or writing

$$g_{pq} = \partial_{a_{pq}} + a_{10}\partial_{a_{p+1,q}} + a_{01}\partial_{a_{p,q+1}} + \ldots,$$

this is

$$\overline{\exp}\left\{(g_{10} + g_{01})h + (g_{20} + 2g_{11} + g_{02})\frac{h^2}{2!} + \ldots\right\}f,$$

the bar over exp denoting that the multiplications of operators, which arise, are symbolic

Now, by the theorem of Art 15, this is

$$\exp\{M_{10}g_{10} + M_{01}g_{01} + \ldots + M_{pq}g_{pq} + \ldots\}f,$$

the multiplication denoting successive operations, and identically

$$\begin{aligned}\exp(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots) \\ = 1 + \xi + \eta + \frac{1}{2!}(\xi+\eta)^2 + \frac{1}{3!}(\xi+\eta)^3 + \ldots \\ = \exp(\xi + \eta)\end{aligned}$$

Hence $M_{10} = M_{01} = 1$ and the other coefficients M are zero

Hence the symmetric function f is converted into

$$\exp(g_{10} + g_{01})h \ \ f,$$

and, if f be a function of the differences

$$\exp(g_{10} + g_{01})h \,.\, f = f.$$

Hence the necessary and sufficient condition, that f may be a function of the differences, is the satisfaction of the linear partial differential equation

$$g_{10} + g_{01} = 0.$$

70. These operators g_{10} and g_{01} have been previously met with in the discussion of the symmetric functions connected with the fundamental identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y)\ldots = 1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^py^q + \ldots,$$

but then they played a different rôle.*

* Two simple cases of this important transformation should be verified by the reader For $n = 2$, $(\overline{11})$ is transformed into $\frac{1}{2}(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$ connected with the ternary quadric For $n = 3$, $(\overline{21})$ is transformed into $\frac{1}{18}\Sigma(\alpha_1 - \alpha_2)\{(\alpha_1 - \alpha_3) + (\alpha_2 - \alpha_3)\}(\beta_1 - \beta_2)$ connected with the ternary cubic

In that case g_{10} and g_{01} were shewn to annihilate all functions in which the biparts $\overline{10}$, $\overline{01}$, respectively, were absent. Hence, expressing all such functions in terms of $a_{10}, a_{01}, \ldots a_{pq}, \ldots$ we have at once a number of symmetric functions of difference of the quantities in the identity

$$(1+\alpha_1 x+\beta_1 y)(1+\alpha_2 x+\beta_2 y)\ldots(1+\alpha_n x+\beta_n y)$$
$$= 1 + na_{10}x + na_{01}y + \ldots + \frac{n!}{(n-p-q)!}a_{pq}x^p y^q \ldots$$

71. The differential equation

$$g_{10} + g_{01} = 0,$$

is, as a particular case, satisfied by the solutions of the simultaneous equations

$$g_{10} = 0, \quad g_{01} = 0.$$

In correspondence therewith we have functions composed of differences $\alpha_s - \alpha_t$, $\beta_s - \beta_t$, but not of differences $\alpha_s - \beta_s$, $\alpha_s - \beta_t$. The functions of differences $\alpha_s - \alpha_t$, $\beta_s - \beta_t$ are represented by the infinite series of monomial symmetric functions whose partitions contain neither of the biparts $\overline{10}$, $\overline{01}$.

The generating function for the number of biweight pq is

$$\frac{1}{(1-x^2)(1-xy)(1-y^2)(1-x^3)(1-x^2y)(1-xy^2)(1-y^3)\ldots}$$

The remaining functions of differences correspond to those solutions of $g_{10}+g_{01}=0$ which are not simultaneous solutions of $g_{10}=0$ and $g_{01}=0$.

Denoting by N any aggregate of biparts from which both $\overline{10}$ and $\overline{01}$ are excluded, we have the system of solutions

$$(\overline{10}\,\mathrm{N}) - (\overline{01}\,\mathrm{N})$$
$$(\overline{10}^2\,\mathrm{N}) - (\overline{10}\,\overline{01}\,\mathrm{N}) + (\overline{01}^2\,\mathrm{N}),$$
$$\ldots$$
$$(\overline{10}^{p+q}\,\mathrm{N}) - (\overline{10}^{p+q-1}\,\overline{01}\,\mathrm{N}) + \ldots + (-)^q(\overline{10}^p\,\overline{01}^q\,\mathrm{N}) + \ldots,$$

for on operating with $g_{10}+g_{01} = \mathrm{G}_{10}+\mathrm{G}_{01}$ the terms destroy each other in pairs.

Observe that these solutions are of the same weight but not of the same biweight in every term.

The number of solutions of a given weight is given by the generating function

$$\frac{x}{(1-x)(1-x^2)^3(1-x^3)^4\ldots(1-x^\mu)^{\mu+1}\ldots}.$$

Hence the *whole* number of asyzygetic functions of differences of a given weight is given by

$$\frac{1}{(1-x)(1-x^2)^3(1-x^3)^4 \cdots (1-x^{\mu})^{\mu+1}}$$

§ 13 *Special Fundamental Identity of Finite Order*

72 By taking the fundamental identity of infinite order syzygetic relations between monomial symmetric functions were avoided. Whenever the fundamental identity is taken of a finite order > 1 certain such relations of necessity arise

Professor CAYLEY ('Collected Papers,' vol. 2, p 454, and 'Phil Trans,' 1857) takes a fundamental identity equivalent to

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) = 1 + 2hx + 2gy^2 + bx^2 + 2fxy + cy^2,$$

and finds identically

$$bc - f^2 - bg^2 - ch^2 + 2fgh = 0,$$

the condition that the expression to the right shall break up into two linear factors

I take as the fundamental identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) = 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$

and observe that the syzygetic relation must connect monomial symmetric functions, each of which is symbolised by a partition containing more than two biparts. The symmetric functions must be of the same biweight of the form pp since the quantities α must occur symmetrically with the quantities β Of the biweight 1, 1, there exists no partition containing more than two biparts Of the biweight 2,2, we have the four partitions

$$(\overline{20}\,\overline{01}^2),\ (\overline{02}\,\overline{10}^2),\ (\overline{11}\,\overline{10}\,\overline{01}),\ (\overline{10}^2\,\overline{01}^2),$$

and if the corresponding symmetric functions can be linearly connected, so that no fundamental symmetric function of weight greater than 2 occurs, the linear function must vanish.

From the tables, *post* § 14, biweight 22, partition $(\overline{10}^2\,\overline{01}^2)$, we find

$$(\overline{20}\,\overline{01}^2) - (\overline{11}\,\overline{10}\,\overline{01}) + (\overline{02}\,\overline{10}^2) = -4a_{20}a_{02} + a_{11}^2 + a_{20}a_{01}^2 + a_{02}a_{10}^2 - a_{11}a_{10}a_{01};$$

the terms involving a_{22}, a_{21}, and a_{12} disappearing.

73 This is right, and shows (changing sign) that the well-known expression (discriminant)

$$bc - f^2 - bg^2 - ch^2 + 2fgh\,,$$

is equal to

$$-(\overline{20}\,\overline{01}^2) + (\overline{11}\,\overline{10}\,\overline{01}) - (\overline{02}\,\overline{10}^2),$$

a form which, for the ternary quadric, vanishes at sight

Another form is

$$-s_{20}a_{02} + s_{11}a_{11} - s_{02}a_{20}.$$

74 The expression

$$4a_{20}a_{02} - a_{11}^2 - a_{20}a_{01}^2 - a_{02}a_{10}^2 + a_{11}a_{10}a_{01}$$

satisfies the partial differential equation which appertains to the differences of the quantities in the relation

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) = 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$$

This equation is

$$2\,\partial_{a_{10}} + a_{10}\,\partial_{a_{20}} + a_{01}\,\partial_{a_{11}} + 2\,\partial_{a_{01}} + a_{10}\,\partial_{a_{11}} + a_{01}\,\partial_{a_{02}} = 0\,.$$

It is not, as a fact, expressible as a function of differences of α_1, β_1, α_2, β_2, because it vanishes altogether for a fundamental identity of the order 2

In relation to a fundamental identity of order greater than 2, the expression does not satisfy the equation of differences. Although it may be regarded from the above as a vanishing function of the differences, it is convertible into a non-vanishing function by the transformation before given The transformed expression is

$$(\overline{20}\,\overline{02}) - 2\,(\overline{11}^2) \quad \text{or} \quad (s_{20}s_{02} - s_{11}^2),$$

which visibly satisfies the differential equation

$$g_{10} + g_{01} \equiv \partial_{s_{10}} + \partial_{s_{01}} = 0$$

§ 14 *The Construction of Symmetrical Tables*

75. From the first law of symmetry it has been established that it is possible to form two symmetrical tables in connexion with every partition of every biweight. As illustrations, I give certain of the results as far as weight 4, inclusive. We have presented for the weight 4 the biweights 40, 31, 22, 13, 04. The theory of the biweights 40 and 04 is precisely the same as that of the weight 4 in the unipartite theory.* The one is, in fact, concerned only with the single system of quantities

* *Vide* 'American Journal of Mathematics,' vol. 11, and succeeding volumes

$\alpha_1, \alpha_2, \alpha_3,$ and the other only with the single system $\beta_1, \beta_2, \beta_3, \ldots$ We may, therefore, suppress altogether the zero elements in the biparts and then proceed to form the tables for the several partitions (unipart) of the weight which I have already set forth in the 'American Journal of Mathematics' (vol 11)

Of the remaining biweights 31, 22, 13, it is merely necessary to calculate the two former, since the tables for the biweight 13 are obviously immediately obtainable from those of the biweight 31 by interchanging the elements of each bipart —e g, by writing $\overline{qp}$ for $\overline{pq}$.

There are seven partitions of biweight 31, viz —

$$(\overline{10}^3\,\overline{01}),\ (\overline{11}\,\overline{10}^2),\ (\overline{20}\,\overline{10}\,\overline{01}),\ (\overline{20}\,\overline{11}),\ (\overline{21}\,\overline{10}),\ (\overline{30}\,\overline{01}),\ (\overline{31}),$$

and nine of the biweight 22, viz —

$$(\overline{10}^2\,\overline{01}^2),\ (\overline{11}\,\overline{10}\,\overline{01}),\ (\overline{11}^2),\ (\overline{02}\,\overline{10}^2),\ (\overline{20}\,\overline{02}),\ (\overline{12}\,\overline{10}),\ (\overline{22})$$
$$(\overline{20}\,\overline{01}^2),\qquad (\overline{21}\,\overline{01}),$$

Of these the table of $(\overline{20}\,\overline{01}^2)$ gives also the table of $(\overline{02}\,\overline{10}^2)$ by transposing the elements of the biparts, and similarly the table of $(\overline{21}\,\overline{01})$ gives that of $(\overline{12}\,\overline{10})$ We have thus 28 tables; but of these, the four corresponding to the partitions $(\overline{31})$ and $(\overline{22})$ are mere identities, so that the number is reduced to 24 The earlier tables which are necessary are those of the partitions $(\overline{10}\,\overline{01})$, $(\overline{20}\,\overline{01})$, $(\overline{11}\,\overline{10})$, $(\overline{10}^2\,\overline{01})$ These are now given Each table is read according to the lines.

Biweight 11

Partition $(\overline{10}\,\overline{01})$

	$(\overline{10}\,\overline{01})$	$(\overline{10})\,(\overline{01})$
$(\overline{11})$	-1	1
$(\overline{10}\,\overline{01})$	1	

± 1

	$(\overline{11})$	$(\overline{10}\,\overline{01})$
$(\overline{10}\,\overline{01})$		1
$(\overline{10})\,(\overline{01})$	1	1

Biweight 21.

Partition $(\overline{20}\,\overline{01})$

	$(\overline{20}\,\overline{01})$	$(\overline{20})\,(\overline{01})$
$(\overline{21})$	-1	1
$(\overline{20}\,\overline{01})$	1	

± 1

	$(\overline{21})$	$(\overline{20}\,\overline{01})$
$(\overline{20}\,\overline{01})$		1
$(\overline{20})\,(\overline{01})$	1	1

Biweight 21

Partition $(\overline{11}\,\overline{01})$

	$(\overline{11}\,\overline{10})$	$(\overline{11})\,(\overline{10})$	
$(\overline{21})$	-1	1	± 1
$(\overline{11}\,\overline{10})$	1		

	$(\overline{21})$	$(\overline{11}\,\overline{10})$
$(\overline{11}\,\overline{10})$		1
$(\overline{11})\,(\overline{10})$	1	1

Biweight 21

Partition $(\overline{10}^2\,\overline{01})$

	a_{21}	$a_{20}a_{01}$	$a_{11}a_{10}$	$a_{10}^2\,a_{01}$	
$(\overline{21})$	1	-1	-1	1	± 2
$(\overline{20}\,\overline{01})$	-1	-1	1		
$(\overline{11}\,\overline{10})$	-1	1			± 1
$(\overline{10}^2\,\overline{01})$	1				

	$(\overline{21})$	$(\overline{20}\,\overline{01})$	$(\overline{11}\,\overline{10})$	$(\overline{10}^2\overline{01})$
a_{21}				1
$a_{20}a_{01}$			1	1
$a_{11}a_{10}$		1	1	2
$a_{10}^2\,a_{01}$	1	1	2	2

Biweight 31

Partition $(\overline{30}\,\overline{01})$

	$(\overline{30}\,\overline{01})$	$(\overline{30})\,(\overline{01})$	
$(\overline{31})$	-1	1	± 1
$(\overline{30}\,\overline{01})$	1		

	$(\overline{31})$	$(\overline{30}\,\overline{01})$
$(\overline{30}\,\overline{01})$		1
$(\overline{30})\,(\overline{01})$	1	1

Partition $(\overline{20}\,\overline{10})$.

	$(\overline{21}\,\overline{10})$	$(\overline{21})\,(\overline{10})$	
$(\overline{21})$	-1	1	± 1
$(\overline{21}\,\overline{10})$	1		

	$(\overline{31})$	$(\overline{21}\,\overline{10})$
$(\overline{21}\,\overline{10})$		1
$(\overline{21})\,(\overline{10})$	1	1

Partition $(\overline{20}\,\overline{11})$

	$(\overline{20}\,\overline{11})$	$(\overline{20})\,(\overline{11})$	
$(\overline{31})$	-1	1	± 1
$(\overline{20}\,\overline{11})$	1		

	$(\overline{31})$	$(\overline{20}\,\overline{11})$
$(\overline{20}\,\overline{11})$		1
$(\overline{20})\,(\overline{11})$	1	1

Partition $(\overline{20}\,\overline{10}\,\overline{01})$.

	$(\overline{20}\,\overline{10}\,\overline{01})$	$(\overline{20}\,\overline{10})\,(\overline{01})$	$(\overline{20}\,\overline{01})\,(\overline{10})$	$(\overline{10}\,\overline{01})\,(\overline{20})$	$(\overline{20})\,(\overline{10})\,(\overline{01})$	
$(\overline{31})$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\pm 1\frac{1}{2}$
$(\overline{30}\,\overline{01})$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		± 1
$(\overline{21}\,\overline{10})$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$		± 1
$(\overline{20}\,\overline{11})$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$		± 1
$(\overline{20}\,\overline{10}\,\overline{01})$	1					

	$(\overline{31})$	$(\overline{30}\,\overline{01})$	$(\overline{21}\,\overline{10})$	$(\overline{20}\,\overline{11})$	$(\overline{20}\,\overline{10}\,\overline{01})$
$(\overline{20}\,\overline{10}\,\overline{01})$					1
$(\overline{20}\,\overline{10})\,(\overline{01})$			1	1	1
$(\overline{20}\,\overline{01})\,(\overline{10})$		1		1	1
$(\overline{10}\,\overline{01})\,(\overline{20})$		1	1		1
$(\overline{20})\,(\overline{10})\,(\overline{01})$	1	1	1	1	1

Partition $(\overline{11}\,\overline{10}^2)$

	$(\overline{11}\,\overline{10}^2)$	$(\overline{10}^2)\,(\overline{11})$	$(\overline{11}\,\overline{10})\,(\overline{10})$	$(\overline{11})\,(\overline{10})^2$	
$(\overline{31})$	1	-1	-1	1	± 2
$(\overline{20}\,\overline{11})$	-1	-1	1		
$(\overline{21}\,\overline{10})$	-1	1			± 1
$(\overline{11}\,\overline{10}^2)$	1				

	$(\overline{31})$	$(\overline{20}\,\overline{11})$	$(\overline{21}\,\overline{10})$	$(\overline{11}\,\overline{10}^2)$
$(\overline{11}\,\overline{10}^2)$				1
$(\overline{10}^2)\,(\overline{11})$			1	1
$(\overline{11}\,\overline{10})\,(\overline{10})$		1	1	2
$(\overline{11})\,(\overline{10})^2$	1	1	2	2

Partition $(\overline{10^3}\ \overline{01})$.

	a_{31}	$a_{30}a_{01}$	$a_{21}a_{10}$	$a_{20}a_{11}$	$a_{20}a_{10}a_{01}$	$a_{11}a_{10}^2$	$a_{10}^3a_{01}$	
$(\overline{31})$	-1	1	1	1	-2	-1	1	± 4
$(\overline{30}\ \overline{01})$	1	2	-1	-1	-1	1		
$(\overline{21}\ \overline{10})$	1	-1	0	-1	1			± 2
$(\overline{20}\ \overline{11})$	1	-1	-1	1				± 2
$(\overline{20}\ \overline{10}\ \overline{01})$	-2	-1	1					
$(\overline{11}\ \overline{10^2})$	-1	1						± 1
$(\overline{10^3}\ \overline{01})$	1							

	$(\overline{31})$	$(\overline{30}\ \overline{01})$	$(\overline{21}\ \overline{10})$	$(\overline{20}\ \overline{11})$	$(\overline{20}\ \overline{10}\ \overline{01})$	$(\overline{11}\ \overline{10^2})$	$(\overline{10^3}\ \overline{01})$
a_{31}							1
$a_{30}a_{01}$						1	1
$a_{21}a_{10}$					1	1	3
$a_{20}a_{11}$				1	1	2	3
$a_{20}a_{10}a_{01}$			1	1	1	3	3
$a_{11}a_{10}^2$		1	1	2	3	4	6
$a_{10}^3a_{01}$	1	1	3	3	3	6	6

BIWEIGHT 22

Partition $(\overline{21}\ \overline{01})$

	$(\overline{21}\ \overline{01})$	$(\overline{21})\ (\overline{01})$	
$(\overline{22})$	-1	1	± 1
$(\overline{21}\ \overline{01})$	1		

	$(\overline{22})$	$(\overline{21}\ \overline{01})$
$(\overline{21}\ \overline{01})$		1
$(\overline{21})\ (\overline{01})$	1	1

Partition $(\overline{20}\ \overline{02})$

	$(\overline{20}\ \overline{02})$	$(\overline{20})\ (\overline{02})$	
$(\overline{22})$	-1	1	± 1
$(\overline{20}\ \overline{02})$	1		

	$(\overline{22})$	$(\overline{20}\ \overline{02})$
$(\overline{20}\ \overline{02})$		1
$(\overline{20})\ (\overline{02})$	1	1

Partition $(\overline{11^2})$

	$(\overline{11^2})$	$(\overline{11})^2$
$(\overline{22})$	-2	1
$(\overline{11^2})$	1	

	$(\overline{22})$	$(\overline{11^2})$
$(\overline{11^2})$		1
$(\overline{11})^2$	1	2

Partition $(\overline{20}\ \overline{01^2})$

	$(\overline{20}\ \overline{01^2})$	$(\overline{20})\ (\overline{01^2})$	$(\overline{20}\ \overline{01})\ (\overline{01})$	$(\overline{20})\ (\overline{01})^2$	
$(\overline{22})$	1	-1	-1	1	± 2
$(\overline{20}\ \overline{02})$	-1	-1	1		
$(\overline{21}\ \overline{01})$	-1	1			± 1
$(\overline{20}\ \overline{01^2})$	1				

	$(\overline{22})$	$(\overline{20}\ \overline{02})$	$(\overline{21}\ \overline{01})$	$(\overline{20}\ \overline{01^2})$
$(\overline{20}\ \overline{01^2})$				1
$(\overline{20})\ (\overline{01^2})$			1	1
$(\overline{20}\ \overline{01})\ (\overline{01})$		1	1	2
$(\overline{20})\ (\overline{01})^2$	1	1	2	2

Partition $(\overline{11}\,\overline{10}\,\overline{01})$

	$(\overline{11}\,\overline{10}\,\overline{01})$	$(\overline{11}\,\overline{10})(\overline{01})$	$(\overline{11}\,\overline{01})(\overline{10})$	$2(\overline{10}\,\overline{01})(\overline{11})$	$(\overline{11})(\overline{10})(\overline{01})$	
$(\overline{22})$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	1	$\pm\frac{3}{2}$
$(\overline{21}\,\overline{01})$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$		± 1
$(\overline{12}\,\overline{10})$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$		± 1
$(\overline{11^2})$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$		$\pm\frac{1}{2}$
$(\overline{11}\,\overline{10}\,\overline{01})$	1					

	$(\overline{22})$	$(\overline{21}\,\overline{01})$	$(\overline{12}\,\overline{10})$	$(\overline{11^2})$	$(\overline{11}\,\overline{10}\,\overline{01})$
$(\overline{11}\,\overline{10}\,\overline{01})$					1
$(\overline{11}\,\overline{10})(\overline{01})$			1	2	1
$(\overline{11}\,\overline{01})(\overline{10})$		1	0	2	1
$2(\overline{10}\,\overline{01})(\overline{11})$		2	2	0	2
$(\overline{11})(\overline{10})(\overline{01})$	1	1	1	2	1

Partition $(\overline{10}^2\,\overline{01}^2)$

	a_{22}	$a_{21}a_{01}$	$a_{12}a_{10}$	$a_{20}a_{02}$	a_{11}^2	$a_{20}a_{01}^2$	$a_{02}a_{10}^2$	$a_{11}a_{10}a_{01}$	$a_{10}^2a_{01}^2$	
$(\overline{22})$	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	1	$\pm\frac{10}{3}$
$(\overline{21}\,\overline{01})$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$		± 2
$(\overline{12}\,\overline{01})$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$		± 2
$(\overline{20}\,\overline{02})$	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{10}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{4}{8}$	$\frac{4}{3}$		
$(\overline{11}^2)$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		$\pm\frac{4}{3}$
$(\overline{20}\,\overline{01}^2)$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		
$(\overline{02}\,\overline{10}^2)$	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		
$(\overline{11}\,\overline{10}\,\overline{01})$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$		$\pm\frac{7}{3}$
$(\overline{10}^2\,\overline{01}^2)$	1									

	$(\overline{22})$	$(\overline{21}\,\overline{01})$	$(\overline{12}\,\overline{10})$	$(\overline{20}\,\overline{02})$	$(\overline{11}^2)$	$(\overline{20}\,\overline{01}^2)$	$(\overline{02}\,\overline{10}^2)$	$(\overline{11}\,\overline{10}\,\overline{01})$	$(\overline{10}^2\,\overline{01}^2)$
a_{22}									1
$a_{21}a_{01}$							1	1	2
$a_{12}a_{10}$						1	0	1	2
$a_{20}a_{02}$					1	0	0	1	1
a_{11}^2				1	2	2	2	2	4
$a_{20}a_{01}^2$			1	0	2	0	1	2	2
$a_{02}a_{10}^2$		1	0	0	2	1	0	2	2
$a_{11}a_{10}a_{01}$		1	1	1	2	2	2	3	4
$a_{10}^2a_{01}^2$	1	2	2	1	4	2	2	4	4

§ 15. *Property of the Coefficients in the Tables*

76 There is in regard to the coefficients a very simple and important property which does not come into view with the unipartite theory so long as the tables are restricted to the particular cases in which the separable partitions are composed entirely of units The property appears the instant we consider a separable partition composed of parts which are *not* all similar. The law is the same whether the symmetric functions are unipartite, bipartite, or in general m-partite It depends upon the possibility of grouping the various separations in a particular manner To make this clear suppose we are presented with a separable partition $(\overline{10}^2\ \overline{01}^2)$ The nine separations may be written down in four groups, as follows —

Group 1	Group 2	Group 3	Group 4
$(\overline{10})^2\ (\overline{01})^2$	$(\overline{10}\ \overline{01}^2)\ (\overline{10})$	$(\overline{10}^2\ \overline{01})\ (\overline{01})$	$(\overline{10}^2\ \overline{01}^2)$
$(\overline{11})\ \ (\overline{10})\ (\overline{01})$	$(\overline{01}^2)\ (\overline{10})^2$	$(\overline{10}^2)\ (\overline{01})^2$	$(\overline{10}^2)\ (\overline{01}^2)$
$(\overline{11})^2$			

In Group 2 it will be seen that the parts $(\overline{10}^2)$ of the separable partition occur in the separation $(\overline{10})^2$, while the parts $(\overline{01}^2)$ occur in the separation $(\overline{01}^2)$, so that the expression $\{(\overline{10})^2, (\overline{01}^2)\}$ may be taken as defining a certain separation property of the separations of the group The group in question may be denoted by $Gr\ \{(\overline{10})^2, (\overline{01}^2)\}$ and on the same principle the Groups 1, 3, and 4 may be denoted by

$$Gr\ \{(\overline{10})^2, (\overline{01})^2\}, \qquad Gr\ \{(\overline{10}^2), (\overline{01})^2\}, \qquad Gr\ \{(\overline{10}^2), (\overline{01}^2)\}$$

respectively. In the separable partition $(\overline{10}^2\ \overline{01}^2)$ the parts $(\overline{10})$ and $(\overline{01})$ occur each twice, and a group results from every combination of a partition of 2 with a partition of 2. If P_2 denote the number of partitions of 2, the number of groups will be $P_2{}^2 = 4$ In general if the different parts of the separable partition occur $a, b\ c, \ldots$ times the number of groups of separations is $P_aP_bP_c \ldots$

77. The leading property that has been adverted to is that in the expression of a single-bipart function by means of separations of a partition composed of dissimilar parts, the algebraic sum of the coefficients in each group of separations is zero. A corollary at once follows which will be given in its proper place.

78. From the identity of Art. 24, viz :—

$$1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots = \Pi_s\,(1 + \alpha_s b_{10}\xi + \ldots + \alpha_s{}^p\beta_s{}^q b_{pq}\xi^p\eta^q + \ldots)$$

is derived a series of relations which express the quantities c in terms of the

quantities b and symmetric functions of the quantities α, β. These are given in Art 24

To put the *group* in evidence it is necessary to modify these relations by writing $b_{\overline{r_1s_1}^{\rho_1}}$ for $b^{\rho_1}_{r_1s_1}$, so that for examples the expressions for c_{20} and c_{11} become

$$(\overline{20})\, b_{20} + (\overline{10}^2)\, b_{\overline{10}^2}$$
$$(\overline{11})\, b_{11} + (\overline{10}\ \overline{01})\, b_{10}b_{01}$$

respectively In any product $c_{p_1q_1}{}^{\pi_1}c_{p_2q_2}{}^{\pi_2}$. the cofactor of the product

$$b^{\sigma_1}_{\overline{r_1s_1}^{\rho_1}}\, b^{\sigma_2}_{\overline{r_1s_1}^{\rho_2}} \ldots\ b^{\upsilon_1}_{\overline{t_1u_1}^{\tau_1}}\, b^{\upsilon_2}_{\overline{t_1u_1}^{\tau_2}},$$

is composed of symmetric function products each of which appertains to the group

$$Gr\left\{(\overline{r_1s_1}^{\rho_1})^{\sigma_1}(\overline{r_1s_1}^{\rho_2})^{\sigma_2} \ldots, (\overline{t_1u_1}^{\tau_1})^{\upsilon_1}(\overline{t_1u_1}^{\tau_2})^{\upsilon_2} \ldots\right\}$$

The sum of the coefficients attached to the members of this group is obtained by putting each monomial symmetric function equal to unity The sum in question then appears as the numerical coefficient of the b product above written

Write then

$$c_{10}{}^1 = b_{10}$$
$$c_{01}{}^1 = b_{01}$$
$$c_{20}{}^1 = b_{20} + b_{\overline{10}^2}$$
$$c_{11}{}^1 = b_{11} + b_{10}b_{01}$$

so that, ξ and η being arbitrary,

$$\begin{aligned} &1 + c_{10}{}^1\xi + c_{01}{}^1\eta + \ldots + c_{pq}{}^1\xi^p\eta^q + \ldots \\ &= (1 + b_{10}\xi + b_{\overline{10}^2}\xi^2 + b_{\overline{10}^3}\xi^3 + \ldots) \ldots (1 + b_{pq}\xi^p\eta^q + b_{\overline{pq}^2}\xi^{2p}\eta^{2q} + \ldots) \end{aligned}$$

a factor appearing on the right for each biweight

To find the sum of the coefficients in each group in the case of the expression of the single-bipart functions we have now merely to take logarithms when (Art 26) the functions being s_{pq} or $(\overline{pq})$, the sum in each case presents itself multiplied by $(-)^{p+q-1}(1/p!q!)(p+q-1)!$. Expanding the right hand side after taking logarithms we see that only terms of the form

$$b^{\sigma_1}_{\overline{rs}^{\rho_1}}\, b^{\sigma_2}_{\overline{rs}^{\rho_2}} \ldots$$

can appear. Hence the theorem :—

"In the expression of symmetric function $(\overline{pq})$ by means of separations of any partition of the same biweight, the partition consisting of dissimilar parts, the algebraic sum of the coefficients in each group of separations is zero."

As regards the remaining cases where the separable partition does not contain dissimilar parts, the *group* obviously contains but a single separation and *quâ group* has no existence.

We have in fact the expression of $(\overline{pq})$ by means of separations of $(\overline{rs}^{\lambda})$ where $\lambda r = p$, $\lambda s = q$.

The result is clearly

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}(\overline{pq}) = \Sigma\,(-)^{\Sigma\sigma-1}\frac{(\Sigma\sigma-1)!}{\sigma_1!\,\sigma_2!\dots}(\overline{rs}^{\rho_1})^{\sigma_1}(\overline{rs}^{\rho_2})^{\sigma_2}\dots$$

79. The law of group of separations may be verified from the tables. It is a very satisfactory aid to calculation, particularly in the detection of missing separations.

Moreover the law embraces symmetric functions other than those symbolised by a single bipart. Suppose the function expressed in terms of single-bipart functions. The latter may be separately expressed in terms of separations of partitions in such wise that the function in question will be represented by means of separations of any given partition of its biweight. The law of the group will hold for the single-bipart functions whenever the separable partition contains dissimilar parts, and moreover, in a product of single-bipart functions the law will hold if one or more of the factors is expressed in terms of separations of a partition containing dissimilar parts. Hence the only exception occurs when we find presented a product of the form

$$s_{(\overline{a_1b_1}^{\rho_1})}\,s_{(\overline{a_2b_2}^{\rho_2})}\,s_{(\overline{a_3b_3}^{\rho_3})}\dots,$$

now if the symmetric function, say $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\dots)$, whose expression we are considering in connection with a given separable partition, say $(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\dots)$ itself possesses a separation of specification

$$(\overline{\alpha_1a_1,\ \alpha_1b_1}\ \overline{\alpha_2a_2,\ \alpha_2b_2},\ \dots)$$

a product of this form will certainly occur, but not otherwise.

Hence the theorem:—

80. "In the expression of symmetric function

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\dots),$$

by means of separations of

$$(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\dots),$$

the algebraic sum of the coefficients in each group of separations is zero if the partition

$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$ possesses no separations of specification $(\overline{\alpha_1a_1, \alpha_1b_1}, \overline{\alpha_2a_2, \alpha_2b_2}\ldots)$ but not otherwise."

The law may be verified in the case of the table of separations of $(\overline{10}^2\,\overline{01}^2)$, for the symmetric functions $(\overline{22})$, $(\overline{21}\,\overline{01})$, $(\overline{12}\,\overline{10})$, $(\overline{11}^2)$, $(\overline{11}\,\overline{10}\,\overline{01})$ for none of these five functions can be separated so that the specification is $(\overline{20}\,\overline{02})$. On the other hand the group law does not hold for $(\overline{20}\,\overline{01}^2)$ because the separation (20) $(\overline{01}^2)$ has a specification $(\overline{20}\,\overline{02})$.

§ 16. *Conclusion.*

81. All the preceding results can be easily extended to the m-partite theory connected with m systems. The weights are m-partite as also the parts of the partitions. As a general rule m suffices appear in the symbols. The laws of symmetry and their consequences, the symmetrical tables, the correspondences between the algebras of quantity and differential operation, the partition linear and obliterating operators, the law of groups of coefficients (and in fact the whole investigation here presented) proceed *pari-passu* with the bipartite theory above set forth. The unipartite or ordinary theory of the single system is also absolutely included in every respect.

In its applications, the results will be chiefly of use in the theory of elimination in the most general case. In this regard SCHLAFLI's memoir (*loc. cit.*) may be consulted.

VIII. *Account of recent Pendulum Operations for determining the relative Force of Gravity at the Kew and Greenwich Observatories.*

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(*Communicated at the request of the Kew Committee.*)

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THE recent pendulum observations for the purpose of determining the gravity connexion between the Royal Observatory at Greenwich and the Royal Society's Observatory at Kew, were undertaken in order to improve and strengthen the connexion between the Indian series of pendulum operations and other series taken in other parts of the world.

The Indian series had been carried out in the years 1865 to 1873, when two invariable pendulums, the property of the Royal Society, which had been designed by Captain KATER for the purpose of investigating the relative force of gravity in different latitudes, were swung at the Kew Observatory, and at various places in and on the way to India, in the course of the operations of the Great Trigonometrical Survey of India. The work was originated at the suggestion of the President of the Royal Society, General Sir E. SABINE.; the greater portion was performed by Captain J. P. BASEVI, R.E., who lost his life from exposure while operating on the high table-lands of the Himalayan Mountains; the remainder was completed by Captain W. J. HEAVISIDE, R.E.; both officers acted under the personal superintendence of General J. T. WALKER, the Superintendent of the Great Trigonometrical Survey.

The points at which the pendulums were swung and the number of vibrations they made in 24 hours were determined, were mostly stations of the Central Meridional Arc of the Survey which extends from Cape Comorin to the Himalayan Mountains; a few stations were added on the East and West Coasts of India, and on neighbouring islands, and also at Aden and Ismailia. The base station of the entire series of operations—that is to say, the one at which they were commenced and concluded—was the Royal Society's Observatory at Kew, near Richmond, Surrey.

With a view to effecting a connexion between the operations in India and similar operations recently completed in Russia, and also for other reasons, two reversible pendulums, the property of the Russian Imperial Academy of Sciences, which had been employed in Russia, were sent out to India and swung at some of the Indian stations, *pari passu*, with the pendulums of the Royal Society.

For the purpose of connecting the Indian operations with those taken by KATER, SABINE, FOSTER, and other observers in various parts of the globe, it was intended to swing the Royal Society's pendulums, on their return to England, at the Royal Observatory, Greenwich, which was a station of various important series of operations But when the time arrived, in 1873, it was found that extensive preparations were being made in the Observatory for several expeditions which were being outfitted for the observation of the approaching transit of Venus, so that no space was left available for the pendulum operations It was therefore decided to make the desired connexion by swinging KATER's convertible pendulum, for determining the absolute length of the seconds' pendulum in any latitude, at Kew, as already it had been swung by SABINE at Greenwich. This was done by Captain HEAVISIDE, and was the last stage of the Indian pendulum operations, the results were published in vol 5 of the 'Account of the Operations of the Great Trigonometrical Survey,' Dehra Dun, 1879, which gives full details of all the operations, including the swings with the Russian pendulums

The absolute length of the seconds' pendulum, *in vacuo*, at Kew, was found to be 39 14008 inches of the British standard yard, in 1873, whereas at Greenwich, SABINE had found it to be 39 13734 inches of Sir GEORGE SHUCKBURGH's standard scale, in 1831 * Thus it would seem that a seconds' pendulum will make about three more vibrations in 24 hours at Kew than at Greenwich But the two Observatories are nearly in the same latitude, and differ very little in height, and are only ten miles apart; thus this difference is much too large to be accepted as trustworthy

In his work on Geodesy, Colonel CLARKE, C.B, R E, of the Ordnance Survey of Great Britain, employs a large number of pendulum determinations in various parts of the globe to investigate the figure of the earth He remarks that the selection of Kew, instead of London or Greenwich, as the base station for the Indian series of swings, was unfortunate, and, disregarding the connexion of Kew with Greenwich by the two determinations of the length of the seconds' pendulum, he employs the ratio of Madras to London from the observations of GOLDINGHAM and SABINE, and that of Kew to St. Petersburgh, by HEAVISIDE and SAWITSCH. Then he makes four different combinations of his data, from which he obtains as many values of the earth's ellipticity; and for each value he finds the corresponding system of quantities, x, indicating the apparent excess or defect of the observed over the theoretical force of gravity at each station of observation; one combination gives Kew an excess of 6 06 vibrations over London, another gives it an excess of 5·12 vibrations over Greenwich, but reduces the excess over London to 3 10 vibrations These figures indicate variations in the vibration numbers such as are usually met with on changes of latitude of 1 to 2 degrees, and they show that the actual relation of Greenwich to

* It has recently been ascertained that very little, if any, of the difference can be due to error of the unit of the SHUCKBURGH scale as compared with the standard yard. See No 288 of the 'Proceedings of the Royal Society' (vol. 47, 1890, p 186)

Kew had not yet been precisely determined, and that special observations were still required for the purpose

In 1881, Colonel HERSCHEL, R E, was deputed by the Secretary of State for India to take pendulum observations at the Greenwich and the Kew Observatories also at some places in America, with a view to making a connexion with the pendulum operations of the Coast and Geodetic Survey of the United States He employed the two pendulums of the Royal Society which had been used in the Indian operations, and also a third pendulum of precisely similar construction which had been deposited in the Kew Observatory by the Admiralty, the experience already gained in India having shown that the employment of a third pendulum was desirable After completing his swings in England and America, he made over the three pendulums to officers of the United States' Survey, who took them round the world, and swung them at Auckland, Sydney, Singapore, Tokio, San Francisco, and finally at Colonel HERSCHEL'S station in Washington *

When the observations came to be finally reduced, it was found that the results between Kew and Greenwich by the three pendulums were largely discordant, one giving Kew an excess of 1 97 vibrations, another an excess of 1 39, while the third gave a defect of 4 98 vibrations It was obviously necessary that the pendulums should be again swung at the two places, in order to obtain a more satisfactory determination of the relative vibration numbers Fresh swings were therefore made at Kew in 1888, and at Greenwich in the following year The operations were performed by members of the Observatory staff at each place, Mr HOLLIS taking the lead and responsibility at Greenwich, under the direction of the Astronomer Royal, and Mr CONSTABLE at Kew, under the Superintendent of the Kew Observatory The final results give a vibration number for Kew which differs by less than one vibration from that at Greenwich, and may be accepted as very fairly probable

It is the object of the present paper to give an abridged account of the above operations, both the primary by Colonel HERSCHEL, and the revisionary by Messrs HOLLIS and CONSTABLE † For this purpose it is necessary, in the first instance, to give brief descriptions of the pendulums, and of the *modus operandi* adopted by the different observers

Description of the Pendulums

All three pendulums are of KATER'S Invariable Pattern, they are made of brass, with a steel knife-edge at the head of each pendulum, and they are of very nearly the same dimensions One is numbered 4 and another 11, the third has

* Full details of the operations and their results are given in Appendix No 14 of the 'Report of the United States Coast and Geodetic Survey' for 1884

† Full details of Colonel HERSCHEL'S operations, in manuscript, were made over to the Royal Society for record, by the Secretary of State for India, the details of the other operations are recorded in the observatories in which they respectively took place

no such distinguishing number, but is marked 1821, presumably the year in which it was constructed, Colonel HERSCHEL believes that it is probably No. 6 of the series, and has so called it. No. 4 was employed by SABINE in his operations between the parallels of 13° South and 80° North Latitude, in 1822–23, and No. 6 (1821) was used by the late Astronomer Royal, Sir GEORGE AIRY, in experiments in the Harton Colliery Pit, in 1854, to determine the earth's mean density; these two are the pendulums of the Royal Society which were employed throughout the operations in India. No. 11 was used by BAILEY, in London, in 1832, and by MACLEAR, at the Cape of Good Hope, in 1839; it was afterwards lent for a while to the Admiralty, and eventually deposited in the Kew Observatory.

Each pendulum is furnished with a pair of agate planes, on which it is intended to be swung. The planes are set on either side of a half-inch opening in a solid brass frame, which is mounted on a plate at the head of the receiver, and is provided with three levelling screws; outside the frame there is a pair of moveable arms carrying Y's, in which the pendulum rests while not vibrating, and on lowering which the knife edge comes in contact with the agate planes for vibration. The pendulum is placed midway between the supporting planes by hand and eye estimate, but it is always brought by the Y's down on to the same line across the planes, in all positions of the pendulum, whether the marked face is pointing towards the observer or towards the clock.

The length of the pendulum is invariable, excepting from change of temperature for which the correction to the vibration-number is known. The shape is that of a flexible bar of plate brass, 62 inches long, 1.7 inch broad, and 0.13 inch thick from the knife edge downwards for a distance of about 40 inches, where a flat circular bob, 6 inches in diameter and 1.3 inch thick, with a bevelled edge, is soldered on to the bar; the tail piece, below the bob, is reduced to a breadth of 0.7 inch, and is about 16 inches long. The bar is necessarily very flexible, its thickness being less than a tenth of its breadth, and this flexibility is greatly in contrast with the rigidity of the German and French pendulums. KATER is believed to have adopted a flexible form of bar in preference to a rigid bar designedly, under the impression that it was less likely to become permanently bent by accident, and more likely to acquire exact verticality when its knife-edge is resting on the agate planes during the course of the vibrations.

The Processes of Manipulation.

When the pendulums were sent out to India, it was intended that they should always be swung as nearly as possible in a vacuum. For this purpose a receiver of sheet copper, mounted on a substantial and well braced wooden stand, was furnished for the pendulums to be swung in; the receiver was closed above by a hemispherical glass cap, which could be removed at pleasure for the insertion or withdrawal of a pendulum. Two thermometers were fixed in a dummy pendulum, of the same size as

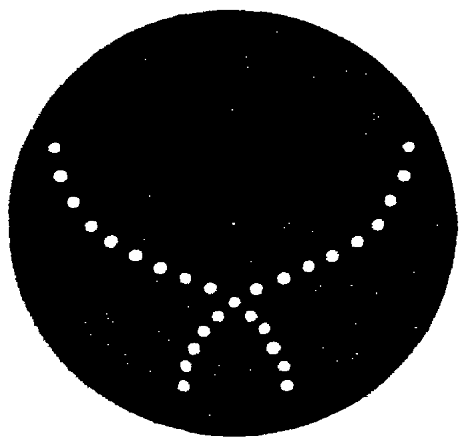
the vibrating pendulum less the tail-piece, which was fixed inside the receiver. In the Indian operations, the air was always exhausted to the lowest pressure attainable and the vibration-number obtained at that pressure was reduced to the vacuum by special corrections, which will be afterwards described. At first the receiver was found to leak slightly, the pressure rising about an inch in the course of a day's swings; but eventually this leakage was traced to the two stuffing boxes, through which one rod passes for lowering the knife-edge on, or raising it off, the agate planes, and another rod for setting the pendulum in motion; the leaks were stopped by fitting cups round the necks of the rods, and keeping them filled with oil. When the apparatus came into Colonel HERSCHEL's hands much leakage was met with at the lower pressures. He continued to use the receiver, but he did not attempt to obtain low pressures, and he abandoned the reduction to the vacuum; his pressures ranged between 26 and 28 inches, and his observations were reduced to an adopted standard pressure of 26 inches, with the temperature of the air at 32° F. The same procedure was adopted when the pendulums were swung by the officers of the United States' Coast and Geodetic Survey. But when the apparatus was returned to England, the receiver was repaired and made thoroughly air-tight before the revisionary swings at Kew and Greenwich were commenced; then half the swings at each place were taken under a pressure of about 2 inches, and the other half at about 27 inches; the results were reduced to a vacuum in both cases, by the same formula, which will be given hereafter.

For each invariable pendulum the vibration-number is determined by swinging it in front of the pendulum of a clock and observing the times of the first and last coincidences of the two pendulums, at the beginning and end of a set of swings; also the time of a coincidence immediately following the first, in order to get an approximate value of the interval between successive coincidences; thus the number of swings made by the invariable pendulum during a given amount of clock time, is ascertained by observation, and from it the number of swings in 24 hours is calculated, after making due allowance for clock rate, arc of vibration, temperature, and pressure. The clock employed in the Indian and subsequent operations was one by SHELTON, which had been used for the same purpose by SABINE. The invariable pendulums make 200 to 300 vibrations daily less than the pendulum of a clock regulated to solar time, and about twice that number to siderial time.

For the observations of coincidence, in the Indian operations, a circular disc of white metal was mounted on the bob of the clock pendulum, and an image of it, made very slightly less in diameter than the breadth of the tail-piece of the invariable pendulum, was produced, by an intermediate lens, in the plane of the tail-piece. The image would disappear for a moment and then reappear behind the tail-piece at every apparent conjunction of the two pendulums; and these conjunctions occur when both pendulums are swinging in the same direction, the intermediate conjunctions, with the swings in opposite directions, being unobservable. The exact moment of coincidence

was at first deduced from observations of the disappearance and reappearance of both edges of the disc, but after a short time, this was considered unnecessary, and one edge only was observed, the same throughout each set of swings

Colonel HERSCHEL substituted for the single large disc a system of multiple discs consisting of several pairs of small circles, arranged symmetrically on opposite sides of a central vertical line, and painted white on a piece of black cardboard which was attached to the bob of the clock pendulum He designed a large variety of systems, one of which is here shown He observed the times of disappearance and re-appearance of several pairs of discs, eventually retaining five pairs only, of which the general



mean was taken as the moment of coincidence The United States' officers adopted one of Colonel HERSCHEL'S discs, but observed on only one side of the tail-piece and not on both sides as he had done In the revisionary operations at Kew and Greenwich, a single large disc, of which the image was made of nearly the same diameter as the tail-piece, was employed, and observations of disappearance and re-appearance were made on one side only, as in India.*

In the Indian and the revisionary operations the times of the three first and the three last coincidences in each set of swings were observed, and the means were employed to indicate the moments of commencement and conclusion of the set, the observed intervals between successive coincidences gave the divisor to the duration of the set to find the total number of intervals which is wanted in calculating the vibration-number. In Colonel HERSCHEL'S operations one or two discs were observed

* Very great precision in the determination of the moment of coincidence is unnecessary If V be the vibration-number of a pendulum, R the clock vibrations in a mean solar day, and N the clock vibrations during a set of swings in which there are n intervals between visible coincidences, then

$$V = R\left(1 - \frac{2n}{N}\right) \quad \text{and} \quad dV = 2R\,\frac{n}{N^2}\,dN$$

Let $R = 86{,}630$, let the duration of the set of swings be 6 hours and the interval between coincidences 6 minutes, giving $n = 60$, then

$$dV = \cdot 022\,dN.$$

Thus an error of 4 seconds in N, which is improbably large, would not affect the vibration-number by as much as 0·1, which is but a fraction of the probable error from other causes

immediately after the initial observations of coincidence, to obtain the interval between successive coincidences

In the Indian operations the agate planes were always carefully levelled before the pendulum was set up on them, and the level readings, as taken before and after the swings by each pendulum, were recorded and published. As the pendulum is necessarily made to vibrate with its tail-piece a short distance in front of the scale for measuring the arc of vibration, this distance was read off on a scale fixed at right angles to the arc-scale, to enable the observed arc reading to be duly corrected, and it was measured in the two positions of the instruments, both with the marked face towards the observer and towards the clock Thus, as the knife-edge was always lowered by the Y's down to the same line on the agate planes, and as the planes were always horizontal, half the difference between the distances of the tail-piece from the arc-scale, in the two positions, indicates the magnitude of any deviation of the bar of the pendulum, at the tail-end, from perpendicularity to the knife-edge These distances were always recorded, and they show that both the pendulums were bent, but that the bending was practically constant throughout the whole of the operations; the marked face of Pendulum No 4 was deflected 5 inch outwards, and that of No 6 (1821) 3 inch backwards, at the tail-end As however, a general constancy was preserved throughout, the whole of the results were truly differential

Colonel HERSCHEL commenced his operations at Kew by swinging No 4 pendulum in the condition in which he found it He soon noticed bends in both the two pendulums, and also found that the knife-edges were somewhat rusted There was reason to suspect that the pendulums might have received some injuries when set up at one of the great Annual Exhibitions in South Kensington which was held a few years after their return from India Consequently both the pendulums were straightened and their knife-edges were re-ground This, of course, causes a break of continuity with the antecedent operations with these pendulums, and destroys the differentiality of the vibration numbers obtained before and afterwards

In the revisionary operations at Kew and Greenwich the pendulums were swung in the same condition as when employed by Colonel HERSCHEL and the officers of the United States

In the Indian operations each set of swings was usually of about 9 hours' duration, from 8 A M to 5 P M, intermediate readings of the thermometers and observations of coincidence being taken at intervals of about $1\frac{1}{2}$ hour apart When all the observations were finally reduced it was seen that, whenever the daily variation of temperature was considerable, the clock rate at different hours of the day varied sensibly from the mean daily rate; thus it was evident that the vibration-number, which depends on the actual clock rate during the set of swings, but is deduced from the mean daily rate which is derived from successive transits of the same stars, might be much influenced by variations of rate occurring during the part of the day when the pendulum is under vibration

Colonel HERSCHEL got over this difficulty, and eliminated the influence of hourly variations of clock rate, by linking successive sets of swings together so as to fill up the whole 24 hours He made the duration of each set of his swings somewhat less than 6 hours, taking all necessary observations of temperature, coincidence, &c at the commencement and conclusion of the set, but without any intermediate observations, immediately after the conclusion of one set he commenced the next set In this way observations were sometimes carried on continuously for two or three days by himself and his assistant In the revisionary operations at Kew a similar procedure was adopted, the temperatures and coincidences in the intervals between the beginning and end of each set were generally observed also, and the temperatures when not observed were recorded on a thermograph At Greenwich observations were made at 10 A.M., 4 P.M., and 10 P.M.; the swings at low pressure were divided into two sets of 6 hours each and one set of 12 hours, so as to fill up the 24 hours, those at high pressure into two sets of 6 hours each, to fill up 12 hours The temperatures between the thermometer readings were registered by a thermograph The daily range of temperature was very small at Kew, and rarely as much as 1° F at Greenwich At both places the time was derived electrically from the sidereal standard clock at Greenwich, which is fixed in the basement of the Observatory, where there is no sensible diurnal variation of temperature Under the actual circumstances there was no real necessity for continuous observations throughout the 24 hours to control the clock rate

The differences in procedure and manipulation which have been pointed out thus far are not such as to have affected the results sensibly, but in one other matter there was a difference of procedure which might have materially influenced the results Colonel HERSCHEL did not maintain the agate planes in exact horizontality, he believed that when the planes were truly level, and the pendulum was set up on them, the knife-edge, if pressed down against them by hand, was found to be not truly in contact with the planes throughout the line of bearing, consequently he dislevelled the planes until the contact, as judged by touch, was thorough, and then he commenced swinging the pendulum. Such imperfect contact was never noticed in the Indian operations; it is possible with pendulums having a rigid bar, when the knife-edge is not truly perpendicular to the bar, but with pendulums such as these, which have a very flexible bar, it seems scarcely possible, at least without a grosser displacement of the knife-edge from the perpendicular than is at all probable The officers of the United States who swung these pendulums at several stations have been questioned on this point, and Mr EDWIN SMITH reports that after levelling he "tested the contact of the knife-edges with the agate planes by touch and found it impracticable to make any change, so the pendulums were always swung with the planes levelled with the spirit level." In the revisionary swings this was done also

Colonel HERSCHEL does not appear to have measured the actual dislevelment of the agate planes which was caused by his method of treatment, had he done so and his surmise been correct, the magnitude of the dislevelment would have been the same

after as it was before each transposition of the pendulum, but its sign would have changed, because its direction would have been reversed The record of the observations gives no level readings, nor does it give the distances between the tail-piece and the arc-scale in the two positions of vibration All that is known is that the agate planes were not maintained in a position of constant horizontality, as in all other operations. Thus the results are not strictly differential and it is to be inferred that the large discrepancies occasionally met with between different groups of results, even when the individual results in each group are highly accordant, are due to this cause

The results of the several operations will now be given

COLONEL HERSCHEL'S RESULTS, REDUCED TO TEMPERATURE OF 62° F, AND TO THE DENSITY OF THE AIR UNDER THE PRESSURE OF 26 INCHES AT THE TEMPERATURE OF 32° F

Determinations at Kew Observatory

PRELIMINARY Vibration-numbers of Pendulum No 4, before straightening the bar and re-grinding the knife-edge

Marked face, M, to front		Plain face, P, to front		Means	
Set	V	Set	V		
2	86158 74	15	86161 44		
3	86158 84	16	86161 02		
4	86158 05	17	86160 87		
5	86159 21	18	86161 25	Face M	86158 38
6	86157 31	19	86160 60	Face P	86160 98
8	86158 41	20	86160 24		
9	86158 25	21	86160 86		86159 68
10	86158 32	22	86161 26		
11	86158 31	23	86160 12		
12	86158 91	24	86160 87		
13	86158 83	25	86161 52		
14	86157 41	26	86161 69		

When this result is reduced to a vacuum, it may be compared with the results of the swings by the same pendulum at the same place which were obtained for the Indian operations. The reduction to the vacuum may be approximately taken as + 8 32, which gives the vibration-number 86168 00, to compare with 86169 45 obtained in 1864, before the pendulum was sent out to India, and 86169 57 obtained in 1873, after its return from India.

Colonel HERSCHEL'S *Determinations at Kew Observatory, continued*

VIBRATION-NUMBERS by all three Pendulums, obtained after the bars were straightened and the knife-edges re-ground.

Pendulum No 4

Face M		Face P		Means	
Set	V	Set	V		
74	86157 07	94	86157 16		
75	86157 62	95	86157 08		
76	86158 21	96	86157 75		
77	86157 05	97	86157 26		
78	86156 75	98	86157 54		
79	86157 74	99	86157 78		
80	86158 13	100	86158 45		
81	86157 88	101	86157 84		
82	86157 44	102	86157 27	Face M	86157 42
83	86157 67	103	86157 52	Face P	86157 87
84	86157 54	104	86157 65		
85	86157 06	105	86157 42		86157 64
86	86157 01	106	86157 62		
87	86156 39	107	86158 47		
88	86159 39	108	86158·87		
89	86158 01	109	86159 01		
90	86155 74	110	86158 63		
91	86157 15	111	86158 07		
92	86157 35	112	86158 21		
93	86156 94				

Colonel HERSCHEL'S *Determinations at Kew Observatory, continued*

Pendulum No 6 (1821)

Face M		Face P		Means	
Set	V	Set	V		
28	86057 21	51	86056 54		
29	86056 64	52	86055 95		
30	86056 28	53	86055 88		
31	86056 00	54	86057 15		
32	86055 05	55	86057 11		
33	86055 25	56	86056 72		
34	86055 87	57	86056 71		
35	86055 64	58	86056 78		
36	86055 82	59	86056 75	Face M	86056 32
38	86058 82	60	86056 42	Face P	86056 74
39	86058 11	61	86056 54		
40	86056 84	62	86057 08		86056 53
41	86056 65	63	86057 18		
42	86053 87	64	86057 18		
43	86054 77	65	86057 19		
44	86056 93	66	86057 33		
45	86056 62	67	86057 01		
46	86056 48	68	86056 93		
47	86056 52	69	86056 68		
48	86056 66	70	86057 11		
49	86056 80	71	86055 21		

Pendulum No 11

Face M		Mean	
Set	V		
115	86099 64		
116	86099 18		
117	86099 11		
118	86099 67		
119	86100 60		
120	86099 83		
121	86101 54		
122	86100 76		
123	86099 87		
124	86100 44	Face M	86100 57
125	86100 17		
126	86100 78		
127	86100 42		
128	86100 83		
129	86100 86		
130	86101 10		
131	86100 81		
132	86101 61		
134	86103 58		

Colonel HERSCHEL'S *Determinations at Greenwich Observatory*

Pendulum No 4

Face M		Face P		Means	
Set	V	Set	V		
135	86153 96	151	86156 34		
136	86153 46	152	86156 04		
137	86153 54	153	86156 24		
138	86153 10	154	86156 38	Face M, 1st Series	86153 39
139	86153 99	155	86156 52	Face M, 2nd and 3rd Series	86156 65
140	86153 13	156	86156 24		
141	86153 20	157	86156 78		86155 02
142	86153 53	158	86156 49	Face P	86156 31
143	86153 35	159	86156 26		
144	86153 40	160	86155 84		86155 67
145	86153 04	161	86156 21		
146	86153 38	162	86156 58		
147	86153 29	163	86155 98		
148	86153 34	164	86156 43		
149	86153 33				
150	86153 12				
165	86156 78				
166	86156 62				
167	86157 18				
168	86156 68				
169	86157 29				
170	86157 06				
171	86157 20				
212	86156 57				
213	86156 42				
214	86156 31				
215	86156 51				
216	86156 24				
217	86156 28				
218	86156 34				
219	86156 30				

Colonel HERSCHEL'S *Determinations at Greenwich Observatory continued*

Pendulum No 6 (1821)

Face M		Face P		Means	
Set	V	Set	V		
172	86054 82	184	86054 89		
173	86055 04	185	86055 91		
174	86054 91	186	86055 70		
175	86054 81	187	86054 97		
776	86054 54	188	86055 82	Face M	86054 87
177	86055 05	189	86055 41	Face P	86055 41
178	86055 17	190	86055 34		
179	86054 49	191	86055 28		86055 14
180	86055 23	192	86055 74		
181	86054 79	193	86055 38		
182	86054 48	194	86055 24		
183	86055 06	195	86055 24		

Pendulum No 11

Face M		Mean	
Set	V		
196	86104 54		
197	86105 40		
198	86105 82		
199	86104 80		
200	86105 29		
201	86105 23		
202	86105 25		
203	86105 15	Face M	86105 55
204	86105 17		
205	86105 80		
206	86106 16		
207	86105 24		
208	86106 04		
209	86105 69		
210	86106 40		
211	86106 87		

Colonel HERSCHEL'S observations give the following values of the differences between the vibration-numbers at Kew and Greenwich.

Kew—Greenwich $= +1\ 97$ by Pendulum No 4.
$= +1{\cdot}39$ „ No. 6 (1821).
$= -4\ 98$ „ No 11

Revisionary Results, Reduced to the Temperature of 62° F, and to a Vacuum

Mr. Constable's Determinations at Kew under Low Pressures, about 2 inches.

Pendulum No. 4

Face M		Face P		Means	
Set	V	Set	V		
4	86167 56	1	86167 12		
5	86167 07	2	86167 47	Face M	86167 31
6	86167 31	3	86167 36	Face P	86166 69
		7	86165 91		
		8	86166 53		86167 00
		9	86165 73		

Pendulum No 6 (1821)

Face M		Face P		Means	
Set	V	Set	V		
10	86066 74	13	86067 15	Face M	86066 25
11	86065 97	14	86067 00	Face P	86067 14
12	86066 04	15	86067 58		
		16	86066 83		86066 70

Pendulum No 11

Face M		Face P		Means	
Set	V	Set	V.		
17	86117 41	20	86116 82	Face M	86117 43
18	86117 11	21	86117 24	Face P	86117 17
19	86117·78	22	86117 44		86117 30

Mr CONSTABLE's Determinations at Kew under High Pressures, about 27 inches

Pendulum No 4

Face M		Face P		Means	
Set	V	Set	V		
39	86165 20	43	86165 75	Face M	86164 80
40	86164 81	44	86165 70	Face P	86165 74
41	86164 44	45	86165 73		
42	86164 75	46	86165 80		86165 27

Pendulum No. 6 (1821)

Face M		Face P		Means	
Set	V	Set	V		
31	86065 98	35	86065 80	Face M	86066 07
32	86066 22	36	86065 57	Face P	86065 54
33	86066 24	37	86065 43		
34	86065 84	38	86065 35		86065 80

Pendulum No. 11.

Face M		Face P		Means	
Set	V	Set	V		
23	86115 88	27	86115 82	Face M .	86116 14
24	86116 25	28	86115 92	Face P	86115 94
25	86116 10	29	86116 04		
26	86116 33	30	86115 97		86116 04

Mr Hollis's Determinations at Greenwich under Low Pressures, about 2 inches

Pendulum No. 4

Face M		Face P		Means	
Set	V	Set	V		
1	86165 08	5	86165 24	Face M	86165 06
2	86165 03	6	86165 25	Face P	86165 16
3	86165 09	7	86165 10		86165 11
4	86165 02	8	86164 98		
		9	86165 25		

Pendulum No. 6 (1821).

Face M		Face P		Means	
Set	V	Set	V		
10	86065 61	16	86065 60		
11	86065 47	17	86065 17	Face M	86065 64
12	86065 55	18	86065 58	Face P	86065 47
13	86065 56	19	86065 54		
14	86065 71	20	86065 51		86065 56
15	86065 91	21	86065 43		

Pendulum No. 11

Face M		Face P		Means	
Set	V	Set	V		
22	86116 73	27	86116 22		
23	86116 65	28	86116 41	Face M .	86116 68
24	86116 72	29	86116 39	Face P	86116 30
25	86116 79	30	86116 18		
26	86116 50				86116 49

MR HOLLIS's determinations at Greenwich under High Pressures, about 27 inches

Pendulum No 4

Face M		Face P		Means	
Set	V	Set	V		
39	86163 84	43	86163 99	Face M	86164 11
40	86163 98	44	86163 78	Face P	86163 97
41	86164 30	45	86164 07		86164 04
42	86164 33	46	86164 00		
		47	86164 03		

Pendulum No 6 (1821)

Face M		Face P		Means	
Set	V	Set	V		
48	86064 47	52	86063 77	Face M	86064 23
49	86064 45	53	86063 74	Face P	86063 65
50	86063 93	54	86063 55		86063 94
51	86064 09	55	86063 54		

Pendulum No 11.

Face M		Face P		Means	
Set	V	Set	V		
35	86116 07	31	86115 20	Face M	86116 14
36	86116 26	32	86115 21	Face P	86115 22
37	86116 17	33	86115 21		86115 68
38	86116 07	34	86115 26		

Thus the revisionary operations give the following values of the differences between the vibration-numbers at Kew and Greenwich —

	Low pressures	High pressures		
Kew—Greenwich	= + 1·89	+ 1·23	by pendulum	No. 4.
	= + 1·14	+ 1·86	,,	No 6 (1821).
	= + 0·81	+ 0 36	..	No 11
General mean	= + 1·22 ± ·16.			

It will be seen that the mean value is fairly in accordance with the values derived from Colonel HERSCHEL's observations with pendulums No 4 and No 6 (1821) His swings with those pendulums, at Kew, were made in the basement of the Kew Observatory, within a few feet of the spot at which the revisionary swings, with all three pendulums, were made, but he swung pendulum No 11 in a shed outside the Observatory, under circumstances of great disadvantage as regards the stability and firmness of the support of the stand of the invariable pendulum and also of the support of the clock. Thus his observations at Kew, with pendulum No 11, though generally very accordant *inter se*, are very probably burdened with a large constant error, and must therefore be rejected

On the Reduction to a Vacuum

In all pendulum experiments—even those of a purely differential character, as with invariable pendulums—it has been generally customary to apply a correction for the retardation which is caused by the air, in order to obtain results such as would have been obtained if the pendulum had been swung in a vacuum. This correction was originally determined by calculating the influence of the buoyancy of the atmosphere in diminishing the weight—and consequently the accelerating force—of the pendulum. Afterwards BESSEL showed that the correction thus obtained was too small, for the pendulum is accompanied in its oscillation by a certain amount of air, varying with its form, which increases the mass in vibration and the moment of inertia. Thus the buoyancy correction has to be multiplied by a factor, $1 + k$, which can be computed mathematically for pendulums of certain simple forms, but must be determined experimentally, by swings at high and low pressures, when the form is not susceptible of being brought under mathematical treatment The buoyancy correction, thus augmented, is usually called the pressure correction

The buoyancy correction and the pressure correction have been investigated for pendulums No 4 and No. 6 (1821) by special and laborious series of operations which are fully set forth in vol. 5 of the 'Account of the Operations of the Great Trigonometrical Survey of India.' Nothing of the kind is known to have been done for No. 11; but the results obtained for the two first pendulums are so closely accordant that they may be applied without objection to the third, which is almost identical with them in form and construction.

The buoyancy correction $= \cdot 23 \dfrac{\beta}{1 + \cdot 0023\,(t - 32^\circ)}$ in which β is the pressure in inches, and t the temperature in degrees of FAHRENHEIT

The pressure correction was found to be $32 \dfrac{\beta}{1 + 0028\,(t - 32^\circ)}$ by experimental swings which were made specifically for the purpose at Kew, under extreme high and low pressures, immediately before the pendulums were sent out to India. Corrections determined by this formula were applied, provisionally, to the whole of the

swings in India, and this has also been done to the revisionary swings at Kew and Greenwich, to produce the vibration-numbers which have already been set forth

But, in the course of the operations in India, Captain BASEVI reinvestigated both the temperature and the pressure corrections of his two pendulums, those of No. 4 with great elaboration A series of several sets of swings was made with it at each of the successive pressures of 0 6, 1 9, 4 2, 10 0, 17 5, and 27 7 inches, at the temperature of about 101° F, another series at the same pressures, at the temperature of about 53°, and a third at the pressures of 1 9 and 4 2 inches and temperature of about 80° He came to the conclusion that the pressure correction is best represented by an empirical formula of three terms,

$$A\beta^{\frac{1}{2}} + B\beta + C\beta^{\frac{3}{2}},$$

in which the second term is the correction for buoyancy Then he assumed A to be $= x\sqrt{461^{\circ} + t}$ and $C = y - \sqrt{461^{\circ} + t} - 461^{\circ}$ being the absolute zero of the air thermometer—and formed a corresponding series of equations for the determination of x and y from his fourteen sets of observations The solution of these equations gave $x = 022 \pm 002$, and $y = 123 \pm 025$, which values satisfied the equations of condition very satisfactorily

But the subject is one of great complexity and difficulty, as will be seen in consulting Chapter VI of the Indian pendulum volume Something appears to be wanting to explain the inconsistencies between vibration-numbers derived from different series of very accordant observations which are occasionally met with Possibly it may be necessary to take cognisance of the atmospheric humidity during the observations, which has never been done hitherto Or it may be that the inconsistencies arise from changes in the relative conditions of the bearing surfaces of the knife-edge and the agate planes, which are met with on successive transpositions of the pendulum, and which the observer cannot control

Transposition is almost invariably attended with a change in the vibration-number, but in the Indian operations it was found that the changes were not constant in either sign or magnitude; it is shown, at page 114 of the volume already cited, that, for the whole of the 34 stations of observation, the mean value of M − P ranges from + ·54 to − 52, and has an average value of + 07 ± 03 for Pendulum No 4, and ranges from + ·67 to − 59, with the average value − 04 ± 03 for Pendulum No 6 (1821)

In reducing the Indian swings for investigating the pressure correction, Captain BASEVI's observations of the vibration-numbers at different pressures were employed directly, without having recourse to his empirical formula The observed vibration-numbers at each pressure were reduced to a vacuum by the Kew formula with the numerical constant 32, and then the results for the higher pressures were compared with the result for the lowest pressure, which, being 0·6 of an inch, was very close to the vacuum; and it was found that the higher pressures required residual positive corrections, increasing with the temperature as well as the pressure; at the highest

pressure, 27·7 inches, the correction amounted to ·53 under the temperature of 53° and to 1·34 under the temperature of 100°. Corresponding corrections were therefore applied to the whole of the Indian swings, as they had already been provisionally reduced by the Kew formula.

This must now be done for the revisionary swings at Kew and Greenwich, the results of which, as hitherto presented, have been reduced to the vacuum by the Kew formula only. It will be seen that the vibration-numbers at the pressure of 27 inches are less than those at the pressure of 2 inches by 1·30 at Kew and 1·16 at Greenwich; but it appears from Captain BASEVI's investigations that the high pressure results at both places should be increased by about 0·7, which will reduce the discordances with the low pressure results to 0·60 and 0·46.

When pendulum operations are differential, and the variations of atmospheric pressure, at different stations, are small, the value of the correction for pressure does not require to be known with much accuracy. Reducing to a vacuum is also then unnecessary, and any convenient standard of atmospheric density may be employed instead. Thus Colonel HERSCHEL has sufficiently provided for the elimination of the effects of variations of pressure at his stations, which were all of low elevation, by reducing his swings to the standard pressure of 26 inches under the temperature of 32° F., instead of to a vacuum.

The Indian swings were invariably reduced to a vacuum, in accordance with previous procedure. They were made in an exhausted receiver, under the lowest pressure attainable, partly because this enabled them to be extended over a longer time, and thus be less influenced by hourly variations of clock-rate, than if taken under full pressure; partly because the receiver would protect the pendulum from the action of currents of air; and partly to obtain as nearly a uniform pressure at all the stations as possible, and thus secure strictly differential results; for it was intended that the pendulums should be swung both at low levels in the neighbourhood of the ocean and at the highest attainable elevations, as on the table-lands of the Himalayan mountains, where the pressure of the atmosphere is halfway down to the vacuum, so that a considerable range of pressure had to be met with and provided for in the best way possible. By exhausting the receiver, the pressures under which the swings were actually taken were generally maintained between 1 and 2 inches, excepting at first, when slight leakage was met with, the locus of which was not immediately detected, and then the pressures ranged from 1 to 4 inches. But these differences of pressure are so small that the uncertainty as to the precise amount of the pressure correction cannot exert an appreciable influence on the differential results which have already been deduced, and which are the ultimate object of the observations. And this is the case also both in Colonel HERSCHEL's and in the revisionary operations, the range of pressure being always under 2 inches, whether the swings were taken under high pressures only or under both high and low pressures.

Thus, a more exact knowledge of the correction for pressure might sensibly affect

the vibration-numbers for Kew and Greenwich which have already been presented, but it would not affect the differences between those numbers, which are what is really wanted, to a degree that is at all comparable with the errors to which pendulum observations are liable from other causes

Reduction to the Sea-level

The pendulums were swung at an elevation above the mean sea-level of about 15 feet at Kew and 157 feet at Greenwich The vibration-numbers must be correspondingly increased

The well-known formula for the correction for height is

$$\text{Correction} = V\frac{h}{r},$$

V being the vibration-number, h the height, and r the radius of the earth. Dr Young has suggested that account should also be taken of the continental mass which is situated between the level of the sea and that of the station of observation, in increasing the force of attraction and consequently the vibration-number Thus, in accordance with his views, the usually-accepted correction takes cognisance of both height and mass, and is

$$= \frac{5}{8}V\frac{h}{r}.$$

Thus for these pendulums, when h is expressed in feet, we have

$$\text{Correction for height only} = \frac{h}{243},$$

$$\text{Correction for height and mass} = \frac{h}{391}$$

It is now, however, very questionable whether any reduction for mass is allowable The pendulum operations in India have thrown much light on the constitution of the earth's crust, and shown that there is a marked deficiency of density under the Himalayan mountains, and an increase of density under the bed of the Indian Ocean. Thus continental matter above the sea level may be conceived as appertaining to the strata underneath, immediately below the sea level, which are correspondingly attenuated. In this case, the excess of matter above would probably compensate for the deficiency of matter below, and would not form an attracting force to be independently allowed for while no cognisance is taken of the deficiency below.

If the vibration-numbers at Kew and Greenwich are corrected for height only, the correction to be applied to their difference will be

$$= \frac{15 - 157}{243} = -0{\cdot}58,$$

whence we obtain

$$\text{Kew—Greenwich} = 1{\cdot}22 - 0{\cdot}58 = 0{\cdot}64$$

as the result of the revisionary operations.

Addendum

Results of the swings at other stations which were taken with the same pendulums, by Colonel Herschel, and by Mr Edwin Smith of the United States Coast and Geodetic Survey They are reduced to the temperature of 62° F, and to the density of air under the pressure of 26 inches at the temperature of 32° F , but they are not reduced to the sea-level.

Swings by Colonel Herschel

Stations	Height in feet above sea	No 4	No 6 (1821)	No 11
London,* Langham Place, Cellar	85	86157 31	86055 97	86109 59
Washington, Smithsonian Institute	34	86109 45	86008 35	86060 93
Hoboken, Stevens Institute	30	86115 23	86014 06	86066 93
Whence				
Greenwich—London =		+ 1 64	+ 0 83	+ 4 05
London—Washington =		+ 47 86	+ 47 62	+ 48 66
Washington—Hoboken =		− 5 78	− 5 71	− 6 00

Swings by Mr Edwin Smith

Stations.	Height in feet	No 4	No 6 (1821)	No 11
Auckland	261	86102 75	86002 11	86054 13
Sydney	140	86090 93	85990 32	86042 08
Singapore	45	86021 13	85919 97	85971 34
Tokio†	20	86099 83	85995 17	86046 91
San Francisco	375	86103 77	86003 13	86055 66
Washington	34	86109 31	86009 29	86061 41
Whence				
Auckland—Sydney =	.	+ 11 82	+ 11 79	+ 12 05
Sydney—Singapore =		+ 69 80	+ 70 35	+ 70 74
Singapore—Tokio =		− 78 70	− 75 20	− 75 57
Tokio—San Francisco =	.	− 3 94	− 7 96	− 8 75
San Francisco—Washington =	.	− 5 54	− 6 16	− 5 75

* This station is at No 1, All Souls Place, Langham Place, it is about 380 feet S E. of, and 14 feet lower in level than, Mr Browne's house in Portland Place, which was Kater's station The value of Greenwich—London (Portland Place) = + 0 48, with Invariable Pendulum No. 12, it was determined by Sabine in 1828, see 'Phil Trans.' for 1829, p. 87.

† Mr. Edwin Smith remarks of the observations at Tokio, that "the vibration-number of Pendulum No 4 is between three and four vibrations too great I can only explain this discrepancy by the supposition that some foreign material was adhering to the pendulum during these observations. Great care was always taken in wiping the pendulums before suspending them"

IX. *On the Alleged Slipping at the Boundary of a Liquid in Motion*

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IN treatises on hydrodynamics, the flow of a liquid through a straight tube is investigated on the supposition that there may be finite slipping between the walls of the tube and the outermost layer of liquid This leads to the introduction of a "slipping coefficient" which vanishes when there is no relative motion between them

Let r denote the radius of the tube,

p_1 the pressure at one end,

p_2 ,, ,, the other,

μ the coefficient of viscosity,

l the length of the tube,

ρ the density of the liquid.

Then it may be shown (LAMB'S 'Hydrodynamics, p 222) that when the motion is linear the flux is given by

$$\tfrac{1}{8}\frac{\pi r^4}{\mu\rho}\frac{p_1 - p_2}{l} + \tfrac{1}{2}\frac{\pi r^3}{\beta}\frac{p_1 - p_2}{l}$$

or

$$\tfrac{1}{8}\frac{\pi (p_1 - p_2)}{\mu\rho l}\left\{r^4 + 4\mu\rho\frac{1}{\beta}r^3\right\},$$

where $1/\beta$ may be defined as the slipping coefficient

The experiments of POISEUILLE* showed that the coefficient was certainly zero for glass tubes, but there was doubt whether this held for all materials.

HELMHOLTZ and PIOTROWSKI† attacked the problem in another way. They suspended bifilarly an accurately worked sphere, whose inner surface was gilded and polished, and by observing the time of swing and the logarithmic decrement when the sphere was filled with water and various other liquids, deduced a value for the

* 'Mémoires des Savants Étrangers,' 1846

† 'Sitzungsber der k Akad in Wien,' vol 40, 1860

coefficient of viscosity and for the slipping coefficient, from the theory of spheres oscillating in a viscous medium, as worked out by STOKES and HELMHOLTZ

For distilled water their value of the factor $\mu\rho\ 1/\beta$ in the above expression is $\lambda = 2\ 3534$ mm

If we apply this to the case of a tube we get a somewhat startling result From equation (i) it follows that the effect of slip varies inversely as the radius of the tube. The smallest tube practicable in the experiments to be presently described had a diameter of about a millimetre It is easy to show that the result of the existence of a slipping coefficient of the magnitude given by HELMHOLTZ would be to produce an increase in the volume of liquid flowing through the tube in a given time, which could not only be detected, but would be of such importance that it could not easily be masked

In HELMHOLTZ's notation equation (i) is written

$$\frac{1}{8}\frac{\pi(p_1 - p_2)}{\mu l}\{r^4 + 4\lambda r^3\},$$

the density of water being taken as unity

Putting $r = 05$ and $\lambda = 23534$, we get

$$\frac{1}{8}\frac{\pi(p_1 - p_2)}{\mu l} \times 117\ 67 \times 10^{-6},$$

whereas if there is no slip, so that λ vanishes, the flux becomes

$$\frac{1}{8}\frac{\pi(p_1 - p_2)}{\mu l} \times 6{\cdot}25 \times 10^{-6}.$$

If we take HELMHOLTZ's coefficient to be correct, the flow through a polished gilt tube of a millimetre in diameter is nearly *twenty times* as fast as through a glass tube of the same size

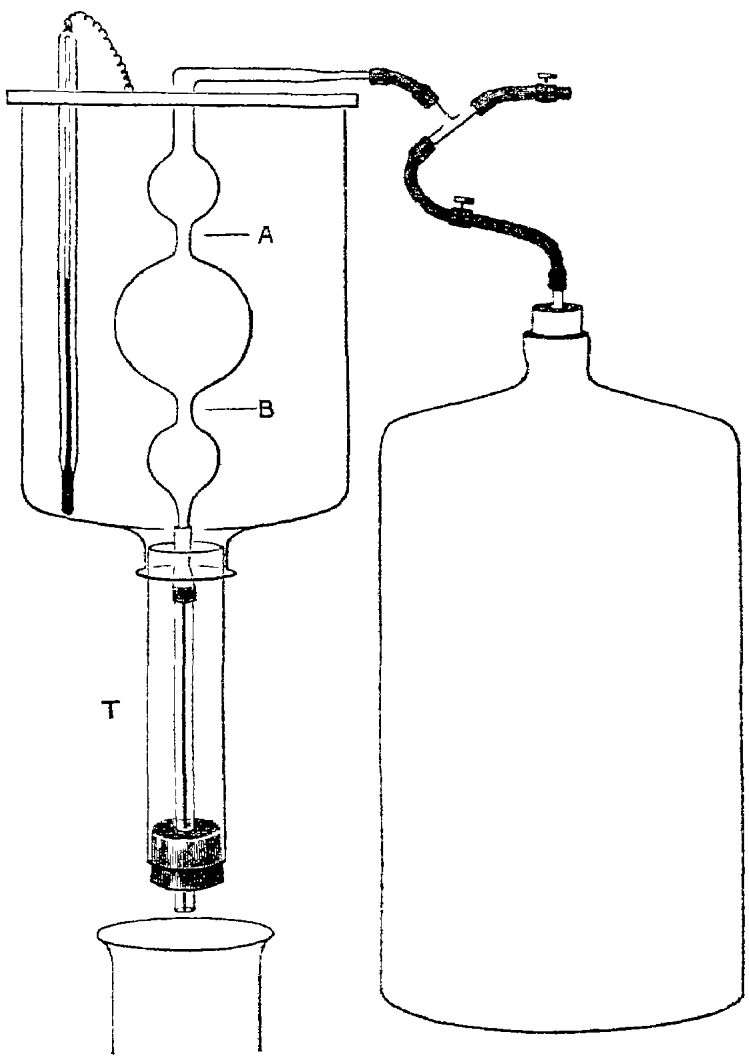
HELMHOLTZ refers to some experiments made by GIRARD with copper tubes* which make the flow some four times faster than do POISEUILLE's formulæ I shall return to the consideration of these observations later. The value of λ which HELMHOLTZ deduces from them is 0·3984 mm.

The discrepancy between these results and the generally received opinion that no slip occurred with any material seemed worthy of further investigation

It is evident that the alleged coefficient could be investigated with much greater advantage by observing the flow of liquid through a small tube than by any experiments on oscillating spheres. In order to avoid all absolute determinations while searching for the existence of such a slip, I decided to observe the time of flow of a

* 'Mémoires de l'Institut,' 1813–1815

given volume of water through a glass tube, and then to deposit a coating of silver on the interior surface of the tube If the time of flow was the same as before, allowing for the (usually very small) change in diameter, it would be conclusive evidence against the existence of the effect. Such evidence I have most satisfactorily obtained



Since the experiments were to be merely comparative there was no object in attempting to keep the pressure constant throughout each observation, and the simplest apparatus could be used The bulb tube AB was fixed in a large glass jar filled with water, and opening below into a wide tube T, also filled with water. The lower end of the bulb tube was attached to the capillary tube by an india-rubber joint, so that the ends of the two glass tubes should just meet The top of the bulb tube could be put into connexion either with the air outside, or with an exhausted bottle which was used to fill the bulbs, as the apparatus was some distance from a pump. The temperature of the water in the glass jar and wide tube could be read off by a delicate thermometer.

The time taken by the upper surface of the water to fall from A to B was observed by a stop watch. Before silvering the tube the whole apparatus was repeatedly taken to pieces and set up again, and the time of flow shown to be unaltered.

The silvering solution was a modification of LIEBIG'S, and was made according to a

recipe kindly given me by Dr A S LEA Each tube was dried and weighed, and the silvering solution then run through till a bright metallic mirror was deposited It was washed out with a current of water, then with air, and finally dried and weighed again The increase gave the weight of silver At the end of the experiments with the tube thus silvered, it was again dried and weighed, and the silver dissolved off with nitric acid The mean of these two results, which usually agreed to one or two tenths of a milligram, was taken to represent the weight of silver adhering to the tube during the experiments Assuming the deposit to be uniform, this at once gave the change in diameter. The correction for temperature was calculated by POISEUILLE's formula

The first tube had a length of 37 57 cms., and an average radius (determined by filling with mercury) of 0 0451 cm

The following series of experiments were made —

Temperature	Time of flow.
18° 2	7′ 50″ 8
18 3	7 50 4
18 3	7 50 6
18 2	7 50 4
18 3	7 51 0
18 3	7 51 0
Means 18 26	7 50 7

The apparatus was then taken completely to pieces as it would be for silvering, and again set up after some hours, with the following results —

Temperature	Time of flow
17° 8	7′ 55″ 4
17 9	7 55 2

If we correct this to 18° 26 by POISEUILLE's empirical formula we get 7′ 51″ 0, a value identical with the above. The apparatus could thus be taken to pieces with safety.

Weight of tube when dry . . . . . . = 16 3916 grams
" with silver . = 16 3932 "

The apparatus was then again set up

Temperature	Time of flow
18° 4	7′ 52″ 6
18 3	7 52 2
18 2	7 52 4
Means 18 30	7 52 4

The tube was then disconnected, dried, and weighed Weight = 16 3931 grms, practically the same as before The silver was then dissolved off, and the tube cleaned and dried Weight = 16 3916

The weight of silver deposited is, therefore, 0015 grm, and its thickness 000014 cm

This gives a change in r^4 equivalent to 0 12 per cent The time of flow for the unsilvered tube, corrected for change of radius, is 7′ 51″ 4, while the observed time for the silvered tube is 7′ 52″ 4 The difference of about 0 2 per cent is probably due to slight irregularities in the thickness of the silver layer

Two objections may be raised to this experiment The first is that with such a thin film, the action between the water and the glass might still be effective, and prevent any slipping When we remember, however, that the sphere of action of molecular forces is only about 10^{-8} cm, we see that no direct action can occur across a distance of 10^{-5} cm, and it is exceedingly unlikely that a layer of silver more than 1000 molecules thick, should be pervious to water, and thus allow of contact with the glass In order, however, to entirely meet this objection, experiments were made with considerably thicker layers

The second objection is that the silver might be deposited so irregularly that the choking effect might mask the quickening due to slip It had been found, in a series of preliminary experiments, that if a tube was used whose diameter was much less than a millimetre, it was exceedingly difficult to get a uniform silver deposit and the time of flow for the silvered tube was always much greater than for the unsilvered To prevent this the tube had to be silvered in a vertical position, and various details, only to be learnt by experience, attended to The time of flow for the silvered tube could, however, never be brought below that for the plain one, although as greater care was taken in the silvering it continually approximated to it, and this is conclusive against the existence of an effect at all comparable with that given by HELMHOLTZ for polished gold, or with that which he deduced from GIRARD's experiments on copper tubes As the choking effect was naturally greater for small tubes, a series of experiments was next made on some of rather greater diameter.

Tube No III.

Unsilvered		Silvered	
Temperature	Time of flow	Temperature	Time of flow
20 0°	1′ 2″2	20 9°	1′ 1″8
20 1	1 2 0	20 9	1 1 3
20 1	1 2 2	20 95	1 1 2
20 2	1 2 0	21 0	1 1 8
20 2	1 1 8	21 0	1 1 4
20 3	1 1 7	21 0	1 1 5
20 4	1 1 7	21 0	1 0 9
20 5	1 1 6	21 05	1 1 3
20 5	1 1 7	21 05	1 1 0
20 5	1 1 5	21 05	1 0 8
20 5	1 1 6		
20 6	1 1 5		
20 3	1 1 79	21 0	1 1 37

Thickness of film = 0000237 cm.
Radius of tube = 0776 cm.
Change in r^4 = 0 12 per cent.
Temperature correction = 7 × 2 22 = 1·55 per cent.
Total correction 1 55 − 12 = 1 43 per cent = 0·87 second
Time for glass tube, corrected = 1′ 0″ 92
,, observed for tube, silvered = 1′ 1″ 37
A change of + 0 7 per cent.

A rather smaller tube was then used, r = 0642.

	Unsilvered			Silvered	
	Temperature	Time		Temperature	Time
1	22 0°	2′ 8″4	1	20 2°	2′ 13″0
2	22 0	2 8 2	2	20 2	2 13 1
3	22 05	2 8 4	3	20 2	2 12 9
4	22 05	2 8 3	4	20 2	2 13 4
5	22 1	2 8·4	5	20 25	2 13 0
6	22·15	2 8 0	6	20 3	2 13 0
7	22 2	2 8 0	7	20 4	2 12 8
8	22 2	2 8 2	8	20 4	2 12 6
9	22·2	2 8 0	9	20 45	2 12 4
10	22 25	2 7 8	10	20 5	2 12 4
			11	20 5	2 12 8
			12	20 5	2 12 4
Means	22·12	2 8·17	Means	20 34	2 12 82

	′	″
Time for unsilvered tube, corrected for changes in temperature and radius .	2	13 59
Time observed for silvered tube . .	2	12 82

A difference of − 0 6 per cent

Another series was made with the same tube which gave as the mean of ten observations for each state —

	′	″
Unsilvered .	2	18 78
„ corrected . .	2	15 92
Silvered, observed .	2	16 51

A difference of + 0 4 per cent The silver was very thin the change in r^4 being 0 10 per cent

Thus, as the result of four series of observations with three different tubes, we have that the difference in the times of flow for the silvered and unsilvered tubes is never greater than 0 7 per cent With both the first and second tubes, and in one series of observations with the third tube, the time of flow is slightly greater (by 0 2, 0 7, and 0 4 per cent) for the silver surface, while in one case—the first series of observations with the third tube—the time is slightly less (by 0 6 per cent)

These differences are all within the limits of experimental error, found by comparing the times of flow for the same tube in the same state on different occasions

On the whole there is some evidence that the time is a little greater for the silvered surface, as would, of course, be the case if the deposit was not quite strictly uniform. In the one case, when the time was less, the temperature difference was largest, and errors likely to be most important.

These experiments may at any rate be considered conclusive against the existence of the large effect, for the existence of which I was searching.

A new series of observations was then undertaken to determine whether any slipping occurred in a silvered tube when the velocity of the water was greater than before, and the gradient of velocity was pushed near the limit beyond which the motion ceased to be linear.

This limit was calculated for each tube by means of a formula given by Professor OSBORNE REYNOLDS,* who found by experiment that in order to insure linear motion,

$$Dv\rho/\mu \text{ must be } < 1400,$$

where D is the diameter of the tube, μ the coefficient of viscosity, ρ the density, and

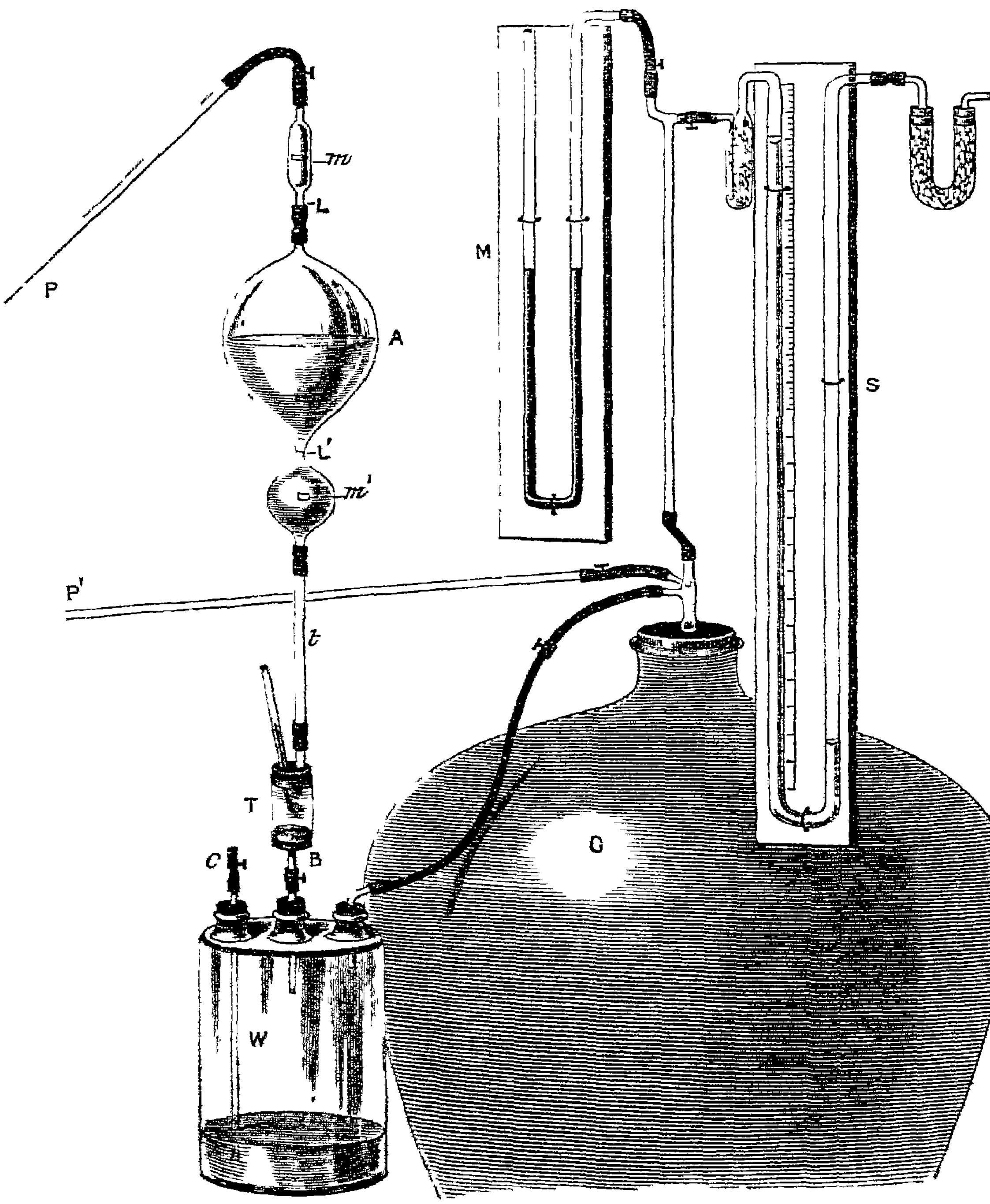
* 'Phil. Trans.,' 1886.

v the velocity of the water calculated from the quantity which flows through in a given time.

It was advantageous to use as small a tube as possible for two reasons firstly, because any slipping effect is inversely proportional to the radius, and secondly, because the gradient of velocity can be pushed farther without exceeding the limits of linear motion, the smaller the tube

A tube, whose diameter was 084 cm, and whose length was 27 30 cms, was therefore taken, as it was the smallest which could conveniently be silvered, and the greatest allowable pressure calculated

From the relation given above, we find that the pressure must not exceed $5600\mu^2 l/r^3 g = 303\ 5$ cms of water column



After many preliminary trials an apparatus was set up, which gave most excellent results.

The bulb had a capacity of about 850 c c, and was fixed to a wooden frame to prevent breakage, this was screwed to the edge of a table. The water bath was eventually abolished, and the temperature of the water read off at intervals during each experiment as it passed through a broad tube, into which the capillary tube opened The temperature of the water was thus read off immediately after it had passed the important place, and by taking readings at equal intervals while the bulb was emptying, a very accurate estimation of the average temperature could be obtained Below the temperature tube was a three-necked WOLFF's bottle W, the second neck was connected to a large carboy C, and the third could be put into communication with the atmosphere The carboy was connected to a three-way tube, the branches of which went, one to the WOLFF's bottle, one to an air pump P′, worked by the water supply, and one to the gauges S and M The gauge S contained sulphuric acid, and M contained mercury The latter was only used for the experiments in which higher pressures were employed* The bulb was filled by disconnecting the joint B, and putting the lower orifice of the temperature tube in communication with c, the tube joining C and W being stopped P was then put to the pump, and the water in W sucked up. When the bulb was filled, B was again connected, the carboy exhausted by putting P′ to the pump and the tube from C to W opened The pump was worked till the requisite pressure, as shown by the gauge, was reached

The apparatus was then left till the temperature had become constant after the disturbances produced by exhausting, and the height of the gauge read off by a kathetometer The pressure could be adjusted to within a millimetre or less by regulating the pumps, and small differences in corresponding experiments, were given by the readings of the kathetometer The correction to be applied to the times of flow for a given small difference of pressure, was determined by observing the actual times of flow for pressures whose difference was considerably greater than that in the experiments to be compared, and keeping all other things unchanged The small pressure correction could then be accurately estimated from the result of this auxiliary experiment. As before, the experiments were to be only comparative, and the pressure was allowed to fall during each observation Any change introduced by this would affect the tube equally whether plain or silvered In order to show the method of working and the degree of accuracy obtained, a complete account of an experiment is given in detail

* All permanent joints, corks, &c, were thickly covered with marine glue, and were quite air-tight

Tube No 3.

9. *Time.*—Start 0′ 0″, finish 15′ 23″ 0 15′ 23″ 0
Pressure —Gauge scale, adjusted at start to 22 00, reading at finish 21 47.

$$\text{Kathetometer readings—Start} \left\{\begin{matrix} 22\ 680 \\ 22\ 583 \end{matrix}\right\} 22\ 682 \left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad 22\ 956$$
$$\text{Finish} \left\{\begin{matrix} 23\ 229 \\ 23\ 230 \end{matrix}\right\} 23\ 230$$

Temperature readings at intervals of two minutes—

13·00 12 88 12·86 12·90 12·92 12 93 12 95 12 98 12° 93.

In order to make the pressure readings strictly comparable, the level of the water before each experiment was adjusted to a mark, *m*, in the top bulb, and after each to a mark *m′* in the lower bulb, and all pressure readings were taken while the water stood at these levels The pressure was adjusted by the pump till the reading on the gauge scale was as nearly as possible 22 00, and the exact height then read off by a telescope in terms of kathetometer scale

Tube No 1.—($r =$ ·042 cm ; $l =$ 27 30 cms)

Total difference of pressure equivalent to about 250 cms. of water column. The limiting pressure is 305 cms.

	Pressure.	Temperature	Time
Unsilvered.	mm		
1. Gauge reading . .	407·1	12°59	12′ 25″ 7
2 " " .	407 0	12 71	12 24 1
Silvered			
3 Gauge reading	407 1	12 79	12 23 5

The temperature correction is 2·22 per cent. for 1° C , and in order to correct (2) to 12°·79, we must subtract 1″·4.

The pressure correction is for 0·1 mm. of sulphuric acid in a total pressure equivalent to about 1400 mm , *i.e.*, 1 in 14,000, and is therefore negligible

The weight of silver deposited is 0 0008 grm , and the thickness 0 00001 cm., a change in r^4 of 0·1 per cent.

The time for the unsilvered tube corrected = 12′ 23″ 4
" " silvered " observed = 12 23·5

At slightly different pressures—

	Pressure	Temperature	Time
	mm	°	′ ″
Silvered	406 4	12 52	12 29 7
Unsilvered	406 5	12 62	12 24 3
Unsilvered, corrected for temperature and change in r^4			12 26 4
Silvered, observed time			12 29 7

Tube No 3—($r =$ 036 cm , $l =$ 19 98 cms)

Total pressure at beginning of each observation 84 15 cms of water + 21 028 cms. of mercury , equivalent to 368 62 cms of water. The critical pressure for this tube is 440 cms of water. The tube was silvered with solutions of half strength very carefully.

	Mean reading of kathetometer	Temperature	Time
Silvered	22 963	12°70	15′ 46″ 5
	22 956	12 93	15 23 0
	22 955	13 41	15 18 0
	22 966	12 76	15 34 0
	22 960	12 95	15 30 4
Unsilvered	22 959	13 04	15 16 5
	23 004	13 69	15 7 3
	22 982	13 37	15 11 9
Silvered, corrected for change in radius — 0 16 per cent , for temperature — 0 93 per cent , and for pressure + 0 14 per cent			15 21·5

In this very small tube there is thus a choking effect which increases the time by about 1 per cent

The same tube was then re-coated with a very thin deposit which was just transparent to blue light.

Pressure	Temperature	Time
mm	°	′ ″
23 026	15 31	14 48 2
23 037	15 27	14 50 8
23 032	15 29	14 49 5

		Time	
		′	″
Unsilvered.—Pressure 23 005 mm, temperature 15° 64		14	43 2
,,	Corrected for temperature and pressure	14	51 7
Silvered.—Observed time		14	49 5

A decrease of 0 25 per cent

The change in radius was inappreciable

Thus the result of four series of experiments at these large differences of pressure is that in three cases the time for the silvered tube comes out slightly greater [by 0 014, 0 44, and 1 0 per cent], and in one case slightly less [by 0 25 per cent]

This agreement may be considered a quite satisfactory proof of identity.

In support of his view that there is a finite slipping coefficient HELMHOLTZ refers to some experiments of GIRARD,* who examined the time of flow of water through copper tubes, found that the motion was linear within limits which agree fairly well with those given by REYNOLDS' formula, but got times of flow much less than those observed by POISEUILLE for glass tubes. Thus with a tube whose diameter was 1 83 mm and length 1790 mm, a quarter of a litre of water flowed through in 624·5 secs under a pressure of 100 mm of water and at a temperature of 0° 5, while POISEUILLE's formula gives 2949 secs

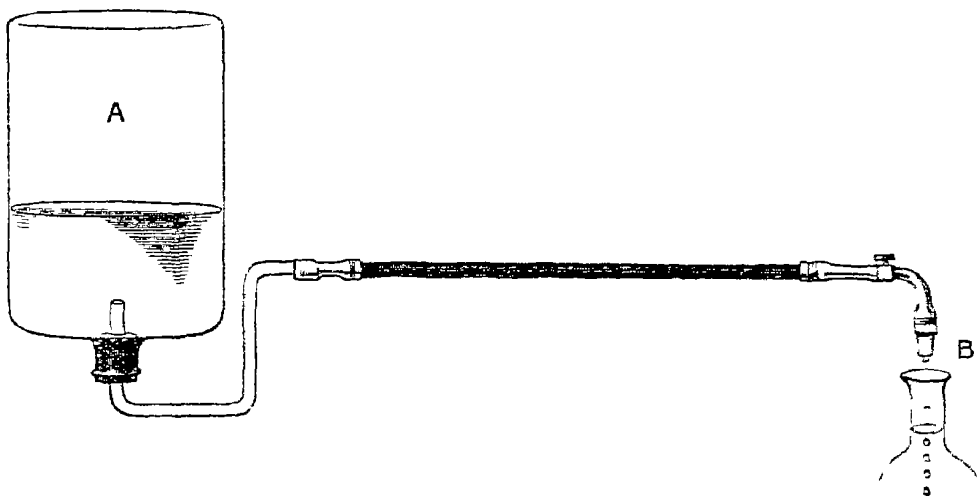
As I could detect no error in GIRARD's account of his experiments I determined to repeat them Messrs ELLIOTT, of Sellyoak, Birmingham, most kindly undertook to manufacture some solid drawn copper tubes of the dimensions required, and they were entirely successful. The apparatus employed was essentially the same as that used by GIRARD, and was of the simplest possible nature The results depended on the value obtained for the diameters of the tubes, as determined by weighing when empty and when full of water, and as no allowance could be made for irregularities or non-uniformities in the bore, calibration being impossible, it was useless to determine the time of flow or the temperature to any great degree of accuracy.

A glass jar was arranged in the manner shown in the figure, and the difference of level between the surface of the water in A and the orifice B determined by a kathetometer to a tenth of a millimetre. Below B was placed a 100 c.c flask, and the time taken to fill this was observed with a stop watch to an accuracy of about 1 sec. in 400 or 600

The temperature was observed in the jar A and also in B, and the two seldom differed by more than a tenth of a degree. In GIRARD's investigations the level of the water in A was allowed to fall during each experiment, and its mean value assumed to represent the effective driving pressure throughout. In order to determine whether such an arrangement was allowable for these small pressures, where the change was a large fraction of the whole, a comparison was made with a series of

* 'Mémoires de l'Institut,' 1813–1815

experiments, during which the level was kept constant by allowing water to flow into A at the same rate as it flowed out.



A glass tube was first used ($r =$ ·0568 cm , $l =$ 22 53 cms)

Pressure constant

Level of reservoir, 19 30 ; „ orifice, 30 23 . . .	10 93 cms
Temperature, 18° 1 , 18° 1	18° 1
Time start, 6′ 0″ , finish, 17′ 28″ .	11′ 28″
Value of the coefficient of viscosity μ deduced from this by POISEUILLE's formula	01339

Pressure varying

Level of reservoir { Start, 18 47 ; Finish, 20 22 } 19 35 ; „ orifice . 30 24 .	10 69 cms
Temperature, 18° 1 , 17° 9 . . .	18° 0
Time . start, 8′ 0″ , finish, 19′ 32″ . .	11′ 32″
Value of μ	01341

A copper tube was then substituted.

Tube No 5.—($r =$ 0836 cm , $l =$ 30 86 cms)

Pressure constant.

Levels, 20 05 , 26 57	6·52 cms.
Temperature, 16° 4 ; 16° 4	16°·4
Time · start, 0′ 0″ ; finish, 5′ 51″	5′ 51″
Value of μ	01391

Pressure varying

Levels $\left\{\begin{matrix}18\ 34\\20\ 06\end{matrix}\right\}$ 26 57	7 37 cms
Temperature, 16° 4 , 16° 6	16° 5
Time, 6′ 0″ , 10′ 37″	4′ 37″
Value of μ	01241
Levels $\left\{\begin{matrix}20\ 06\\21\ 78\end{matrix}\right\}$ 26 56	5 64 cms.
Temperature, 16° 5 , 16°·5	16° 5
Time, 1′ 0″ , 7′ 7″	6′ 7″
Value of μ	·01258

Pressure constant.

Levels, 23 42 ; 26 56	3 07 cms
Temperature, 16° 5 , 16° 6	16° 6
Time, 7′ 0″ , 18′ 33″	11′ 33″
Value of μ	01293
Levels, 21 56 , 26 56	5 00 cms.
Temperature, 16° 6 ; 16° 6	16° 6
Time, 0′ 0″ ; 6′ 54″	6′ 54″
Value of μ	01258

Pressure varying

Levels $\left\{\begin{matrix}21\ 78\\23\ 48\end{matrix}\right\}$ 26 56	3 93 cms.
Temperature, 16°·6 ; 16°·6	16° 6
Time, 7′ 0″ ; 15′ 59″	8′ 59″
Value of μ	·01288

Thus the experiments, both with the glass and with the copper tube, show that the time of flow is the same if the pressure be allowed to fall, as it is if the pressure be kept constant at the mean value of the falling pressure.

The results also show that the copper tube gives a value for μ practically identical with that given by the glass tube, and a little greater than that given by POISEUILLE's experiments, instead of about five times less

The effect of modifying the interior surface was then investigated Tubes were cleaned with acids and alkalies, polished with emery powder, coated with a film of oil, and amalgamated with mercury.

Tube No 1 —($r =$ ·0803 cm , $l =$ 30 9 cms).

Average pressure	Temperature	Time of flow	Value of μ
cms.	°	′ ″	
16 825	13 2	2 46 0	01448
15 320	16 6	3 3 3	01456

The tube was then cleaned with dilute nitric acid The radius was re-determined, but was only changed by 0 08 per cent

Average pressure	Temperature	Time of flow	Value of μ
cms	°	′ ″	
17 215	14 0	2 37 5	01406
16 37	14 6	2 42 8	01382

Cleaned with hydrochloric acid and potash

Average pressure.	Temperature	Time of flow	Value of μ
cms	°	′ ″	
17 86	13 7	2 32 5	01412

Another tube was then taken and polished inside by working it along a stretched string, covered with fine emery powder.

Tube No 2, polished with emery powder —($r =$ ·08094 cm , $l =$ 23 30 cms.)

Average pressure	Temperature	Time of flow	Value of μ
cms.	°	′ ″	
16 43	13 0	2 9 5	·01510
14 69	13 0	2 23 6	01497

The inside was then coated with a film of oil The radius was re-determined, $r =$ ·08003 cm.

Average pressure	Temperature	Time of flow	Value of μ.
cms.	°	′ ″	
16 56	13 6	2 13	01494
15 07	13·6	2 27	01502

Tube No 5 was then cleaned with acid and amalgamated by leaving it for some time filled with mercury, and running a stream of mercury through several times

The radius was re-determined, $r = 0833$ cm

Average pressure	Temperature	Time of flow	Value of μ
cms.	°	′ ″	
3 36	15 4	10 56	01324
3 35	15 5	10 54 5	01317
4 94	15 7	7 12	01282

In the experiments with the same tube described above, when the surface was copper, the following results were obtained at similar pressures —

Average pressure	Temperature.	Time of flow	Value of μ
cms	°	′ ″	
3 07	16 6	11 33	01293
3 93	16 6	8 59	01288
5 00	16 6	6 54	01258

Thus in none of these experiments does the value of μ differ much from that given by POISEUILLE for glass tubes, but, like his, agrees with the formula deduced from the supposition that no slip occurs. In all cases it is slightly greater, which is readily explained by irregularities in the tubes, owing to the difficulty of drawing them. According to GIRARD's results, the value of μ should have about a quarter of the value given by POISEUILLE, but in none of the experiments described in this paper did it fall below POISEUILLE's value, and more decisive still, no change in the nature of the surface changed the rate of flow, this is purely a comparative method, and seems much more reliable than the absolute method of GIRARD, which depends on accurate measurements of the radii of the tubes, differences in pressure, &c GIRARD only used tubes of two sizes, and gives no account of the means he employed to estimate their radii. At the same time it should be noticed that his values for the two sizes agree fairly between themselves with the supposition that a slipping coefficient exists, whose value is about 0 4 mm. Any constant error in the estimation of the radius, would however be naturally of greater importance in the smaller tube and may have led to the apparent agreement with the results of an effect, inversely proportional to the radius, and due to the existence of a finite slipping coefficient.

We must now return to the consideration of the experiments of HELMHOLTZ and PIOTROWSKI.

The discussion of the body of their paper I must leave to those with the requisite mathematical knowledge, merely observing in passing that it is remarkable that the value they deduce for the coefficient of viscosity of the liquid itself is considerably

greater (by about one-fourth) than that given by POISEUILLE's experiments. This seems to suggest that some slight modification in the application of the formulæ may be necessary, which will reduce the value deduced for the viscosity of the liquid, and increase that for its adhesion to the vessel to the value requisite for the condition of no slip.

By a preliminary series of experiments PIOTROWSKI claims to have shown that the friction on a body oscillating in contact with a liquid depends on the nature of the surface. He suspended a glass flask bifilarly, filled it with water, and observed the time of swing and the logarithmic decrement. He then silvered the inner surface and repeated his observations. The results are as follows:—

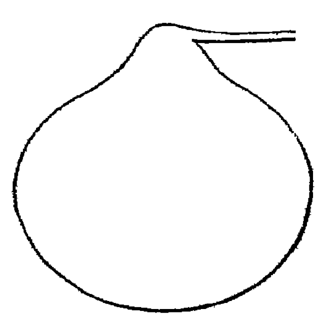
	Time of swing	Logarithmic decrement
Unsilvered	23·9333	0·0622182
	23·9333	0·0628467
	23·9333	0·0625325
Silvered	24·0088	0·0600305
	24·0076	0·0599622
	24·0082	0·0599964

As the result of these observations, PIOTROWSKI calculates that the ratio of the friction on glass to the friction on silver is as 1 · 95645.

Independently of the fact that no account is given of any precautions to keep the temperature constant, or even to measure it, it is evident that the above determination is liable to errors due to changes in the suspension, which are very apt to occur, and that the agreement between the pairs of readings is not very close.

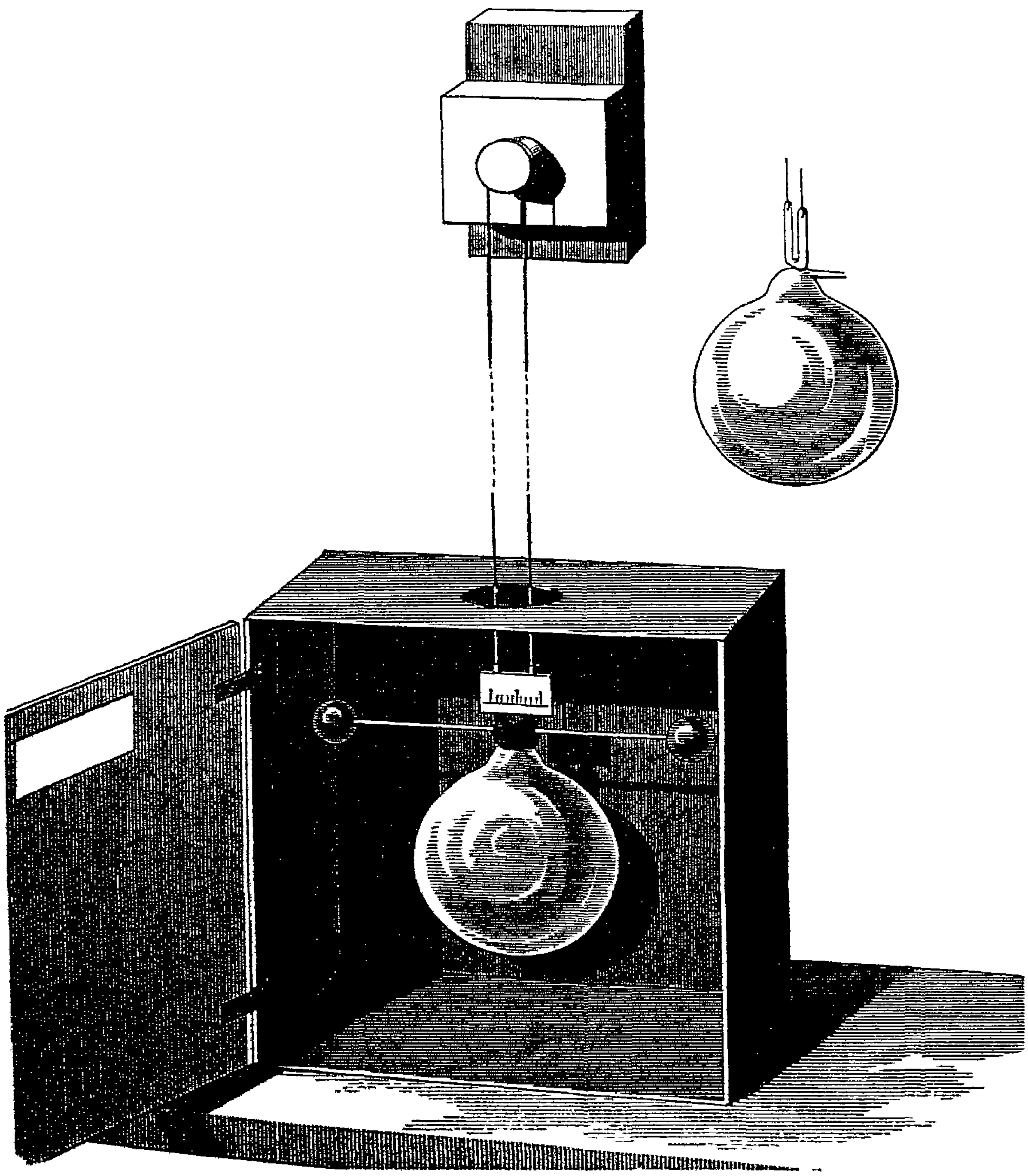
However, in order to test whether such an effect were appreciable, I undertook a series of experiments with an apparatus similar to that used by PIOTROWSKI.

A glass bulb was blown as nearly as possible spherical, and the neck drawn off



sideways into a fine tube. It was filled with water by means of an air pump, and always kept completely full; when left, a piece of india-rubber tubing filled with water was attached, so that if the temperature of the room sank, water, and not air was drawn in. During working the temperature was always slowly rising, and before

each observation the drop of water was removed by blotting paper At any given temperature the apparatus was therefore in a definite state, and this was obtained at much less expenditure of time and patience than if the bulb had been always filled to a certain mark at a certain temperature, or the same mass of water always put in by adjusting the weight A mirror was attached to the bulb by sealing wax, and the whole suspended bifilarly by a fine copper wire. The logarithmic decrement had to be reduced by suspending two brass balls at the end of a long bar magnet which was fixed to the bulb and by means of which the apparatus was set in oscillation



A series of preliminary observations gave as the ratio of the frictions 1 : 1·0001, an accuracy in the proof of identity which was not justified by the roughness of the observations, but which, at any rate, showed that the difference could not be very great.

After a week spent in preliminary investigation, the apparatus was set up in the

manner which had been found to answer best, and a series of observations taken one of which is given in detail —

TIMES OF TRANSIT

32' 1''·5	10''·5	19	28	37	45·5	54·5	3·5	12	21	30	38·5
35' 6''·5	15''·5	24	32·5	41·5	50	59	8	16·5	25·5	34	43

Time of 1 vibration = 8''·794

Logarithmic decrement.—Zero 1·0, at end 1·0 Readings at one end of swing —

42·2	36·7	32·0	27·9	24·3	21·2	18·5	16·16
14·17	12·40	10·90	9·58	8·44	7·44	6·60	5·85
5·20	4·63	4·16	3·74	3·37	3·06	2·77	

∴ Logarithmic decrement = ·14273

The bulb was kept in a beaker of water whose temperature could be easily observed, till just before each observation, when it was rapidly dried with blotting paper, allowed to come to rest, and set oscillating by means of a strong bar magnet. The limit of each swing was read by means of a telescope mounted at the centre of a curved scale at about two metres distance, and the times of transit over the centre of the scale taken by a chronometer

The following determinations were made —

	Temperature	Time of swing	Logarithmic decrement
Unsilvered	13·5	8″·781	·14224
	13·6	8·783	·14063
	13·6	8·787	·14115
	12·8	8·801	·14269
	12·9	8·796	·14328
		8·806	·14276
	13·0		·14204
			·14278
	13·1	8·798	·14305
			·14307
			·14289
	13·2	8·794	·14273
			·14192
			·14125
	13·8	8·732	·14066
	14·0	8·736	·14052
	14·0	8·740	·14094
	14·2	8·732	·14106
	14·2	8·750	·14093
			·14090
Silvered	12·6	8·764	·14338
			·14310
	12·6	8·768	·14274
			·14380
	12·9	8·837	·14304
	13·1	8·810	·14270
	14·1	8·829	·14232
	13·9	8·828	·14125
	13·8		·14191
	14·0	8·806	·14143
	14·1	8·808	·14115
			·14121
Unsilvered	14·0	8·823	·14167
	14·3	8·830	·14126
	14·4		·14101

If we take the means of these observations we get —

	Temperature	Time	Logarithmic decrement
Unsilvered	13°·66	8″·779	·141801
Silvered	13·40	8·806	·142335

From the series of observations with the unsilvered flask we find that the alteration in the logarithmic decrement for a change in temperature of 1° C is about ·00160. For 0°·26 the change will be ·000416, and the logarithmic decrement of the unsilvered bulb corrected to a temperature of 13°·40 is ·142217

By HELMHOLTZ and PIOTROWSKI's paper we see that the ratio of the friction on silver to the friction on glass is as 1 $\frac{8 \cdot 806 \times 142217}{8 \cdot 779 \times 142335} = 1 \quad 1 \cdot 0022$

The change, if it exists at all, is according to these experiments less than 0·3 per cent

A modification of PIOTROWSKI's experiment was then tried Instead of filling the oscillating flask with water, it was filled with sand and oscillated as a rigid body in a large beaker of water The temperature could then be accurately observed and the ordinary investigation of the oscillations of a rigid body in a resisting medium, and acted on by a force proportional to the displacement, will hold

Let k denote the frictional force proportional to the velocity,

M the moment of inertia,

λ the logarithmic decrement.

Then it is easily shown that

$$k = \frac{\lambda M}{\pi} p,$$

where

$$p = \sqrt{\left(\frac{\mu}{M} - \frac{k^2}{4M^2}\right)},$$

μ being the force of restitution for unit displacement

$$k^2 = \frac{4\lambda^2 \mu M}{4\pi^2 + \lambda^2}$$

Now in our case λ has a value of about 0·2, and λ^2 can therefore be neglected in comparison with $4\pi^2$,

$$k = \frac{\lambda}{\pi} \sqrt{(\mu M)} \quad \text{(approximately)}$$

When the flask is silvered μ is unchanged, as the weight of silver is much too small to appreciably alter the bifilar couple, and, therefore, we get for the ratio of the frictions

$$\frac{k}{k'} = \frac{\lambda \sqrt{M}}{\lambda' \sqrt{M'}} = \frac{\lambda T}{\lambda' T'},$$

where T and T′ are the respective times of vibration

A thick platinum wire was attached to the bulb, and the bifilar arrangement fixed to this above the surface of the water.

At the conclusion of the experiments, the bulb filled with sand was oscillated in air and the logarithmic decrement found to be a very small fraction of that observed when the bulb was in water. This meets the objection that the chief resistance

might be due to the suspension, and a small change in that part due to the water be inappreciable

Logarithmic decrement in air $=$ ·00284
Time of swing $=$ 9·952 seconds

The logarithmic decrement due to the air, suspension, &c., is only about 2 per cent. of that due to the water.

	Temperature	Time of swing	Logarithmic decrement	
	°	s		
Unsilvered	12·6	9·766	·19745	(1)
	12·6	9·769	·19882	
	12·7	9·772	·19830	
			·19795	
	12·9	9·741	·19694	(2)
	13·0	9·722	·19729	
			·19685	
	13·0	9·737	·19392	(3)
	13·7	9·686	·19331	
			·19330	
	13·8	9·725	·19374	(4)
	13·7	9·739	·19333	
	13·8	9·731	·19412	
	13·18	9·739	·19579	
Silvered	13·6	9·884	·19588	(5)
	13·5	9·891	·19717	
			·19622	
	13·55	9·887	·19642	
Resilvered	12·5	9·892	·19945	(6)
	12·6	9·892	·19943	
			·19943	
	12·9	9·882	·19811	(7)
	13·0	9·907	·19793	
			·19775	
	14·5	9·894	·19406	(8)
	14·6	9·885	·19363	
			·19331	
	9·75	9·891	·20688	(9)
	9·75	9·904	·20695	
			·20772	
Unsilvered	10·0	9·939	·20687	(10)
	10·0	9·936	·20721	
			·20636	
	12·7	9·917	·19881	(11)
	12·8	9·920	·19862	
			·19860	

Between each of the series of observations marked (1), (2) . . . the suspending wire was re-adjusted, but an inspection of the results shows that they agree well among themselves, and that therefore the bulb might safely be moved for silvering. After

the observations marked (4) the bulb was suspended in the silvering solution for about an hour, and then removed. It was then found that the top hemisphere was covered with a black sediment from the solution on top of the silver, while the under hemisphere was silvered as usual. Nevertheless a series of three determinations of the logarithmic decrement was taken. The mean of these shows an increase in both the logarithmic decrement and in the time of swing, even though the temperature was actually higher. It was thought that this might be due to the black sediment, which could not be removed alone, so the whole deposit was dissolved off and the bulb resilvered, the solution being kept stirred and frequently changed. This time the deposit was bright and uniform. The observations were, however, identical with the last. The only explanation of this (unless we suppose that the friction is about 3 per cent. greater for silver, instead of 4 per cent. less as PIOTROWSKI deduced) is to suppose that a change had occurred in the suspension. To test this, a series of observations (6) to (9) were taken with the silver on to get the temperature correction, and immediately after (9) the beaker of water was removed from under the bulb, and one of nitric acid of the same temperature put in its place without disturbing the suspension. As soon as the silver was dissolved the bulb was washed, and the beaker of water replaced.

A series of observations (10) were at once taken with the glass surface. Thus by comparing (9) with (10) we get a comparison of the friction on glass with the friction on silver, free from all possible errors due to change of suspension, and at temperatures whose difference is only 0°·25. The means are—

	Temperature.	Time of swing.	Logarithmic decrement.
(9) Silvered	9·75	9·898	·20718
(10) Unsilvered	10·00	9·938	·20681

The temperature correction for the logarithmic decrement is by (8) and (9) ·002815 for 1° C. or ·00070 for 0°·25.

Therefore the logarithmic decrement for the glass surface corrected to 9°·75 is ·20751.

The ratio of the frictions is $\frac{9 \cdot 938 \times \cdot 20751}{9 \cdot 898 \times \cdot 20718} : 1 = 1 \cdot 00564 : 1$.

Thus the effect is, if it exists at all, less than 0·6 per cent. instead of 4 per cent.

Another independent comparison can be taken between the series marked (6) and (7) and the series marked (11). The means are—

	Temperature	Time of swing	Logarithmic decrement
(6) and (7)	12°75	9 893	19868
(11)	12 75	9 922	19867

The ratio of the frictions is 1 00288 1, the change being less than 0·3 per cent greater for glass Thus within the limits of experimental error the friction on silver is the same as the friction on glass, and this part of PIOTROWSKI's paper is certainly misleading

It is evident that the oscillation method is much inferior to that in which the flow of water through tubes is observed The experiments described in the early part of this paper show that the difference in the time of flow for a glass and silver tube is less than *one-half per cent*, in a case where the existence of a slipping coefficient of only one-half the magnitude of that deduced by HELMHOLTZ for gold, would make the time of flow for the silver tube about *twelve times* less than the time of flow for the glass tube

The arguments sometimes used in favour of the contact theory of electromotive force, based on the differences in friction of a liquid on different solid surfaces, must now be admitted to be without value It is certain that no slip occurs, at any rate in the case of substances which are wetted by the liquid

In conclusion I must offer my most sincere thanks to Professor J J THOMSON and to Mr. GLAZEBROOK for the help they have given me, and the many valuable suggestions they have made.

X. A Determination of "v," the Ratio of the Electromagnetic Unit of Electricity to the Electrostatic Unit.

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THE experiments made by one of us in 1883 having given a value of "v" considerably smaller than the one found by several recent researches, it was thought desirable to repeat those experiments. The method used in 1883 was to find the electrostatic and electromagnetic measures of the capacity of a condenser, the electrostatic measure being calculated from the dimensions of the condenser, the electromagnetic measure determined by finding the resistance which would produce the same effect as that produced by the repeated charging of the condenser placed in one arm of a Wheatstone's Bridge. In the experiments of 1883 the condenser used in determining the electromagnetic measure of the capacity was not the same as the one for which the electrostatic measure had been calculated, but an auxiliary one, without a guard ring, the equality of the capacity of this condenser and that of the guard ring condenser being tested by the method given in MAXWELL's 'Electricity and Magnetism,' vol. 1, p. 324.

In repeating the experiment we adopted at first the method used before, using, however, a key of different design for testing the equality of the capacity of the two condensers by MAXWELL's method. We got very consistent results, practically identical with the previous ones. We may mention here, since it has been suggested that the capacity of the leads might account for the small values of "v" obtained, that this capacity is allowed for by the way the comparison between the capacities of the auxiliary and guard ring condensers is made, for the same leads are used both in this comparison and in the determination of the electromagnetic measure of the capacity of the auxiliary condenser; the capacity of the auxiliary condenser, plus that of its leads, is made equal to the capacity of the guard ring condenser, and it is the capacity of the auxiliary condenser, plus its leads, which is determined in electromagnetic measure. As the introduction of the auxiliary condenser introduced increased possibilities of error, we endeavoured to determine directly the electromagnetic measure of

the capacity of the guard ring condenser, by using a complicated commutator which worked both the guard ring and the condenser At first we tried one where the contacts were made by platinum styles attached to a tuning fork, but as the results were not so regular as we desired, we replaced the tuning fork commutator by a rotating one driven by a water motor A stroboscopic arrangement was fixed to this commutator so that its speed might be kept regular and measured With this arrangement, which worked perfectly, we got values for the electromagnetic measure of the capacity of the condenser distinctly less than those obtained by the old method We then endeavoured to find out the cause of this difference, and after a good deal of trouble discovered that in the experiments by which the equality of the capacities of the guard ring and auxiliary condensers was tested by MAXWELL's method, the guard ring did not produce its full effect When the guard ring of the standard condenser was taken off, and its capacity made equal by MAXWELL's method to the capacity of the auxiliary condenser, the two methods gave identical results, but the effect of adding the guard ring was less in the old method than in the new We found also, by calculation, that the effect produced by the guard ring in the old method was distinctly too small, while that determined by the new method agreed well with its calculated value As the new method was working perfectly satisfactorily, and as it possesses great advantages over the old one, inasmuch as we get rid entirely of the auxiliary condenser, and can also alter the speed of the rotating commutator with very much greater ease and considerably greater accuracy than in any arrangement where the speed is governed by a tuning fork, we discarded the old method and adopted the new one which we now proceed to describe, beginning by considering the errors to which this method is liable

Advantages of the Method of Determining "v"

The best way of discussing the advantages of this method is to consider the quantities which have to be measured and the accuracy which can be obtained in their measurement. The investigation naturally divides into two parts (1) the determination of the capacity of a condenser in electrostatic measure, (2) the determination of the capacity of the same condenser in electromagnetic measure Let us begin by considering the first part. The condenser consisted of two co-axial cylinders, the inner cylinder being provided with a guard ring If the distribution of electricity on the middle part of the inner cylinder were the same as that on an equal length, l, of an infinite cylinder whose radius is a, surrounded by a co-axial infinite cylinder of radius b, the electrostatic measure of the capacity would be $\frac{1}{2} l/\log b/a$ The actual case may differ from this ideal one in some or all of the following ways. (1) The two cylinders may not be quite co-axial; this, however, is not important if we know the distance betwen the axes, as we can find the capacity of the system got by placing one cylinder anywhere inside another. (2) The cross sections of the cylinders may

not be accurately circles The effect on the capacity of a slight departure from circularity is calculated below, so that this effect may be corrected (3) The conductors may not be true cylinders but swell or contract slightly as we proceed along their lengths, we show, however, below, how to correct for an effect of this kind (4) The existence of the air space between the guard ring and the middle cylinder will cause the distribution of electricity near the ends of this cylinder to be irregular, and there will also be some electricity on the cross section of the cylinder, we have, therefore, found the distribution of electricity in a case so nearly resembling this as to allow us to use the result as a correction In the arrangement we used the potential of the guard ring differed slightly from that of the middle cylinder, the very small correction due to this is, however, easily calculated Since we know the corrections, the capacity of the condenser can be calculated in terms of its dimensions, and the only errors to which we are liable are those which may be made in the determination of these dimensions The lengths which have to be measured with great accuracy are the length of the middle cylinder, its radius and that of the outer cylinder, and the distance between the cylinders The first three of these are long enough to be measured by the ordinary methods of measuring length, without danger of an error greater than one part in 3000, the fourth, however, is too small to be measured with so great an accuracy by these methods, it was determined, therefore, by finding v, the volume of water required to fill the space between the two cylinders, then d the distance between the cylinders is given by the formula

$$d = \frac{v}{\pi l (a + b)}$$

where l is the length of the middle cylinder and a and b the radii of the two cylinders In this way the percentage error of d was not greater than those of a, b, and l Since an accuracy of one part in 3000 can be obtained in the measurements of the dimensions of the cylinders, and since the electrostatic measure of the capacity is of the dimensions of a length, this measure of the capacity can be obtained correct to one part in 3000

We now pass on to the determination of the capacity in electromagnetic measure This was determined by balancing, in a Wheatstone's bridge, a discontinuous current produced by rapidly charging the condenser against a steady current derived from the battery which charged the condenser. In order to calculate the electromagnetic measure of the capacity it is necessary to know accurately the number of times per second the condenser is charged, and to keep this number constant. The charging and discharging of the condenser were effected by a commutator driven by a Thirlmere Water Motor, the water being obtained, not from the main, but from a cistern at the top of the Laboratory. The number of revolutions per second made by the commutator was compared by a stroboscopic arrangement with the frequency of an electrically driven tuning fork. The observer (G F C S) was able, after practice, to govern the

speed of the commutator so efficiently that when the condenser was in action the spot of light reflected from the mirror of the galvanometer did not move over more than half a millimetre.

The accuracy of measurement of the number of times the condenser was charged per second is thus practically the same as the accuracy of the determination of the frequency of the tuning-fork, this frequency could be determined (see *infra*) to less than one part in 10,000

The limit which is practically put on the determination of the electromagnetic measure of the capacity of the condenser is that imposed by the galvanometer With the galvanometer we employed, which was one made in the laboratory, having about 30,000 turns and a resistance of 17,400 legal ohms, when the resistance of the variable arm of the Wheatstone's bridge was 2500 ohms, an alteration of 2 ohms could be detected, thus the measurement of the resistance equivalent to the repeatedly charged condenser could be made to one part in 1250 an error of this magnitude would cause an error of one part in 2500 in the value of "v" and as all the other measurements were more accurate than this, there seems no reason why this method should not give as accurate a value of "v" as that obtained for the ohm.

The electromagnetic way of measuring the capacity affords us the means of testing the accuracy of the corrections applied to the electrostatic measure of the capacity, we availed ourselves of this in the case of the correction for the effect of the air space between the middle cylinder and the guard-ring, we altered the thickness of this air space and found that the effect of this alteration was accurately represented by the correction we employed One great advantage of the method is the ease with which the number of times per second the condenser is charged can be altered, this affords a valuable means of detecting any leakage or any effect due to self-induction.

Calculation of the Electrostatic Measure of the Capacity of the Condenser.

Description of the Condenser—The condenser, which was designed some years ago by Lord RAYLEIGH, is represented in section in fig. 1, and in plan in fig 2 BHPD is a thick ebonite board, placed in an approximately horizontal position, in this board two concentric circular grooves are cut. A cylindrical brass ring, HP, whose external diameter is about 23 cm, and whose height is about 10 cm., fits into the smaller of these grooves. Three pieces of ebonite carefully ground down to the same thickness (about 3 mm in most of the experiments), with V-shaped grooves cut in them to increase the distance over which the electricity would have to leak are placed at equal intervals on the top of this ring. On these the brass cylinder FGMN is placed, this cylinder is of exactly the same diameter as the cylindrical ring HP, and is about 60 cm. long. The cylinders GFMN and HP are placed so that their axes are coincident. On the top of this cylinder three pieces of ebonite similar to those on HP are placed, and

upon the top of these a cylindrical ring EL, similar to the ring at the bottom. Another brass cylinder, ABCD, made in three pieces, two rings somewhat similar in height to the rings HP, EL, and a long middle piece of the same length as the cylinder FGMN, is then fitted over the other cylinders, the bottom ring fitting into the outer groove in the ebonite board, the internal diameter of this cylinder is about 25 cm.

Fig 1

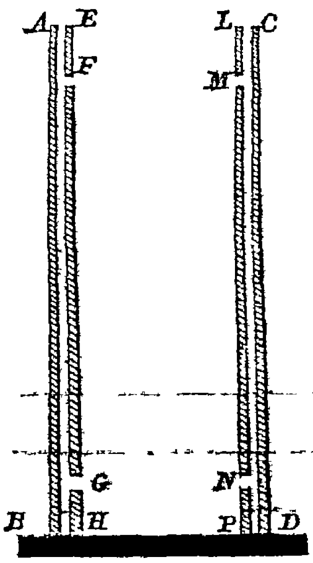
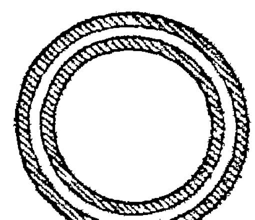


Fig 2.



The cylinders are made co-axial by means of three pieces of ebonite worked down to the same thickness (the difference between the radii of the cylinders) pushed by rods attached to them down between the cylinders, the cylinders are adjusted until these three pieces of ebonite arranged symmetrically round the cylinder are each just in contact with the two cylinders, the rods were then removed. The insulation between the inner and outer cylinders and between the inner cylinder and its guard rings was tested by connecting one of these to earth, and the other to a charged gold leaf electroscope, the condenser was not used unless there was no appreciable loss of electricity shown by the electroscope in five minutes.

Calculation of the Capacity.—The capacity of the system regarded as two co-axial cylinders of circular section with a uniform distribution of electricity over them is

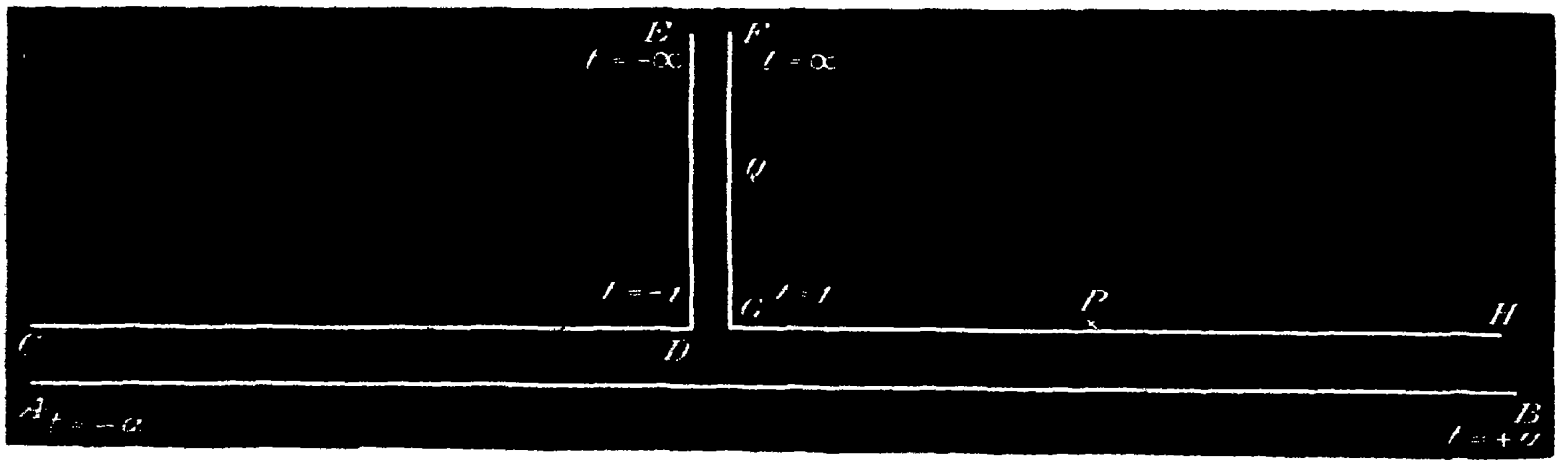
$$\tfrac{1}{2}\, l/\log\frac{a}{b},$$

where a is the radius of the outer cylinder, b that of the inner, and l the length of the cylinder FGMN.

Correction for want of coincidence between the Axes.—It is shown in a paper by J. J. Thomson, "On the Determination of the number of Electrostatic units in the Electromagnetic unit of Electricity" ('Phil. Trans.,' 1883, p. 714), that if ϵ be the small distance between the axes of the cylinders, the capacity is

$$\frac{1}{2}\,\frac{l}{\log\frac{a}{b}}\left\{1+\frac{\epsilon^2}{(a^2-b^2)\log\frac{a}{b}}\right\}.$$

Correction for the want of equality in the distribution produced by the air spaces between the inner cylinder and the guard rings —To find this correction we shall find the distribution of electricity on the system represented in the figure, when AB is the section by the plane of the paper of an infinite horizontal metal plane, and CDE, FGH sections of conductors CD and GH being horizontal and at the same distance from AD, and DE and FG vertical Let h be the distance between the planes CD, and AB, and $2c$ the breadth of the slit DE FG



Let us take AB as the axis of x and the vertical line midway between ED and FG as the axis of y Then writing z for $x + \iota y$, and supposing that ϕ and ψ are the stream and potential functions respectively, we find by using SCHWARZ's method that the solution of the problem is given by the equations

$$dz = -A\frac{\{1 - t^2\}^{\frac{1}{2}}}{t^2 - a^2}\,dt \qquad a < 1 \tag{1}$$

$$\phi + \iota\psi = B\log\frac{t - a}{t + a} \quad . \quad . \tag{2}$$

t being supposed to have all real values from $-\infty$ to $+\infty$.

For putting $t = \sin\theta$, $a = \sin\alpha$, and integrating (1) we find

$$z = A\left(\theta - \tfrac{1}{2}\cot\alpha\log\frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}\right)$$

or

$$x + \iota y = A\left(\theta - \tfrac{1}{2}\cot\alpha\log\frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}\right) \quad . \quad . \quad . \tag{3}$$

as θ goes from 0 to α, the right hand side of this equation is real so that $y = 0$ and x ranges from 0 to $+\infty$, this gives the positive half of the plane AB As θ goes from α to $\frac{1}{2}\pi$

$$x + \iota y = A\left(\theta - \tfrac{1}{2}\cot\alpha\log\frac{\sin(\theta - \alpha)}{\sin(\alpha + \theta)} + \tfrac{1}{2}\iota\pi\cot\alpha\right)$$

so that $y = \frac{1}{2} A\pi \cot \alpha$, and x ranges from ∞ to $\frac{1}{2} A\pi$, so if

$$h = \frac{1}{2} A\pi \cot \alpha \quad . \tag{4}$$

$$c = \frac{1}{2} A\pi \quad . \tag{5}$$

this will give the portion GH of the diagram. When $\sin \theta$ is greater than unity we may put

$$\theta = \frac{1}{2}\pi + \iota\vartheta,$$

the right hand side of (3) now equals

$$A\left\{\frac{\pi}{2} + \iota\vartheta - \frac{1}{2} \cot \alpha \log \frac{\cos \alpha (\epsilon^{\vartheta} + \epsilon^{-\vartheta}) + \iota \sin \alpha (\epsilon^{\vartheta} - \epsilon^{-\vartheta})}{\cos \alpha (\epsilon^{\vartheta} + \epsilon^{-\vartheta}) - \iota \sin \alpha (\epsilon^{\vartheta} - \epsilon^{-\vartheta})} + \frac{1}{2} \iota\pi \cot \alpha\right\}$$

calling the quantity under the logarithm $P + \iota Q$, we see since

$$P + \iota Q = \frac{\cos \alpha (\epsilon^{\vartheta} + \epsilon^{-\vartheta}) + \iota \sin \alpha (\epsilon^{\vartheta} - \epsilon^{-\vartheta})}{\cos \alpha (\epsilon^{\vartheta} + \epsilon^{-\vartheta}) - \iota \sin \alpha (\epsilon^{\vartheta} - \epsilon^{-\vartheta})}$$

that

$$P - \iota Q = \frac{\cos \alpha (\epsilon^{\vartheta} + \epsilon^{-\vartheta}) - \iota \sin \alpha (\epsilon^{\vartheta} - \epsilon^{-\vartheta})}{\cos \alpha (\epsilon^{\vartheta} + \epsilon^{-\vartheta}) + \iota \sin \alpha (\epsilon^{\vartheta} - \epsilon^{-\vartheta})}$$

multiplying these together we see that $P^2 + Q^2 = 1$. So that $\log (P + \iota Q)$ is wholly imaginary. Thus we see that as ϑ ranges from 0 to ∞, and t therefore from 1 to ∞, $x = \frac{1}{2}A\pi$ and y ranges from $\frac{1}{2}A\pi \cot \alpha$ to ∞, thus this range of values of t gives the portion GF of the figure. Since the real part of the right hand side of the equation (3) changes sign with θ or t, we see that the other portions of the figure are given by the negative values of t.

Since

$$\phi + \iota\psi = B \log \frac{t - a}{t + a}$$

we see that as long as t is between $-a$ and a, that is for the portion AB of the figure,

$$\phi + \iota\psi = \iota\pi B + \text{real quantities},$$

so that $\psi = \pi B$, and, therefore, the potential is constant over AB; when t is not between these values, that is for the other portion of the figure $B \log \{(t - a)/(t + a)\}$ is real, and therefore $\psi = 0$.

Thus these equations give us the solution of the problem when AB is maintained at the potential πB and CDE, FGH are at zero potential.

The quantity of electricity on the conductor FGP when P is a point on GH

$$= \frac{1}{4\pi}(\phi_P - \phi_F)$$
$$= \frac{B}{4\pi}\log\frac{t-a}{t+a},$$

where t is the value of t at P. If we represent the increase in the quantity of electricity, due to the irregularity of the distribution, by supposing a strip of breadth d to be added to the conductor GH, and the distribution of electricity to be regular, and the same as if the air space were not present, the equation to find d is, if x is the value of x at P and V, the difference of potential between AB and GH

$$\frac{V}{4\pi h}\{x - c + d\} = -\frac{B}{4\pi}\log\frac{t-a}{t+a},$$

substituting for x and t their values in terms of θ, and remembering that

$$V = \pi B, \qquad c = \tfrac{1}{2}A\pi \qquad h = \tfrac{1}{2}A\pi\cot\alpha$$

we get

$$A\left(\theta - \tfrac{1}{2}\cot\alpha\log\frac{\sin(\theta-\alpha)}{\sin(\alpha+\theta)}\right) - c + d = -\tfrac{1}{2}A\cot\alpha\log\frac{(\sin\theta - \sin\alpha)}{(\sin\alpha + \sin\theta)},$$

or

$$d = c\left\{1 - \frac{2}{\pi}\theta\right\} - \frac{h}{\pi}\log\frac{(\sin\theta - \sin\alpha)\sin(\alpha+\theta)}{(\sin\alpha + \sin\theta)\sin(\theta-\alpha)}$$

Now if P be some distance from G we may put $\theta = \alpha$ and we get

$$d = c\left\{1 - \frac{2}{\pi}\alpha\right\} - \frac{h}{\pi}\log\cos^2\alpha,$$

from equations (4) and (5) we see that $\tan\alpha = c/h$, so that

$$d = c\left\{1 - \frac{2}{\pi}\tan^{-1}\frac{c}{h}\right\} + \frac{1}{\pi}h\log\left\{1 + \frac{c^2}{h^2}\right\}$$

To deduce the corresponding solution for the cylinders from this we must multiply by the correction for curvature $1 + \frac{1}{4}h/a$, where a is here the radius of the inner cylinder, so that we have finally, if D be the whole breadth to be added for the two air spaces,

$$D = 2\left[c\left\{1 - \frac{2}{\pi}\tan^{-1}\frac{c}{h}\right\} + \frac{1}{\pi}h\log\left(1 + \frac{c^2}{h^2}\right)\right]\left(1 + \frac{1}{4}\frac{h}{a}\right).$$

Now in our condenser l was about 60, $2c = 3$, and $h = 1$, so that if we put $D = 2c$ the value of the capacity will be correct to 1 part in 2000.

Correction for a small difference of potential between the guard ring and the middle cylinder —The arrangement we used necessitates the existence of a small difference of potential between the cylinder and the guard ring, a slight modification of the preceding investigation will enable us to find the correction for this If we put

$$\phi + \iota\psi = \frac{V}{\pi}\log\frac{t-a}{t+a} + \frac{\delta V}{\pi}\log(t+a),$$

then the potential over FGH $= 0$, that over EDC $= \delta V$, and over AB $= V$

The breadth of the strip which must be added to compensate for the electricity on the portion QGH due to the difference of potential δV between CD and GH is

$$\frac{h}{\pi}\frac{\delta V}{V}\log\frac{(t_Q+a)}{(t_H+a)}$$

Now, if QG is large,

$$GQ = A\log\frac{2t_Q}{\epsilon},$$

where ϵ is the base of the Napierian logarithms. Hence, since $t_H = a = \sin\alpha$, the breadth of the additional strip is

$$\frac{h}{\pi}\frac{\delta V}{V}\left\{\frac{QG}{A} - \log\frac{4\sin\alpha}{\epsilon}\right\},$$

but $A = c\,2/\pi$ and $\sin\alpha = c/h$ approximately, hence the breadth of the strip for the two guard rings is

$$h\frac{\delta V}{V}\left\{\frac{QG}{c} - \frac{2}{\pi}\log\frac{4c}{h\epsilon}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

In our experiments $h/c = 6\ 6$, QG $= 1$, so that the correction amounts to a strip whose breadth is about

$$7\ 5\frac{\delta V}{V}$$

The value of $\delta V/V$ depended on the speed, the usual value was about $\frac{1}{183}$ In this case the breadth of the strip would be about $\frac{1}{30}$ of a centimetre, and, since the length of the cylinder was about 60 cm , the correction amounts to about 1 part in 1800.

To test the accuracy of these corrections, determinations of the capacity of the condenser were made when the top guard ring was separated from the middle cylinder (1) by pieces of ebonite 504 cm. thick, (2) by pieces 067 cm. thick. The capacity of the cylinders with the thick ebonite was greater than that with the thin by about 11 parts in 2760. According to the results we have obtained for these

corrections the effect of increasing the thickness of the ebonite would be to add a breadth 218 cm to the cylinder in consequence of the increased air space, and 06 in consequence of the difference of potential, thus the two would add 28 to the length of the cylinder, and would increase the capacity by $\frac{28}{61} \times 2760$, or 12 parts in 2760. Thus the observed and calculated results agree well together

Correction for ellipticity of the cross section—Let us consider the case of a cylinder whose cross section is represented by the equation

$$r = b\{1 + e \cos 2\theta\},$$

placed inside one whose cross section is represented by

$$r = a\{1 + \alpha \cos 2\theta + \beta \sin 2\theta\},$$

where, since the measurements of the cylinder show that e, α, β are less than $\frac{1}{2000}$, we can neglect the squares of these quantities

Let the potential between the cylinders be given by

$$V = A \log r + \frac{C \cos 2\theta}{r^2} + Dr^2 \cos 2\theta + \frac{E \sin 2\theta}{r^2} + Fr^2 \sin 2\theta.$$

Then, neglecting the squares of e, α, β, the difference of potential between the cylinders is

$$A \log \frac{a}{b},$$

and to the same approximation the charge per unit length is $\frac{1}{2}A$, thus the capacity per unit length is $\frac{1}{2} \log a/b$. Here a and b are the means of any two radii of the cylinders at right angles to one another. If we take these values as the radii of the cylinders the only correction required will be one of the order of one part in $(2000)^2$, which may be neglected

Correction of Conicality—We may see how to get rid of this correction by considering the electrical distribution on two infinite conductors, the one a plane perpendicular to the axis of y, the other a corrugated plane represented by the equation

$$y = h + \beta \sin \frac{2\pi x}{l},$$

the other plane being taken as the plane of xz. Let V the potential between the planes be given by

$$V = Ay + C \sin \frac{2\pi x}{l} \{\epsilon^{2\pi y/l} - \epsilon^{-2\pi y/l}\}$$

putting $y = h + \beta \sin 2\pi x/l$, and making the potential constant and equal to V_0

$$\left.\begin{array}{l} V_0 = Ah \\ C = -\dfrac{A\beta}{\epsilon^{2\pi h/l} - \epsilon^{-2\pi h/l}} \end{array}\right\} \text{neglecting } \beta^2$$

Thus σ, the surface density on the plane of xz,

$$= \frac{V_0}{4\pi h}\left\{1 - \frac{\beta \sin \frac{2\pi x}{l} \frac{2\pi}{l}}{\epsilon^{2\pi h/l} - \epsilon^{-2\pi h/l}}\right\}$$

$$= \frac{V_0}{4\pi h}\left\{1 - \frac{(y - h)\frac{2\pi}{l}}{\epsilon^{2\pi h/l} - \epsilon^{-2\pi h/l}}\right\}$$

Thus, if we choose h so that it is the *mean* distance between the plates, for the breadth on which we wish to find the charge, the second term will vanish in our integration, and we get for Q the quantity of electricity on a breadth x

$$Q = \frac{V_0}{4\pi h} x$$

Thus we can use the ordinary formula even when the plates are slightly inclined, provided h is the mean distance. Any correction to this will be the order of the square of the inclination at least, and in our case may be neglected.

Measurement of Dimensions of Condenser

The dimensions are all referred to the standard metre of the Cavendish Laboratory which has been compared with the standard of the Board of Trade. The errors of the divisions are too small to affect the measurements given below. The comparison of the lengths with the standard metre was made by means of a pair of reading microscopes with micrometer screws. The pitch of the screws is accurately $\frac{1}{50}$th of an inch, and the head of the screw is divided into 100 parts, so that one division of the screw-head corresponds to .0002 inch. The tenths of divisions are easily read and are recorded. The screws were tested by Mr FITZPATRICK when working with Mr GLAZEBROOK at the Specific Resistance of Mercury, and were found to be free from sensible error in either pitch or uniformity.

The standard metre is correct at 0° C., and its temperature coefficient is .000017 per 1° C.

We require the dimensions of the condenser at 16° C. The metal of which the condenser is made is much the same as that of the standard metre, so that if we assume that the temperatures of the condenser and standard metre are the same at

the time of comparison we shall simply have to correct the metre to 16° C The temperature of the room never differed from 16° by more than 2°, so that no appreciable error can be introduced on this account

External Diameter of Inner Cylinder

The sliding calipers of the laboratory were used to measure this. The bar of the calipers rested on the flat top of the cylinder, so that the calipers could be moved backwards and forwards along the top The jaws are supposed to be at right angles to the bar along which the sliding one moves, but this was found not to be exactly the case To obviate this difficulty a small piece of brass was fastened to the end of one jaw, so that the contacts were made at the ends of both jaws The calipers were then placed under the microscope and two definite marks read off The standard metre was then placed beneath the microscopes and treated in the same way The distance between the marks when the jaws of the calipers were in contact was determined by the micrometer screw alone.

The readings of the screws are given in terms of $\frac{1}{5}$ inch

MAXIMUM Diameter of top end of Cylinder.

	Calipers		23 8 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
1	1 1765	6643	1 2308	5970
2	1 1930	6420	1 1939	6280
3	1 8100	5642	1 7980	5648
4	1 8676	5134	1 8793	4965

The numbers in (4) are the mean of three readings

These measurements gave as the distance between the marks—

	23 8 cm. . .	— 00260 in.	(1)
	,, . . .	— 00262 ,,	(2)
	,, ...	—·00220 ,,	(3)
	,, . . .	—·00104 ,,	(4)
Mean	23·8 cm . .	— 00211 in.	
		= 23 7946 cm.	

Distance between the marks with jaws of calipers closed Read with left-hand screw—

(1)	(2)	(3)
1·2779	1·5778	2·0561
·7538	1·0430	1·5192
·5241	·5348	·5369

(4)	(5)	(6)
2·3553	2·2183	2·2187
1·8136	1·6891	1·6895
·5417	·5292	·5292

(7)	(8)	(9)
2·3550	2·5410	2·5330
1·8172	2·0096	1·9998
·5378	·5314	·5332

The mean of these is

$$\frac{\cdot 53314}{5} \text{ in} = \cdot 10662 \text{ in} = \cdot 2708 \text{ cm.}$$

Thus we find that the maximum diameter at the top of the cylinder is

$$23 \cdot 7946 - \cdot 2708 = 23 \cdot 5238 \text{ cm},$$

referred to the standard at 16° C

To reduce the standard to 0°·C we must multiply by $(1 + 16 \times \cdot 000017)$ and we get as the true diameter

$$23 \cdot 5302 \text{ cm}$$

Minimum diameter of top end of Cylinder

	Calipers		23·8 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
1 (Mean of 4 readings)	2·2158	0·4338	2·1822	0·4279
2 (Mean of 4 ,,)	1·9271	0·7133	1·8625	0·7406

Giving as minimum diameter of top end

(1.) 23·8 cm − ·2708 cm. − ·00790 in.
(2.) 23·8 cm. − ·2708 cm. − ·00746 in.

Correcting for temperature, we find

Minimum diameter of top end = 23·5161 cm.

BOTTOM end of Cylinder

MAXIMUM DIAMETER

	Calipers		23 8 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
1 (Mean of 4 readings)	1 4078	1 2042	1 4171	1 1950
2 (Mean of 4 ,,)	1 5469	1 0671	1 5910	1 0196

MINIMUM DIAMETER

	Calipers		23 8 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
1 (Mean of 4 readings)	1 6915	9584	1 6814	0 9318
2 (Mean of 4 ,,)	1 6916	9581	1 6495	0 9634

Thus, maximum diameter of bottom end equals

$$(1.)\ 23{\cdot}8\text{ cm.} - {\cdot}2708\text{ cm} + 00002\text{ in}$$
$$(2.)\ 23{\cdot}8\text{ cm} - 2708\text{ cm} - {\cdot}00068\text{ in}$$

Correcting for temperature, we find

Maximum diameter of bottom end = 23 5348 cm

The minimum diameter of bottom end equals

$$(1.)\ 23\ 8\text{ cm.} - 2708\text{ cm} - {\cdot}00734\text{ in.}$$
$$(2.)\ 23\ 8\text{ cm} - {\cdot}2708\text{ cm} - {\cdot}00736\text{ in}$$

Correcting for temperature

Minimum diameter of bottom end = 23 5169 cm

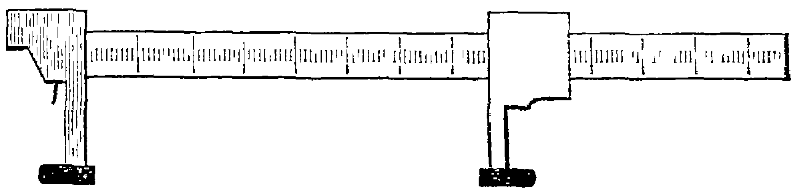
Collecting these results we have for the inner cylinder

Top end .	Maximum diameter	= 23·5302
	Minimum diameter	= 23 5161.
Bottom end	Maximum diameter	= 23·5348.
	Minimum diameter	= 23·5169.
Mean of these		= 23·5245.

The corresponding ends of the measured diameters were found to be almost exactly on the same generating line, so that though the cylinder is slightly elliptical and conical, it is free from anything of the nature of helicality

Measurement of the Internal Diameter of the Outer Cylinder

This was found to be a good deal more troublesome than the measurement of the external diameter of the inner cylinder, the plan finally adopted was to fix two pieces of hardened steel to the ends of the jaws of the sliding calipers, thus



One side of each piece was polished, and the end was then ground and polished on a fine oilstone so as to form a good edge with the polished face The shape of the end was semi-circular In this way the edges made contact with the cylinder, and the cross wires (one of which was set carefully perpendicular to the line of travel of the microscopes) could be easily focussed on to the end of the steel.

It was not found practicable to determine exactly when contact was made in the same way as was done for the inner cylinder, since when the calipers were set to nearly the size of the cylinder scarcely any movement was possible The sharp edges were also an impediment to the motion. We found, however, that by insulating one of the steel contact pieces we could determine accurately by the aid of a telephone when contact was made As the cylinder was found to be nearly circular, and the formula for a slightly elliptical cylinder outside a circular one indicates that the lengths of two diameters at right angles to each other are required, two such diameters were measured The following are the details of the measurements each of the numbers being the mean of four observations —

TOP end

DIAMETER A

	Calipers		25 4 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
(1)	1 3821	1 1721	1 3859	1 1793
(2)	1 3319	1 2060	1 3255	1 2324
(3)	1 1822	1 3445	1 1462	1 3993

DIAMETER B

	Calipers		25 4 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
(1)	1 1701	1 3843	1 1793	1 3787
(2)	1 1873	1 3762	1 1614	1 4032
(3)	1 1268	1 4415	1 1150	1 4485
(4)	1 8703	0 6891	1 8875	0 6620

BOTTOM end

DIAMETER A

	Calipers		25 4 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
(1)	1 4352	1 0999	1 4289	1 1209
(2)	1 4734	1 0699	1 4018	1 0586

DIAMETER B

	Calipers		25 4 cm	
	Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw
(1)	1 6109	0 9270	1 6131	0 9385
	1 6978	0 8427	1 7106	0 8371

Taking the mean of these and correcting for temperature we find

Top end . Diameter A = 25 4154 cm.
Diameter B = 25 4056 cm

Bottom end Diameter A = 25 4125 cm
Diameter B = 25 4122 cm

Mean of these . . . . = 25·4114 cm.

Measurement of the Length of the Cylinder

The length of the cylinder was transferred from the cylinder to the reading microscopes by means of the beam compasses of the laboratory; care being taken to keep the bar of the compasses parallel to the length of the cylinder while setting the compasses to the length of the cylinder.

On account of the length of the cylinder it was found difficult to ascertain by

moving the beam compasses just when contact was complete A small piece of thin sheet steel (about 03 cm thick) was interposed between the end of the cylinder and the point of the beam compasses The compasses were considered adjusted when a slight resistance to the motion of the feeling piece was perceived The beam compasses were then placed under the microscopes, and the distance between two definite marks on their points determined The points were then placed close together, so that the same resistance to the motion of the sliding piece was felt as in the former case The distance between the marks was then ascertained by means of one of the microscopes and its screw The distance being so small it seems unnecessary to compare it with the divisions of the standard metre

DISTANCE between the Marks when the Compasses were Closed

	Right-hand mark	Left-hand mark	
(1)	2 3156	1 7220	Mean of 4 observations
(2)	2 1283	1 5308	,, 4 ,,
(3)	2 1287	1 5290	,, 5 ,,
(4)	2 9120	2 3154	,, 6 ,,

Giving, as the distance between the marks, 11937 in , or 3032 cm

DISTANCE between the Marks when the Compasses were Open

Calipers		61 3 cm		
Left-hand screw	Right-hand screw	Left-hand screw	Right-hand screw	
2 1297	5698	2 0656	5532	Mean of 4 observations
2 4270	2537	2 3819	2415	,, ,,

Giving as the distance betwen the marks when open, 61 3 cm — 0138 in

Hence the length of the cylinder when corrected for temperature equals 60 9784 cm.

The Distance between the Inner and Outer Cylinders.

Since the difference of the mean diameters is only about 1 09 cm., and since, on account of the difficulties of measurement and the irregularities in the shape of the cylinders, it is impossible to arrive at any satisfactory result by subtracting the mean diameter of one cylinder from that of the other, we had to apply some other method. We adopted that used in the experiment of 1883, which was to ascertain the amount of water required to fill the space between the two cylinders This amount was determined by weighing The water employed was distilled, and was boiled a few hours previous to its use to enable it to absorb air bubbles more readily

A 500 c c flask was filled with water and weighed, its contents were then transferred to the cylinders, and it was then weighed again The difference in weight gives the weight of water transferred to the cylinder This process was repeated until the space between the cylinders was full

The weights of the full and empty flasks were determined to 1 centigrm

The 500 grm weight used to balance the water was compared with the standard 500 grm weight of the Laboratory and found too heavy by 055 grm. This has been allowed for The equality of the arms of the balance was also tested

The two cylinders were fastened down to a flat metal plate with a thin layer of cement so as to be quite water-tight To get any accurate estimate of the volume of water required to fill the space it was necessary to provide some means to ascertain when the space was exactly full The effects of capillarity, grease, &c , preclude any very accurate result being obtained when there is no top or cover fitted to the top of the cylinder, and as it was necessary to see whether any air bubbles were left inside a glass top had to be used Two holes were bored through the glass, and tubes were fixed into these The water was introduced through one tube, and the air escaped through the other Although no difficulty was experienced in making the joint at the bottom quite water-tight with any of the cements employed, it took us several days to make a satisfactory joint at the top The ease with which tightness at the bottom was secured was probably owing to the great weight of the cylinders

The top gave us all the more trouble, because we could not tell whether it was watertight or not until we had almost completed the filling in of the water If, then, the joint proved bad the whole of the time spent in weighing the water poured in was wasted.

We tried Prout's elastic glue, then gutta-percha dissolved in benzene, but both of these failed. The water seemed to loosen the hold of the glue upon the glass, so that although the system seemed air-tight it would not remain water-tight for more than a minute or two.

We finally tried some red wax which had been sent to the Laboratory by Professor THRELFALL, who obtained it in Germany, and this answered very well. It looks somewhat like a mixture of bees'-wax and sealing wax, and as it never gets quite hard it never cracks. It possessed another property which was also useful for our purpose, viz., that of melting at a comparatively low temperature. To apply the other cements in a satisfactory manner the glass had to be heated to a somewhat high temperature, and this frequently cracked it

The wax when melted became very fluid, so that only an extremely thin layer was included between either the top or bottom plate and the cylinders.

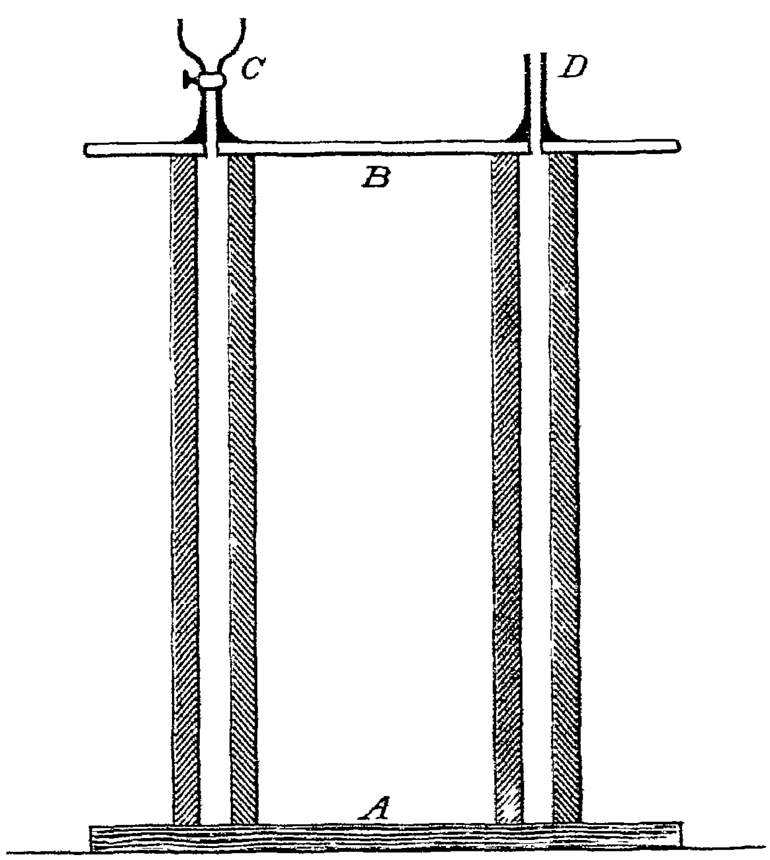
At first the water was poured in through a funnel inserted into one of the tubes, but with this arrangement it was found impossible to get rid of the last air bubble, since fresh quantities of air were continually carried down the tube After some experiments with different arrangements for filling, we finally adopted as the means

of introducing the water a tube with a funnel top provided with a tap. The funnel could always be kept partially full of water by means of the tap, so that no air was introduced. The cylinders were slightly tilted so that the exit tube was at the highest place. As the last few grammes were poured in the air was gradually swept along by the advancing water and driven up the exit tube. The small bubble which sometimes remained under the exit tube was easily removed by agitating it by a fine wire introduced through the exit tube.

The amount of water remaining in the tap and the amount which rose up the exit tube were ascertained afterwards by detaching the tap and tube, and finding the weight of water required to fill them to the same extent as when the cylinders were filled.

While the water was still about 3 cm. from the glass the air was exhausted from the cylinders by a water pump; the pressure of the air was reduced to about 50 mm. of mercury. This had the effect of removing a good many bubbles, and we may hope that the experiment was free from error in this respect. The wax employed for fixing the top on to the cylinders stood this difference of pressure perfectly. A drying tube was placed between the cylinders and the tube leading to the pump in order to catch any water which might be carried off in the pumping. The increase in weight of the drying tube was found to be not more than ·02 or ·03 gramme, and this has been allowed for. The weighings have also been corrected to a vacuum.

The annexed sketch shows a section of the cylinders by a plane through their axis, and through the tap and exit tube.



A is the flat metal plate. C the funnel and tap.
B the glass plate. D the exit tube.

The following are the results of the weighings on two separate days :—

	[1.] Temp. 17°.	[2.] Temp. 15·3°.
Weight of water put into funnel	4403·15	4403·53
Weight of water left in tap and tube	1·33	1·35
Volume of a piece of wax underneath the glass top	·70	0
Weight of water in space between the cylinders is therefore	4402·52	4402 18
Correction to vacuum	4·66	4·66
Correction for temperature	5·19	3·91
Correction for error in 500-grm. weight	·50	·50
Correction for inequality of arms	·09	·04
Volume between the cylinders is	4412·87	4411·29

The mean of these is

$$4412·08 \text{ c.c.}$$

which we take as the value of the volume between the cylinders.

If d is the mean distance between the cylinders, l the length of the inner cylinder, a and b the radii of the outer and inner cylinders respectively

$$d = \frac{4412·08}{\pi(a+b)l},$$

so that

$$d = ·94128 \text{ cm.}$$

$$\frac{a}{b} = 1 + \frac{a-b}{b} = 1·0800262,$$

$$2 \log \frac{a}{b} = ·15397063.$$

The thicknesses of the pieces of ebonite between the guard rings and the cylinder at the top and bottom were respectively ·2934 and ·288. Correcting for the air space the effective length of the cylinder is

$$60·9784 + ·2907 = 61·2691 \text{ cm.}$$

Hence the electrostatic measure of the capacity

$$= \frac{61·2691}{·15397063};$$

$$= \mathbf{397·927} \text{ cm.}$$

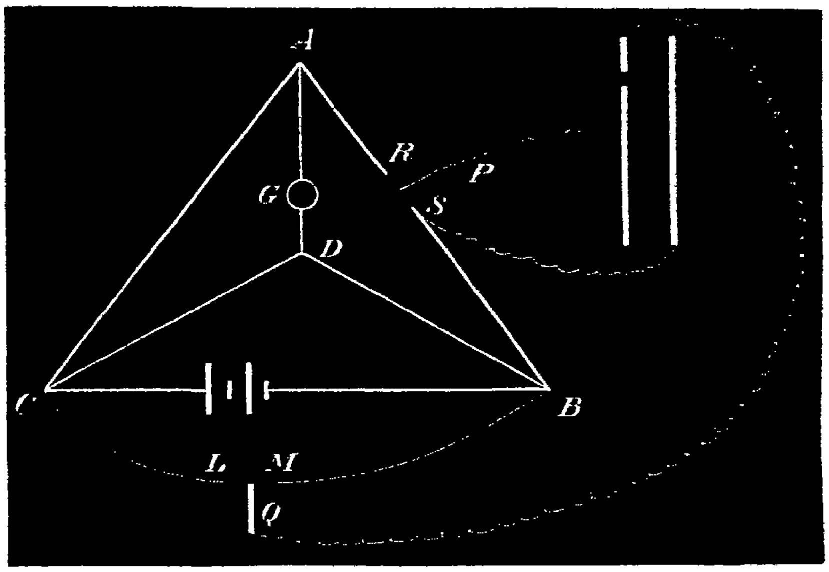
No correction is required for want of coincidence between the axes of the cylinders, for if c be the distance between the axes the correction is proportional to $c^2/(a^2 - b^2)$. In our experiments c was less than ·01, so that the correction only amounts to one part in more than 30,000.

The difference of potential between the guard ring and the cylinder depends upon the speed of the commutator, so that the correction on this account is made on the electromagnetic measure of the capacity

The Electromagnetic Measure of the Capacity

The arrangement employed in this measurement is represented in fig 3

Fig 3



ABCD is a Wheatstone's bridge with the galvanometer at G and the battery between B and C. The arm AB is broken at R and S, which are two poles of a commutator, which alternately come into contact with a spring P, connected with the middle part of the inner cylinder of the condenser The outer cylinder is connected to S The points C and B are connected respectively with L and M, the two poles of a commutator, which alternately come into contact with a spring Q, attached to the guard ring of the condenser. The system is arranged so that when the commutators are working the order of events is as follows —

I	P on S	Condenser discharged
	Q on M	Guard ring discharged
II	P on R	Condenser begins to charge
	Q on M	
III	P on R	Condenser completely charged to potential (A)-(B)
	Q on L.	Guard ring charged to potential (C)-(B).
IV	P on S.	Condenser begins discharging
	Q on L	
V	P on S.	Condenser discharged.
	Q on M	Guard ring discharged

Thus when the commutators are working, there will, owing to the flow of electricity to the condenser be a succession of momentary currents through the galvanometer The resistances are so adjusted that the effect of these momentary currents on the galvanometer just balances the effect due to the steady current, and there is no deflection of the galvanometer

To investigate the relation between the resistances when this is the case, let us suppose that when the guard ring and condenser are charging

$\dot{x}$ = current through BC

$\dot{y}$ = current through AR.

$\dot{z}$ = current through AD.

$\dot{w}$ = current through CL.

Thus, if a, b, α, β, γ are the resistances in the arms BC, AC, AD, BD, CD respectively, L the coefficient of self induction of the galvanometer, and E the electromotive force of the battery, we have

$$L\ddot{z} + (b + \gamma + \alpha)\dot{z} + (b + \gamma)\dot{y} + \gamma\dot{w} - \gamma\dot{x} = 0 \qquad (1)$$

$$(a + \gamma + \beta)\dot{x} - (\gamma + \beta)\dot{y} - \gamma\dot{z} - (\gamma + \beta)\dot{w} - E = 0 \qquad . \quad (2)$$

Now it is evident that the currents are expressed by equations of the following kind

$$\dot{x} = \dot{x}_1 + \dot{x}_2,$$
$$\dot{z} = \dot{z}_1 + \dot{z}_2,$$

where $\dot{x}_1$ and $\dot{z}_1$ express the steady currents when no electricity is flowing into the condenser, and $\dot{x}_2$, $\dot{z}_2$ are of the form $A\epsilon^{-\lambda t}$, $B\epsilon^{-\lambda t}$, and express the variable parts of the currents due to the charging of the condenser, $\dot{y}$ and $\dot{w}$ will be of the form $C\epsilon^{-\lambda t}$, $D\epsilon^{-\lambda t}$; t in all these equations is the time which has elapsed since the condenser commenced to charge

Equations (1) and (2) will thus contain constant terms, and terms multiplied by $\epsilon^{-\lambda t}$, the latter must separately vanish, hence we have

$$L\ddot{z}_2 + (b + \gamma + \alpha)\dot{z}_2 + (b + \gamma)\dot{y} + \gamma\dot{w} - \gamma\dot{x}_2 = 0 \,. \qquad . \quad (3)$$

$$(a + \gamma + \beta)\dot{x}_2 - (\gamma + \beta)\dot{y} - \gamma\dot{z}_2 - (\gamma + \beta)\dot{w} = 0 \,. \quad . \quad . \quad . \quad . \quad (4)$$

Let Z, X be the quantities of electricity which have passed through the galvanometer and battery respectively, in consequence of the charging of the condenser, and

Y and W the charges in the condenser and guard ring Then integrating equations (3) and (4), over a time extending from just before the condenser began to charge until it is fully charged, remembering that at each of these times $z_2 = 0$, we get

$$(b + \gamma + \alpha) Z + (b + \gamma) Y + \gamma W - \gamma X = 0$$

$$(a + \gamma + \beta) X - (\gamma + \beta) Y - \gamma Z - (\gamma + \beta) W = 0,$$

hence eliminating X

$$Z\left(b + \gamma + \alpha - \frac{\gamma^2}{a + \gamma + \beta}\right) + Y\left(b + \gamma - \frac{\gamma(\gamma + \beta)}{a + \gamma + \beta}\right) + W\gamma \frac{a}{a + \gamma + \beta} = 0$$

In our experiments, the battery resistance is very small, being less than 1 ohm, while β is 500,000 ohms, b 200,000 ohms, and γ 3000 ohms, thus the third term is less than $\frac{1}{5,000,000}$th part of the second, and may be neglected, and we get, neglecting the battery resistance

$$Z = -\frac{b}{b + \gamma + \alpha - \dfrac{\gamma^2}{\gamma + \beta}} Y$$

If $\{A\}$ $\{B\}$ $\{D\}$ denote the potentials of A, B, D when the condenser is fully charged, C the capacity of the condenser, then

$$Y = C[\{A\} - \{B\}]$$

But

$$\frac{\{A\} - \{B\}}{\alpha + \beta \dfrac{(b + \alpha + \gamma)}{\gamma}} = \frac{\{A\} - \{D\}}{\alpha}$$

The right-hand side of this equation is z_1, the steady current through the galvanometer, so that

$$Y = -Cz_1\left(\alpha + \beta \frac{(b + \alpha + \gamma)}{\gamma}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$Z = z_1 bC \frac{\left\{\alpha + \beta \dfrac{(b + \alpha + \gamma)}{\gamma}\right\}}{b + \gamma + \alpha - \dfrac{\gamma^2}{\gamma + \beta}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the condenser is charged n times per second, the quantity of electricity which passes in consequence through the galvanometer per second is nZ If the galvanometer

needle remains undeflected, the quantity of electricity which passes through the galvanometer in unit time must be zero But this quantity is $nZ + z_1$, so that

$$nZ + z_1 = 0.$$

Substituting this relation in equation (6), we get

$$C = \frac{1}{n}\frac{\gamma}{b\beta}\frac{\left\{1 - \dfrac{\gamma^2}{(\gamma+\beta)(b+\gamma+\alpha)}\right\}}{1 + \dfrac{\gamma\alpha}{(b+\alpha+\gamma)\beta}} \qquad (7)$$

From this equation, if we know the resistances and the speed, we can calculate the capacity.

In order to apply the correction for the difference of potential between the guard ring and the inner cylinder, we require $\{A\} - \{C\}/\{A\} - \{B\}$, now

$$\frac{\{A\} - \{C\}}{\{A\} - \{B\}} = \frac{b}{\alpha}\,\frac{\{A\} - \{D\}}{\{A\} - \{B\}}$$

$$= \frac{b}{\alpha + \beta\dfrac{(b+\alpha+\gamma)}{\gamma}}$$

Where n was equal to 64 the approximate values of the resistances were

$$\begin{aligned} b &= 200{,}000 \text{ ohms} \\ \alpha &= 20{,}000 \quad ,, \\ \beta &= 500{,}000 \quad ,, \\ \gamma &= 3{,}000 \quad ,, \end{aligned}$$

substituting the values

$$\frac{\{A\} - \{C\}}{\{A\} - \{B\}} = \frac{1}{183}$$

We shall now go on to discuss the details of the method whose theory we have just given.

The Commutator

The general view of this is shown in figs. (4), (5), (6). The framework is strongly made of cast iron, and somewhat resembles the headstock of a lathe, it is provided with two hardened steel centres, capable of adjustment, between which runs the axle of the commutator. This was made of tool steel but not hardened. The centres were very good, and the apparatus ran with very little friction and wear

We will now describe the different pieces fixed to the axle beginning from the left of fig 4

Fig 4

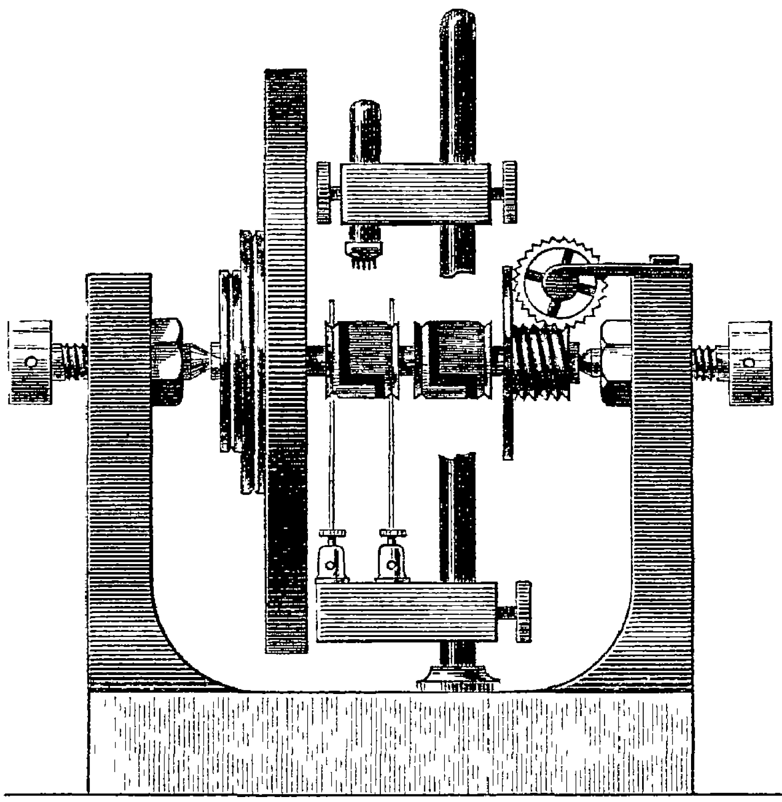
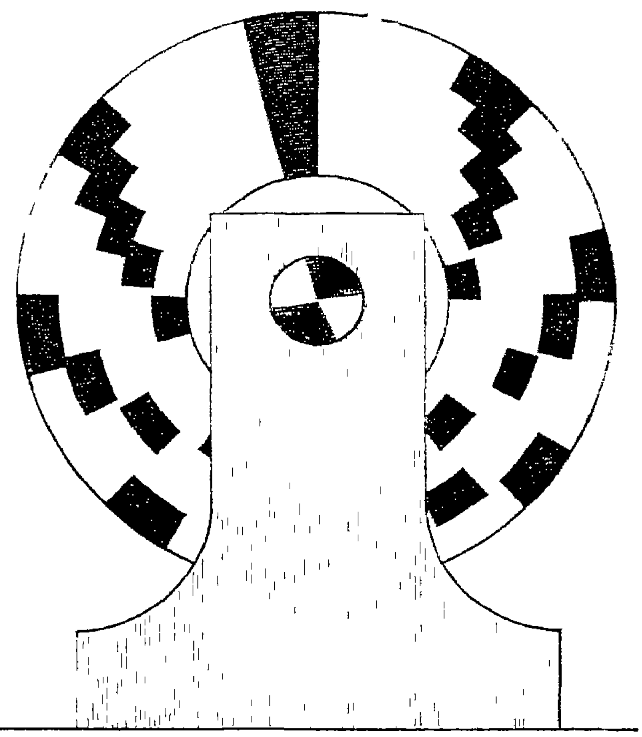


Fig 5



0 5 10 15 20

SCALE OF CENTIMETRES

First there is a wooden disc with two grooves for the driving string, next comes the wooden disc on which the stroboscopic pattern is painted. The end view of this is shown in fig 5

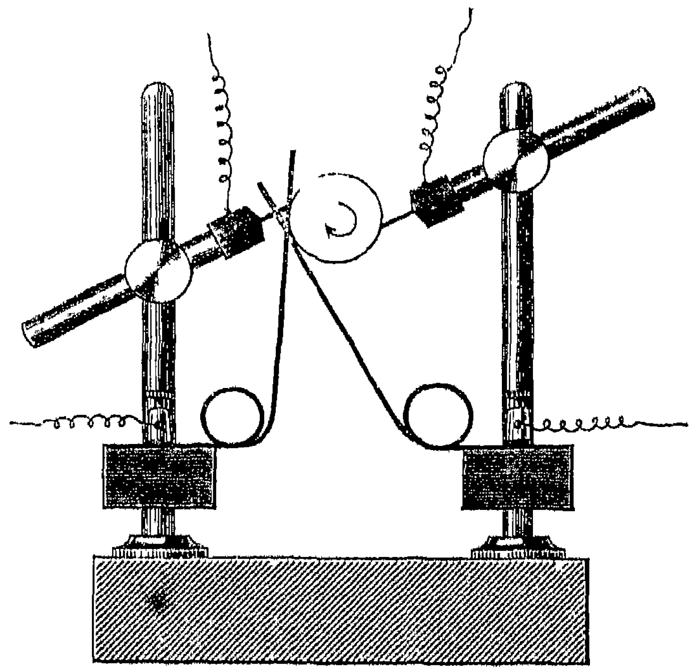
Beyond this are the two commutators, which are exactly alike, each is made of two portions of brass tube fixed to an ebonite bush Two grooves are turned in the continuous portion of the tube, in which rest two wires by means of which electrical connection is made with the parts of the commutator Although the slits in the commutator were not more than about 2 mm. wide yet no trouble at all was experienced through want of insulation The commutator could be easily cleaned by scraping the ebonite between the parts of the commutator with a sharp tool There was also very good insulation between the commutator and the axle The insulation was tested several times by means of the gold leaf electroscope, and was found practically perfect.

Following the two commutators is an ebonite disc to prevent oil from the centres getting to the commutators

The last thing on the axle is an endless screw in which gears a worm-wheel of 30 teeth. This wheel is furnished with a pin which makes contact with a spring once in every revolution of the wheel, *i e*, every 30 revolutions of the commutator. The spring is insulated from the framework, and the contact of the pin with it completes the circuit required for exciting one of the electromagnets of the recording apparatus The spring is not shown in the figure.

The supports for the contact wires of the commutator and for the brushes were two pillars screwing into the base of the framework. One of these is shown in the figure, it is shown broken, and the two parts separated in order to show both parts of the commutator Fig 6 shows the arrangement as it would look in an end view if the

Fig 6



stroboscopic disc were absent. The wires for making contact with the commutator are clamped under the two binding screws and are made with a curl Q so as to be very elastic and to ensure contact with the commutator. Before the curl was put into the wires it was found that the jarring of the apparatus gradually caused the wires (though made of hard drawn brass) to relax their pressure on the commutator and to make uncertain contacts. With the curled wires no trouble was found in this respect, and the pressure could be made much less, saving both friction and wear The binding screws clamping the wires are fixed to a piece of ebonite which can slide on the pillar and be fixed in any position by a screw. The pillar also carries a support for the brush. The brush itself is made of very fine hard drawn brass wire, soldered into a brass piece of a suitable shape. The way in which the brush acted depended a good deal upon the regularity and straightness of its wires. We found it best not to have a thick bunch of wire, but a thin layer only a few wires thick. The brass piece into which the brush wires were soldered, fits on the end of an ebonite rod passing through the brush holder and capable of being fixed in its proper position by means of a screw.

The whole arrangement was clamped down to a thick iron slab, resting on a strong table. In this way the vibrations, which would otherwise have been set up, were avoided.

The tuning fork, by means of which the speed was observed and regulated, was placed on a separate table. At first the two were on the same table, but it was found that at the speed at which the commutator made one revolution for each

complete vibration of the fork, the vibrations set up by the fork were sufficient to make the contacts uncertain. As finally fixed, the apparatus worked extremely satisfactorily, and would run for several hours without either the brushes or the springs requiring any adjustment

The commutator was driven from a water motor which was supplied from a tank at the top of the laboratory to secure a constant head of water It was driven by a band of fine fishing line joined with a "long splice", any rougher method of joining produced a joint, the effect of whose passage over the pulley of the commutator was plainly seen by the observer at the galvanometer. A second band went from a small pulley on the motor to a pulley fixed within easy reach of the observer stationed at the tuning fork The regulation of the speed was done by letting the auxiliary band run through the fingers, and slightly pressing it. This was found to be a much better plan than regulating by the band driving the commutator But, in spite of this, the speed of the commutator as judged by the steadiness of the pattern seen through the slits of the tuning fork was subject to incessant small agitations, and it required considerable vigilance on the part of the observer at the fork to keep the pattern quite at rest A heavier disc on the commutator would no doubt have made this easier.

The supply of water was so adjusted that it was able to drive the commutator slightly faster then the speed required The necessary fine adjustment was made by slightly pressing the regulating band.

Determination of the Speed of the Commutator.

To ascertain the speed at which the commutator was being driven its stroboscopic disc was observed through a pair of narrow slits fastened to the prongs of an electrically maintained fork. This fork made approximately 64 complete vibrations per second The disc was provided with circles containing 4, 5, 6, 7, 8 spots at equal intervals, so that when a distinct pattern was observed through the slits on the fork the commutator made one of the following numbers of revolutions per second —

16·0, 18·3, 21·35, 25·6, 32·0, 36·6 42·7, 48·0,
51·2, 54·9, 64, 73·2, 76·9, 80·1, 85·4.

These numbers are respectively $\frac{1}{4}$, $\frac{2}{7}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{4}{7}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{6}{7}$, 1, $\frac{8}{7}$, $\frac{6}{5}$, $\frac{5}{4}$, and $\frac{4}{3}$ of 64.

Any one of these speeds could be obtained by simply regulating the supply of water to the motor. Higher speeds could, of course, have been observed, but the motor would not drive the commutator much faster than 80 revolutions per second.

Experiments at most of these speeds will be found below.

It will be observed that this method gives a great choice of speeds, all of which can be determined with the same accuracy, and whose relation one to another is known with absolute accuracy.

The standard to which the speed was referred during the experiments was the

standard fork used by Lord RAYLEIGH This makes about 128 complete vibrations per second The electrically driven fork maintained another fork whose natural period is about half its own This gave beats with the standard, and by counting the beats the speed of the fork through which the stroboscopic disc was observed could be determined in terms of that of the standard fork.

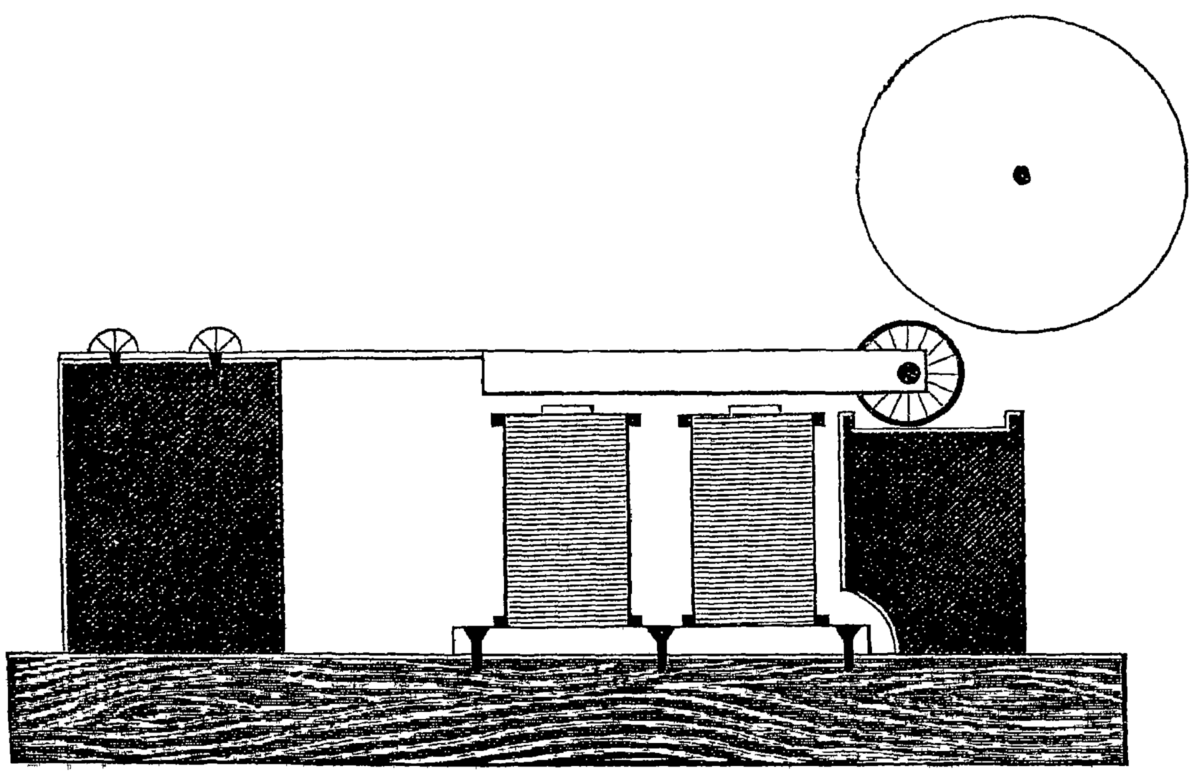
Determination of the Speed of the Standard Fork

It was considered advisable to make a new determination of the standard fork Lord RAYLEIGH ('Phil Trans,' 1883, p 320) found that its speed at t° C was

$$128\ 140\ \{1 - (t - 16) \times \cdot 00011\}$$

We have taken Lord RAYLEIGH's value of the temperature coefficient

We determined the speed of the standard in the following way. At one end of the commutator an endless screw was fixed with a cog-wheel geared into it The cog-wheel makes one revolution for 30 made by the commutator. A pin fixed to the wheel touches a spring once in every revolution, and completes an electric circuit



which causes a mark to be made on the tape of the recording apparatus. The laboratory clock is also arranged so as to complete a circuit once every second A paper tape is pulled along at an approximately uniform rate between guides on a block of wood. Two electro-magnets are fixed at right angles to the guides. One of these is excited when contact is made by the cog-wheel of the commutator, the other when contact is made by the clock. The armatures are kept away from their magnets by springs fixed to one end.

The other end of the armature carries a fork in which a small disc can rotate. When the magnet is not excited the disc presses against a roller covered with printer's ink. When the magnet is excited the disc presses on the paper tape as it passes underneath it. The disc is kept inked by revolving the inking roller.

The general disposition of the apparatus will be seen from the figure.

As the tape is drawn along we get two series of marks, one due to the commutator, the other to the clock. By comparing the position of the marks made by the commutator with those made by the clock we get a very accurate method of timing the commutator. We can find the time at which say 15 commutator contacts are made, and then, after the lapse of 5 or 10 minutes, find the time at which another 15 commutator contacts are made. Subtracting the one set of times from the others we get 15 intervals which should, of course, be equal. Taking the average we get a very accurate value for the time occupied by (say) 600 revolutions of the cog wheel.

The paper tape was at first drawn along by a Morse receiver, but as this did not act very uniformly it was replaced by a small winding arrangement driven by the same motor as the commutator. This ran perfectly uniformly.

The method of experimenting was as follows: when the observing forks had been running for a few minutes, so that everything had got steady, the beats between the standard fork and the auxiliary fork were counted. The forks were so arranged that 20 beats occurred in about 65 seconds. The motor was then started and regulated so that the commutator could be kept at half the speed of the fork, *i.e.*, about 32 revolutions per second.

The recording apparatus was then started and allowed to run for 5 or 10 minutes, the commutator being kept at a constant speed by observing the stroboscopic pattern through the fork. The apparatus was then stopped and the beats again counted.

Our reason for adopting this method was that we had originally intended to measure the speed of the commutator while we were making the electrical observations. This was, however, found to be too laborious, and had to be given up. The Laboratory clock was compared with the clock belonging to the Cambridge Philosophical Society, which is regularly rated from the Observatory.

The last three observations, when the apparatus was working very satisfactorily give for the rate of the standard fork at 16° C.

December 19 (tape running for 6 minutes)	128·1081
February 14 (" " 10 ")	128·0909
" 15 " " " "	128·1146
Mean	128·1045

Thus according to our observations the fork is slightly slower than when used by Lord RAYLEIGH. This is what might be expected from the secular softening of the steel.

The Galvanometer

This was one made in the Laboratory It has two coils, each with about 16,000 turns of wire The resistance of the coils when in series is 17,380 legal ohms Great care was taken with the insulation, which, when tested by means of a gold leaf electroscope, was found to be practically perfect

The Battery

We used 36 very small storage cells, two sets of 18 being placed in parallel The battery had thus an electromotive force of about 36 volts The small size of the battery enabled us to insulate it with ease The insulation was tested by a gold leaf electroscope, but no leak could be detected

The Resistances

The resistances used were contained in three boxes

I A Wheatstone's bridge box, No. 1256, ELLIOTT Legal ohms, with coils ranging from 1 to 5000 ohms, and proportional arms, each containing 10, 100, 1000 ohms

II. A box by ELLIOTT, containing 4 coils, 10,000, 20,000, 30,000, 40,000 B A units.

III A box by MUIRHEAD, containing originally 10 coils, each 100,000 B A. units.

The box I was provided with an aperture for a thermometer. The other boxes had no such provision, but as they were always kept permanently on the same table as box I we may hope that their temperature did not differ much from that of I

The resistances of the coils in II. and III. were always ascertained by means of I. It was, therefore, necessary to obtain some definite knowledge of I with reference to some coils whose values are accurately known

The standard coils used were

No. 141, C.L C No 102	. .	10 00103 legal ohms at	16° 7 C.
No. 143, C L C No 104	. .	99 9977 ,, ,,	16° 05 C.
No 145, C.L C No 106	. . .	1000 306 ,, ,,	17° 4 C.

The Legal ohm being assumed to be 1/ 9889 B A unit

See "British Association Report,' 1885.

Mr GLAZEBROOK informs us that the temperature coefficient of these coils may be taken as 0003 per 1° C We shall take the same temperature coefficient for the coils in the boxes.

Calling the coils in the "proportional arm" which joins the rest of the box 10_a, 100_a, 1000_a, those in the other arm 10_b, 100_b, 1000_b we found

$$\frac{1000_b}{1000_a} = 1 - \frac{3}{10000}$$

$$\frac{1000_b}{100_a} = 10\left\{1 - \frac{18}{10000}\right\}$$

$$\frac{1000_b}{10_a} = 100\left\{1 + \frac{76}{10000}\right\}.$$

Correcting for the inequalities of 1000_a and 1000_b we found by comparison with the Standard coils the following values at 16° C, for the various coils in box I.

	Legal ohms
100_α	100 144
100_β	100 144
100	100 113

The last resistance was made up from all the coils from 1 to 50,

	Legal ohms
1000_α	1001 06
1000_β	1001 01
1000	1001 06

The last was made up of all the coils from 1 to 500,

	Legal ohms
2000	2002 22
5000	5004 88

We see from this that the coils are very consistent, and on the average greater than legal ohms, in the ratio of 1 0011 to 1

The box II. was measured as a whole, using the "proportional" coils 1000_b and 10_a

The apparent value was 98765. Hence the true value is

$$98765 \times 1{\cdot}0011\left(1 + \frac{76}{10000}\right) = 98870 \text{ legal ohms at } 16^\circ \text{ C.}$$

The coils of box III were tested in the same way and were found to be

	Resistance as measured by box I	True resistance in legal ohms
1	98723	98878
2.	98617	98732
3.	98690	98805
4	98727	98842
5	98717	98832
6	98768	98883

The box III had one of its coils wrong when we began to use it, and during part of our work when we were attempting to use much greater battery power than we finally adopted, three more of the coils gave way. For this reason we could only use six out of the ten coils These six coils, as well as box II, kept their resistances very constant during the whole of the investigation. They were tested several times during a whole year, and no perceptible change was detected

The resistance boxes were placed on blocks of paraffin, and the insulation tested by a gold leaf electroscope

Method of making the Observations

The battery, resistance boxes, galvanometer, and condenser were connected as in the diagram (3), the commutator being placed so near to the condenser that a very short wire sufficed to make the connection. The insulation of the whole system when connected up was tested from time to time by a gold leaf electroscope.

The electrically driven fork was set going and the beats of the auxiliary fork with the standard fork observed The water supply was then adjusted so that the motor drove the commutator at the required speed One observer (G. F C. S) observed the stroboscopic disc of the commutator through the slits of the fork, and kept the speed steady by means of the controlling arrangement already described The other observer (J J T) observed the galvanometer. When the speed of the commutator had got steady, resistances were taken out of the Wheatstone's bridge box in the arm CD until no deflection was produced when the galvanometer circuit was broken. The sensitiveness of the galvanometer was such that the effect produced by altering the resistance in CD by 2 ohms in 3000 could be detected, and the speed of the commutator was kept so steady that the light reflected from the galvanometer mirror did not move over more than half a division, the deflection produced by the condenser when not balanced was more than 500 scale divisions.

The battery was then reversed and the operation repeated The wire connecting the condenser to the commutator was then detached from the condenser, and the same operation repeated. In this way the capacity of the wire was determined The temperatures of the fork and resistances were then read, and the beats of the auxiliary fork with the standard fork again determined

The results of these observations are exhibited in the annexed Table.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Date	Speed	Beats per minute	Temperature of fork	b	β	Value of γ for condenser + wire	Value of γ for wire	Value of γ for condenser	Reduced to 64	Temperature of resistances	Correction to 16° for resistances	Correction to 21 beats per minute	Correction to 16° for fork	Correction to $nC = \frac{\gamma}{b\beta}$	Correction to legal ohms	Correction for difference of potential between g 1 and condenser	Corrected value of γ	Electromagnetic measure of capacity, $\times 10^{-21}$
Dec 17	64	21.4	17.5	197753	494089	2797.5	39.5	2758	2758	16.5	− .41	+ .14	+ .45	− 1.37	+ 3.04	+ 1.72	2761.6	443 448
„	32	21.4	17.5	„	„	1399.5	20	1379.5	2759	16.5	− .41	+ .14	+ .45	− .68	+ 3.04	+ .86	2762.4	443 576
„	48	21.5	17.5	„	„	2099	30	2069	2758.7	16.5	− .41	+ .18	+ .45	− 1.07	+ 3.04	+ 1.29	2762.1	443 528
„	80	21.5	17.5	„	„	3498	50.25	3447.75	2758.2	16.5	− .41	+ .18	+ .45	− 1.80	+ 3.04	+ 2.15	2761.8	443 480
„	64	21.8	18	„	„	2798	40.5	2757.5	2757.5	17.2	− 1	+ .29	+ .60	− 1.37	+ 3.04	+ 1.72	2760.8	443 319
„	55	21.8	18	„	„	2399	34.5	2364.5	2758.6	17.2	− 1	+ .29	+ .60	− 1.15	+ 3.04	+ 1.49	2761.9	443 496
„	42	21.8	18	„	„	1866	27	1839	2758.5	17.2	− 1	+ .29	+ .60	− .91	+ 3.04	+ 1.14	2761.6	443 448
Dec 18	64	21.7	17.5	296585	397257	3355	49	3306	3306	17.2	− 1.2	+ .29	+ .54	− 1.66	+ 3.64	+ 3.20	3310.8	443 112
„	48	21.7	17.5	„	„	2517.5	37.5	2480	3306.7	17.2	− 1.2	+ .29	+ .54	− 1.32	+ 3.64	+ 2.40	3311.1	443 153
„	32	21.7	17.5	„	„	1679.5	25	1654.5	3309	17.2	− 1.2	+ .29	+ .54	− .84	+ 3.64	+ 1.60	3313.0	443 407
„	80	21.7	17.5	„	„	4193	60.5	4132.5	3306	17.4	− 1.4	+ .29	+ .54	− 2.38	+ 3.64	+ 4.00	3310.2	443 032
„	64	21.8	18.2	„	„	3359.5	49	3310.5	3310.5	17.2	− 1.2	+ .35	+ .79	− 1.66	+ 3.64	+ 3.20	3315.6	443 754
„	55	21.8	18.2	„	„	2881.5	43	2838.5	3311.6	17.5	− 1.5	+ .35	+ .79	− 1.54	+ 3.64	+ 2.74	3315.7	443 768
„	48	21.8	18.2	„	„	2520	38	2482	3309.3	18	− 2	+ .35	+ .79	− 1.32	+ 3.64	+ 2.40	3313.2	443 434
„	32	21.6	18.2	„	„	1680.5	25	1655.5	3311	18	− 2	+ .26	+ .79	− .81	+ 3.64	+ 1.60	3314.4	443 594
„	16	21.6	18.2	„	„	841	13	828	3312	18	− 2	+ .26	+ .79	− .41	+ 3.64	+ .80	3315.0	443 675
„	80	21.6	18.2	„	„	4196	61.5	4134.5	3307.6	18.1	− 2.1	+ .26	+ .79	− 2.38	+ 3.64	+ 4.00	3311.8	443 246
Dec 20	64	21.6	16.6	197748	494094	2799	41.5	2757.5	2757.5	16.2	− .16	+ .22	+ .18	− 1.37	+ 3.04	+ 1.72	2761.2	443 390
„	32	21.8	16.8	„	„	1399.5	21	1378.5	2757	16.4	− .33	+ .29	+ .24	− .68	+ 3.04	+ .86	2760.4	443 262
„	48	21.8	16.8	„	„	2101	30.75	2070.25	2760.3	16.4	− .33	+ .29	+ .24	− 1.07	+ 3.04	+ 1.29	2763.7	443 792
„	80	21.6	16.6	„	„	3500	51.5	3448.5	2758.8	16.4	− .33	+ .22	+ .18	− 1.80	+ 3.04	+ 2.15	2762.2	443 551
„	55	21.6	16.6	„	„	2401.5	35.25	2366.25	2760.6	16.4	− .33	+ .22	+ .18	− 1.15	+ 3.04	+ 1.49	2764.0	443 840
„	64	21.6	16.6	„	„	2800.5	41.3	2759.2	2759.2	16.7	− .57	+ .22	+ .18	− 1.37	+ 3.04	+ 1.72	2762.4	443 583

Explanation of Table of Results and Methods of Reduction

Column 2 gives the approximate speed of the commutator.

Column 3 gives the beats per minute between the standard fork and the auxiliary fork driven by the electrically maintained fork The auxiliary fork vibrated twice as fast as the driving fork, and was slightly slower than the standard.

Column 4 The temperature of the fork given by a thermometer hung a short distance from the fork

Columns 5 and 6 The resistances of the arms AC and BD in legal ohms at 16° C

Column 7 gives the value of the resistance in CD required for the balance when both the condenser and the connecting wire were joined to the key of the commutator We shall denote this by γ_1 Each number is the mean of two observations made with the current from the battery flowing first in one direction and then in the opposite The difference between the two readings very seldom amounted to more than 1 ohm.

Column 8 gives the value of the resistance in CD required for the balance when the wire alone is joined to the key of the commutator We shall denote this by γ_2 Each is the mean of two readings corresponding to the two directions of the battery current

Column 9 gives the difference between the last two columns We must remark that, since the formula $n\text{C} = \gamma/\beta b$ is not sufficiently accurate for our purpose, that this difference does not strictly represent the value of γ, which would be required to balance the condenser alone. With the resistances employed we may write the formula (7)

$$n\text{C} = \frac{\gamma}{\beta b}\left\{1 - \frac{\gamma(\gamma + \alpha)}{(\alpha + b)\beta}\right\},$$

so that if, as in our case, we keep β and b constant, $n\text{C}$ is not quite proportional to γ Strictly, we should find the capacity of the combination of the condenser and wire, and then subtract the capacity of the wire. What we have done is to calculate as if $\gamma_1 - \gamma_2$ represented the capacity of the condenser. The difference amounts to writing in the small term $\gamma(\gamma + \alpha)/(\gamma + b)\beta$, $\gamma_1 - \gamma_2$ instead of γ_1, since the correcting term inside the bracket, in the case of the wire, is too small to be appreciable Since, however, this term is already very small, and does not affect the result by much more than one part in 2000, and the change we have made only alters its value by 1 per cent at most, it is evident that it will produce no appreciable effect on the value of the capacity

Column 10 is headed "(reduced to 64)." Since the value of the capacity is given by

$$\text{C} = \frac{\gamma}{n\beta b}\left\{1 - \frac{\gamma(\gamma + \alpha)}{(\alpha + b)\beta}\right\},$$

or with sufficient accuracy for our purpose by

$$C = \frac{\gamma}{nb\beta}\left\{1 - \frac{\gamma\alpha}{b\beta}\right\},$$

and if γ_a, γ_b are the values of γ corresponding to the speeds n_a, n_b, then

$$\frac{\gamma_a}{n_a}\left\{1 - \frac{\gamma_a\alpha}{b\beta}\right\} = \frac{\gamma_b}{n_b}\left\{1 - \frac{\gamma_b\alpha}{b\beta}\right\},$$

or

$$\gamma_a\left(1 - \frac{\gamma_a\alpha}{b\beta}\right) = \frac{n_a}{n_b}\gamma_b\left\{1 - \frac{\alpha}{b\beta}\gamma_b\right\}$$

Hence we see that we may take as the corresponding uncorrected value of γ_a the value $\gamma_b n_a/n_b$, if we apply to this the correction $1 - \frac{\alpha}{b\beta}\gamma_b$ instead of the correction $\left(1 - \frac{\alpha}{b\beta}\gamma_a\right)$, this is what we have done in compiling this table, and this column contains the values of $\gamma_b n_a/n_b$

Column 11 gives the temperature of the box from which the resistance was taken. The temperature of the other two boxes are assumed to be the same. Since they were always on the same table as the γ box, and the temperature of this box varied very little, this assumption will not lead to any serious error

Column 12. "Correction to 16° for resistances" from the formula

$$nC = \frac{\gamma}{b\beta}$$

we see that if we take the temperature coefficients to be the same for the three resistances and equal to 0003, then we may throw all the effect of the variation on γ if for each degree above 16 we *subtract* from γ 0003 γ. In the first and third sets this amounts to 82 ohm for each degree, in the second to 1 ohm per degree

Column 13 "Correction to 21 beats per minute" The auxiliary fork always made rather more than 21 beats per minute with the standard When making 21 beats the speed of the auxiliary fork is 127 7545 complete vibrations per second, and 1 additional beat per minute indicates a diminution in the speed in the ratio $1 - \frac{1}{128\times 60}$ to 1. Since we wish to treat the observations as if the speed were constant we must for each additional beat increase γ in the ratio $1 + \frac{1}{128\times 60}$ to 1, *i e*, in the first and third set observations we must add 36 ohm to γ, and in the second set ·43 ohm

Column 14 "Correction to 16° for fork." The fork has a temperature coefficient of − ·00011, so that if we regard the speed as constant we must increase γ in the proportion 1 00011 to 1 for each degree the temperature of the fork exceeds 16° C. In the first and third sets this amounts to ·30, and in the second set to 36 ohm for each degree of excess of temperature above 16° C.

Column 15 "Correction to $nC = \gamma/b\beta$" We see from equation (7), page 606 that this is only an approximation to the correct value for nC, which is

$$nC = \frac{\gamma}{b\beta}\frac{\left\{1 - \frac{\gamma^2}{(\gamma+\beta)(b+\gamma+\alpha)}\right\}}{\left\{1 + \frac{\gamma\alpha}{(b+\alpha+\gamma)\beta}\right\}}$$

or with sufficient accuracy for our purpose

$$nC = \frac{\gamma}{b\beta}\left\{1 - \frac{\gamma(\gamma+\alpha)}{\beta(b+\alpha+\gamma)}\right\},$$

thus, if we wish to use the formula $nC = \gamma/b\beta$ we must diminish γ in the ratio $1 - \frac{\gamma(\gamma+\alpha)}{\beta(b+\alpha+\gamma)}$ to 1 The amount by which γ is to be diminished is given in column 15

Column 16. "Correction to legal ohms" The values of b and β given in columns (5) and (6) are already expressed in terms of legal ohms, but the values of γ are expressed in terms of the resistances in the Wheatstone bridge box, which are greater than legal ohms in the proportion 1 0011 to 1 We have then to add 3 04 ohms to γ in the first and third sets of experiments, and 3·64 ohms in the second

Column 17 "Correction for the difference of potential between the middle cylinder and the guard ring" By equation (6), page 9, if the difference of potential between the outer cylinder and the guard ring is less by δV than that between the outer cylinder and the inner cylinder, the capacity is greater than it would be if $\delta V = 0$ in the ratio

$$1 + \frac{\delta V}{V}\frac{h}{l}\left\{\frac{t}{c} - \frac{2}{\pi}\log\frac{4c}{h\epsilon}\right\} \text{ to } 1$$

where V is the difference of potential between the cylinders, t the thickness of the guard ring, $2c$ the thickness of the pieces of ebonite, and h the distance between the cylinders

Since $t = 1$, $h = 1$, $c = 145$ and $\delta V/V = -b/\left\{\alpha + \beta\frac{(b+\alpha+\gamma)}{\gamma}\right\}$ the capacity is less than it would be if the guard ring and the cylinder were at the same potential in the ratio of

$$1 - \frac{7\,5}{61}\frac{b\gamma}{\alpha\gamma + \beta(b+\alpha+\gamma)} \text{ to } 1,$$

so that in order to get the corresponding value of γ when the guard ring and cylinder are at the same potential, we must add to γ

$$\frac{7\,5}{61}\frac{b\gamma^2}{a\gamma + \beta(b+a+\gamma)},$$

and it is this correction which is given in column 17.

Column 18 contains the values of γ which result when all these corrections are made.

Column 19 contains the values of the electromagnetic measure of the capacity calculated from the formula

$$C = \frac{\gamma}{n\beta b}$$

Since the auxiliary fork makes 21 beats per minute with the standard fork, which makes 128 1045 vibrations per second, and since the observing fork makes half the number of vibrations of the auxiliary fork

$$n = 63\ 8773$$

Assuming that the B A unit = 9867 × 10^9 in absolute measure which corresponds to the legal ohm being 9978 × 10^9, we have

$$C = \frac{\gamma \times 10^{-9}}{b\beta \times 63\ 8773 \times 9977}$$

We have supposed that the resistance to the variable current which passes through the resistance coils during the charging of the condenser, is the same as the resistance to a steady current This is justified by the following investigation. Though in our experiments the resistances were so large that the charging was not oscillatory, let us suppose that it is oscillatory and has the maximum frequency $1/\sqrt{LC}$ Then if r' is the resistance to this oscillatory current, r the resistance to a steady current

$$r' = r\left\{1 + \lambda n^2\right\},$$

where λ is a small numerical coefficient and

$$n = \frac{\pi}{\sqrt{LC}}\frac{a^2}{\sigma},$$

where a is the radius of the cross section of the wire, and σ its specific resistance The coefficient of self induction of a galvanometer similar to the one we used was found some time ago to be 5 × 10^9, $\sigma = 2 \times 10^4$ and $a =$ 023 for 32 B W G , substituting these numbers we find $n = 10^{-3}$ about, and

$$n^2 = 10^{-6}$$

so that the correction will only amount to one part in a million, and may be neglected

The mean of the first set of observations		= 443 471 × 10^{-21}.
,,	second ,,	= 443 417 × 10^{-21}.
,,	third ..	= 443 569 × 10^{-21}.
The mean of all the observations	.	= 443 486 × 10^{-21}

The means of the observations at different speeds are given in the following table

Speed	Electromagnetic measure of capacity $\times 10^{-21}$
80	443 327
64	443 434
55	443 701
48	443 478
42	443 448
32	443 460
16	443 675

The means for the different speeds thus agree very well together, the greatest difference from the mean being about one part in 2000. There does not seem any indication of an effect depending on the number of times the condenser is charged per second, such as was very marked in Professor ROWLAND'S experiments ('Phil Mag,' vol 28, 1889)

Since the electrostatic measure of the capacity is 397·927, and the electromagnetic measure $443\ 454 \times 10^{-21}$

$$\text{“}v\text{”} = \sqrt{\left(\frac{397\ 927}{443{\cdot}486 \times 10^{-21}}\right)}$$
$$= \mathbf{2{\cdot}9955 \times 10^{10}} \text{ cm sec}^{-1}.$$

The value of the B A unit is taken to be 9867×10^{9} in absolute measure. This value of "v" agrees very nearly indeed with the value obtained by the most recent experiments for the velocity of light in air, these are

CORNU (1878)	$3\ 003 \times 10^{10}$ cm sec^{-1}
MICHELSON (1879) .	$2{\cdot}9982 \times 10^{10}$
MICHELSON (1882)	$2\ 9976 \times 10^{10}$.
NEWCOMB (1885) .	$\left.\begin{matrix}2\ 99615\\ 2{\cdot}99682\\ 2\ 99766\end{matrix}\right\} \times 10^{10}$
,, ,, . .	
,, ,, .	

In conclusion we desire to express our thanks to Mr. R S COLE, of Emmanuel College, who has given us valuable assistance on several occasions.

The following table taken from Mr E B Rosa's paper on the Determination of "v" ('Phil Mag,' vol 28, 1889, p 315), gives the results of previous determinations of "v" :—

1856	Weber and Kohlrausch	$3\ 107 \times 10^{10}$
1869.	W. Thomson and King	$2\ 808 \times 10^{10}$
1868.	Maxwell .	$2\ 842 \times 10^{10}$.
1872	M'Kichan	$2\ 896 \times 10^{10}$
1879	Ayrton and Perry	$2\ 960 \times 10^{10}$
1880	Shida	$2\ 955 \times 10^{10}$
1883.	J J Thomson	$2\ 963 \times 10^{10}$
1884	Klemenčič	$3\ 019 \times 10^{10}$
1888	Himstedt	$3\ 009 \times 10^{10}$
1889	W. Thomson . .	$3\ 004 \times 10^{10}$
1889	E B Rosa	$2\ 9993 \times 10^{10}$

INDEX

TO THE

PHILOSOPHICAL TRANSACTIONS (A)

FOR THE YEAR 1890

B

C

D

E

G.

I

K

L.

M

P

R.

S

T

V

W

LONDON
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PRINTERS IN ORDINARY TO HER MAJESTY

[To face Index Map.

List of Stations

No	Name
1	Aberdeen
2	Arinagower (Coll)
3	L Aylort
4	Ayr
5	Ballater
6	Banavie
7	Banff
8	Berwick
9	Boat of Garten
10	L Boisdale
11	Bunnahabhain
12	Callernish
13	Campbelton
14	Canna
15	Car-tairs
16	Crianlarich
17	Crieff
18	Cumbrae
19	Dalwhinnie
20	Dumfries
21	Dundee.
22	Edinburgh
23	Elgin
24	L Eriboll
25	Fairlie
26	Fort Augustus
27	Gairloch
28	Glasgow
29	Golspie
30	Hawick
31	L Inver
32	Inverness
33	Iona
34	Kirkwall
35	Kyle Akin
36	Lairg
37	Lerwick
38	Lochgoilhead
39	L Maddy
40A	Oban
40B	,, (Kerrera)
41	Pitlochrie
42	Port Askaig
43	Portree
44	Row (Gairloch)
45	Scarnish (Tiree)
46	Soa
47	Stirling
48A	Stornoway (Ard Point)
48B	,, (Castle)
49	Strachur
50	Stranraer
51	Stromness
52	E L Tarbert
53	Thurso
54	Wick
55	Aberystwith
56	Alderney.
57	Alnwick.
58	Alresford
59	Appleby
60	Barrow
61	Bedford
62	Birkenhead
63	Birmingham.
64	Braintree
65	Brecon
66	Bude Haven.
67	Cambridge.
68	Cardiff.
69	Cardigan
70	Carlisle
71	Chesterfield
72	Chichester
73	Clenchwarton
74	Clifton
75	Clovelly
76	Coalville
77	Colchester
78	Cromer
79	Dover
80	Falmouth
81	Gainsborough
82	Giggleswick
83	Gloucester
84	Grantham.
85	Guernsey (L'Erée)
86	,, (Peter Port)
87	Harwich
88	Harpenden
89	Haslemere
90	Holyhead
91	Horsham
92	Hull
93	Ilfracombe
94	Jersey (Grouville)
95	,, (S Louis)
96	,, (S Owen)
97	Kenilworth
98	Kettering
99	Kew
100A	King's Lynn
100B	,, ,, (Gaywood)
101	King's Sutton
102	Lampeter
103	Leeds
104	Leicester
105	Lincoln
106	Llandudno
107	Llangollen
108	Llanidloes
109	Loughborough
110	Lowestoft
111	Mablethorpe
112A	Malvern (Colwall Green)
112B	,, (Great Malvern)
112C	,, (Malvern Wells)
112D	,, (Mathon)
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114	Manton
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120	Northampton
121	Nottingham
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171	Cork
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174	Dublin
175	Enniskillen
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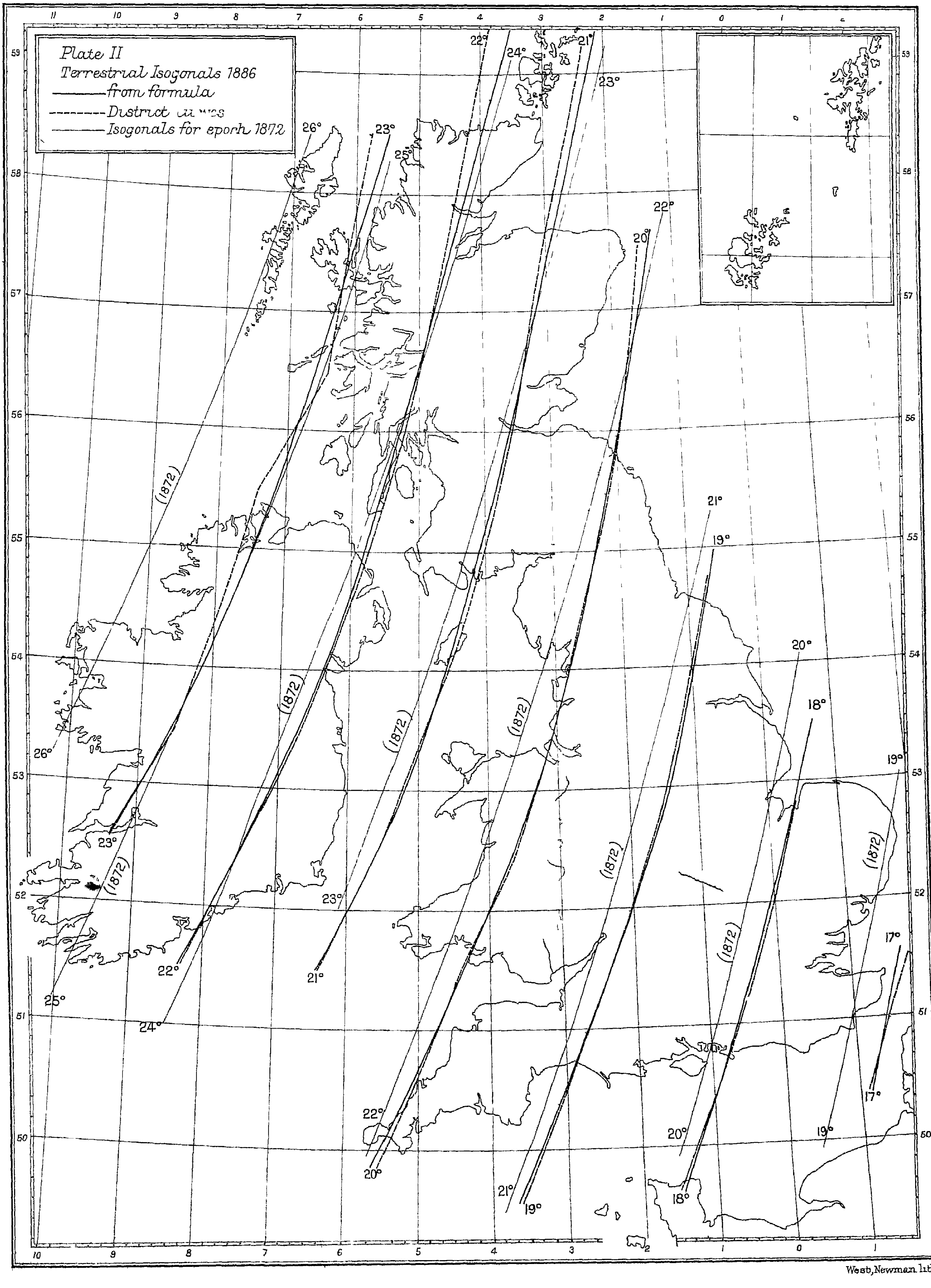
177	Gort
178	Greenore
179	Kells
180	Kildare
181	Kilkenny
182	Killarney
183	Kilrush
184	Leenane
185	Limerick
186	Lisdoonvarna
187	Lismore
188	Londonderry
189	Oughterard
190	Parsonstown
191	Sligo
192	Strabane
193	Tipperary
194	Tralee
195	Valentia
196	Waterfoot
197	Waterford
198	Westport
199	Wexford
200	Wicklow
201	Chepstow
202	Goodrich
203	Hereford
204	Ross
205	Tintern



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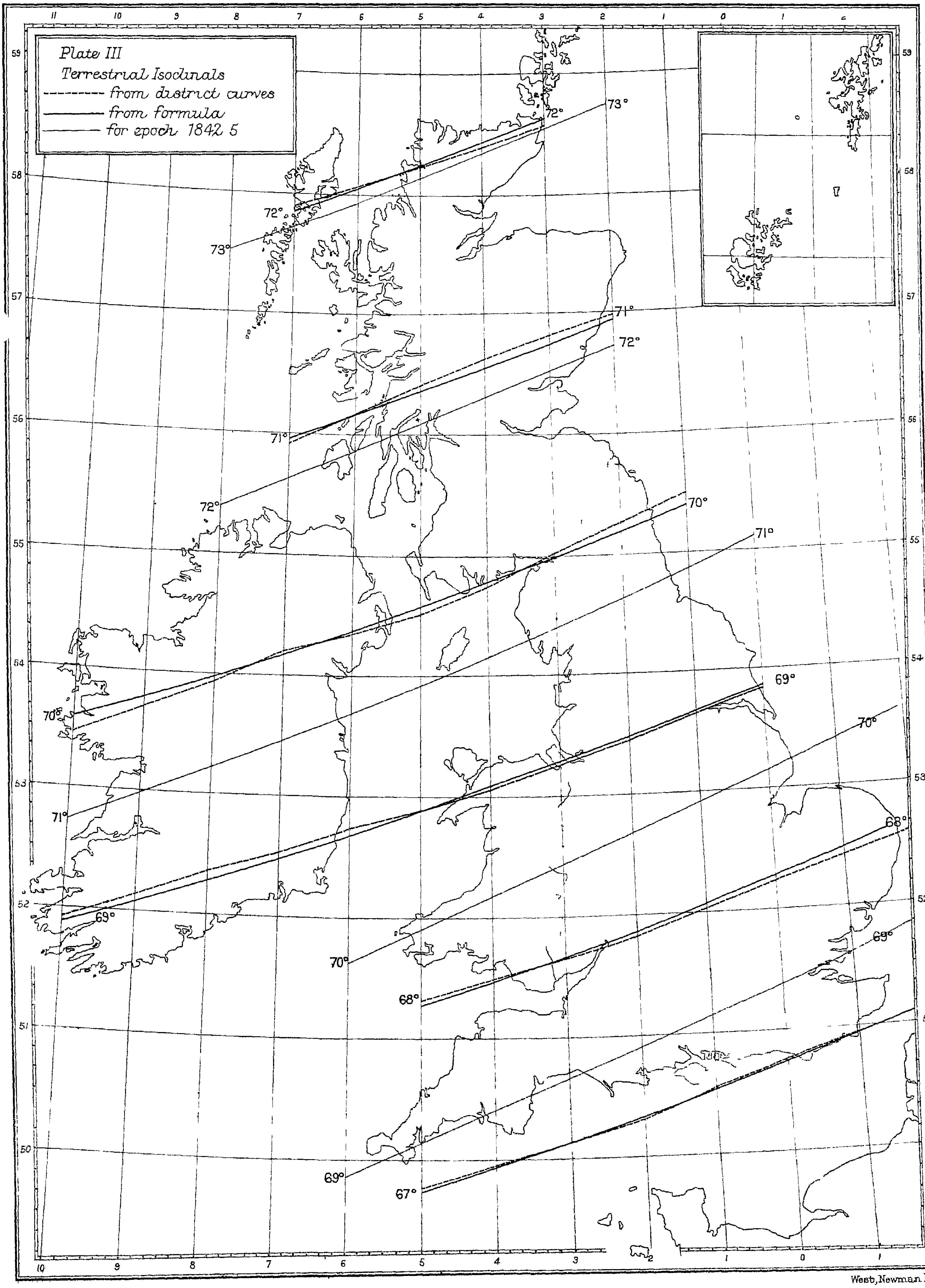

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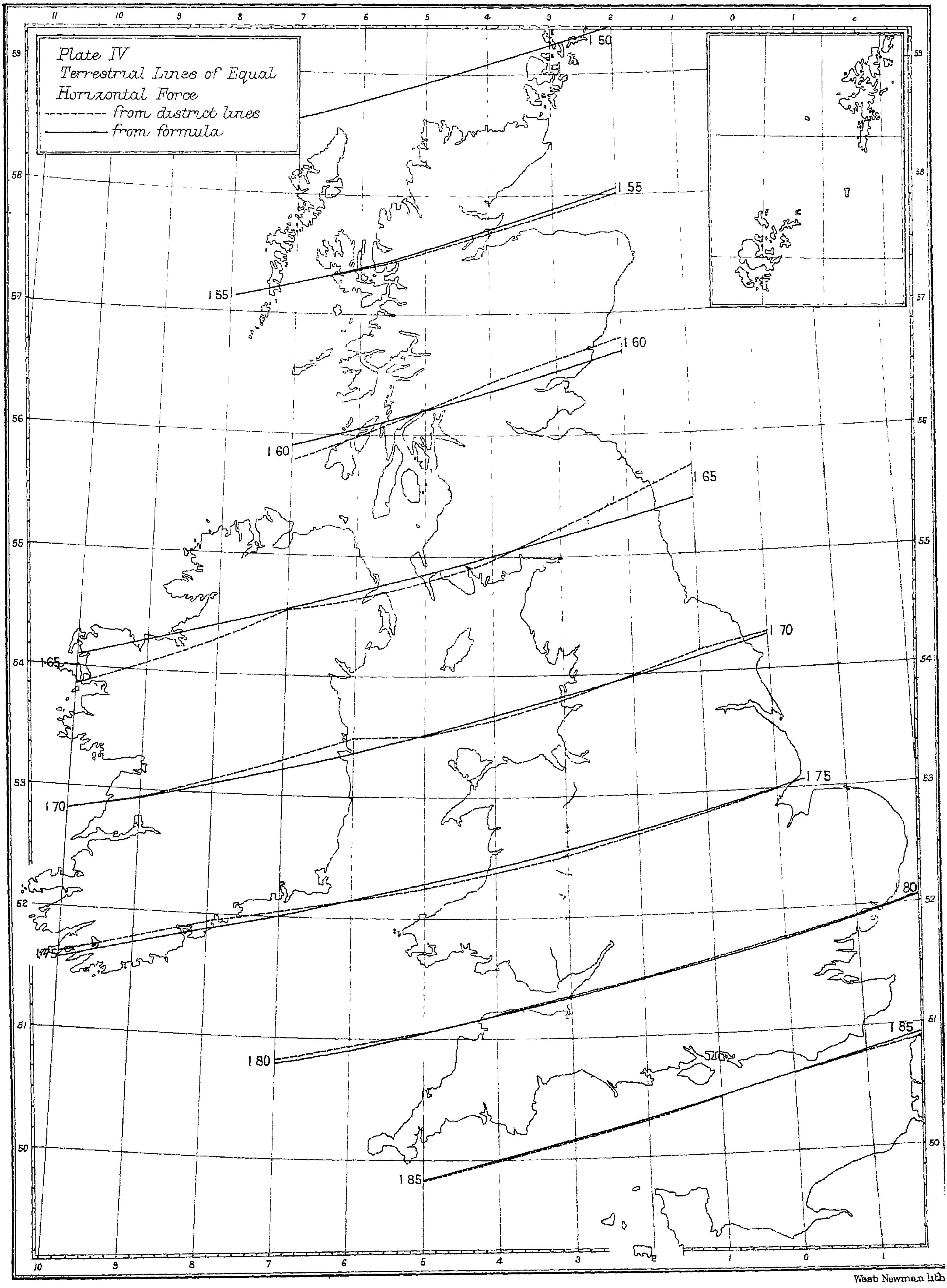

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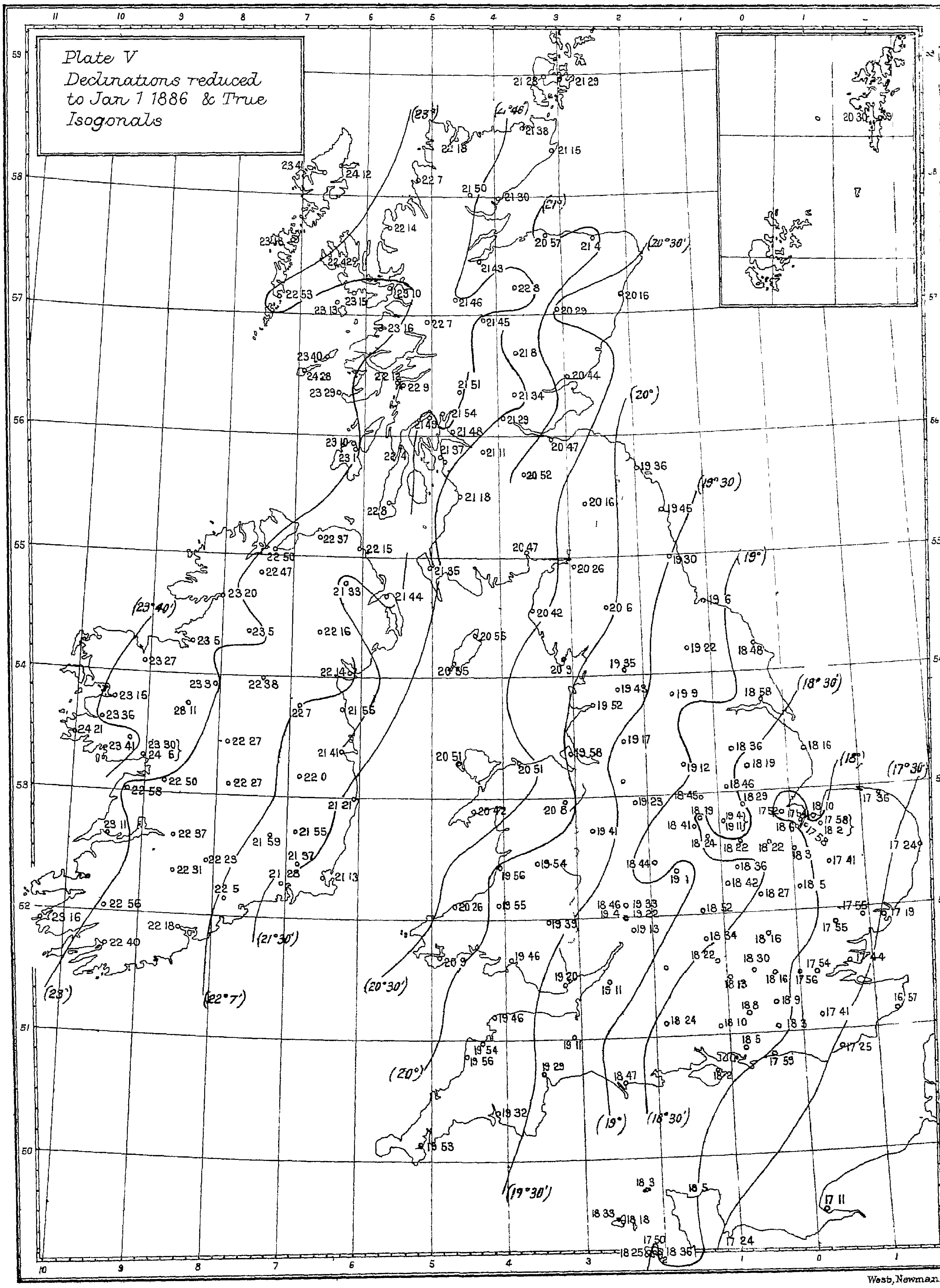
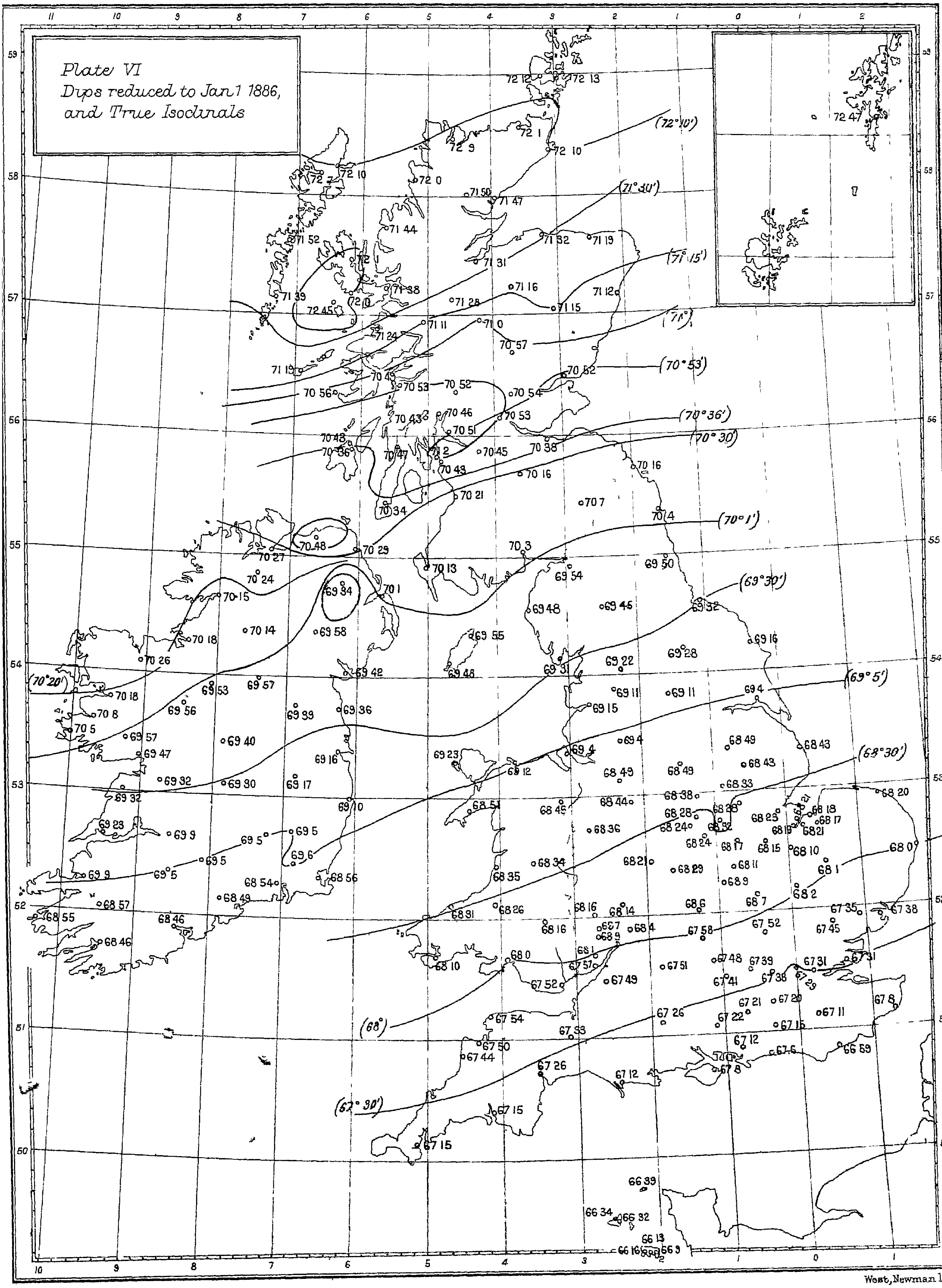


Webb, Newman lith



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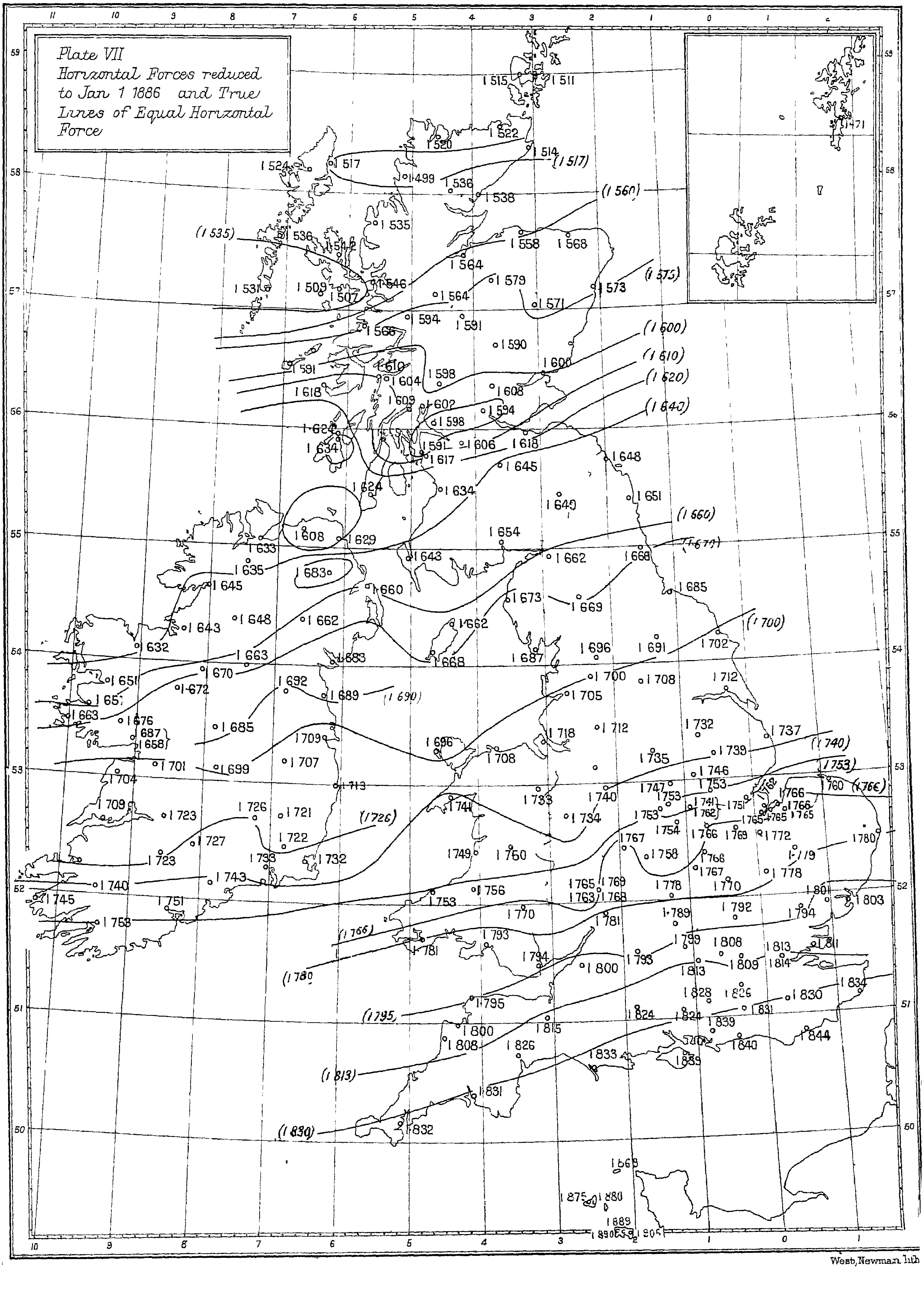

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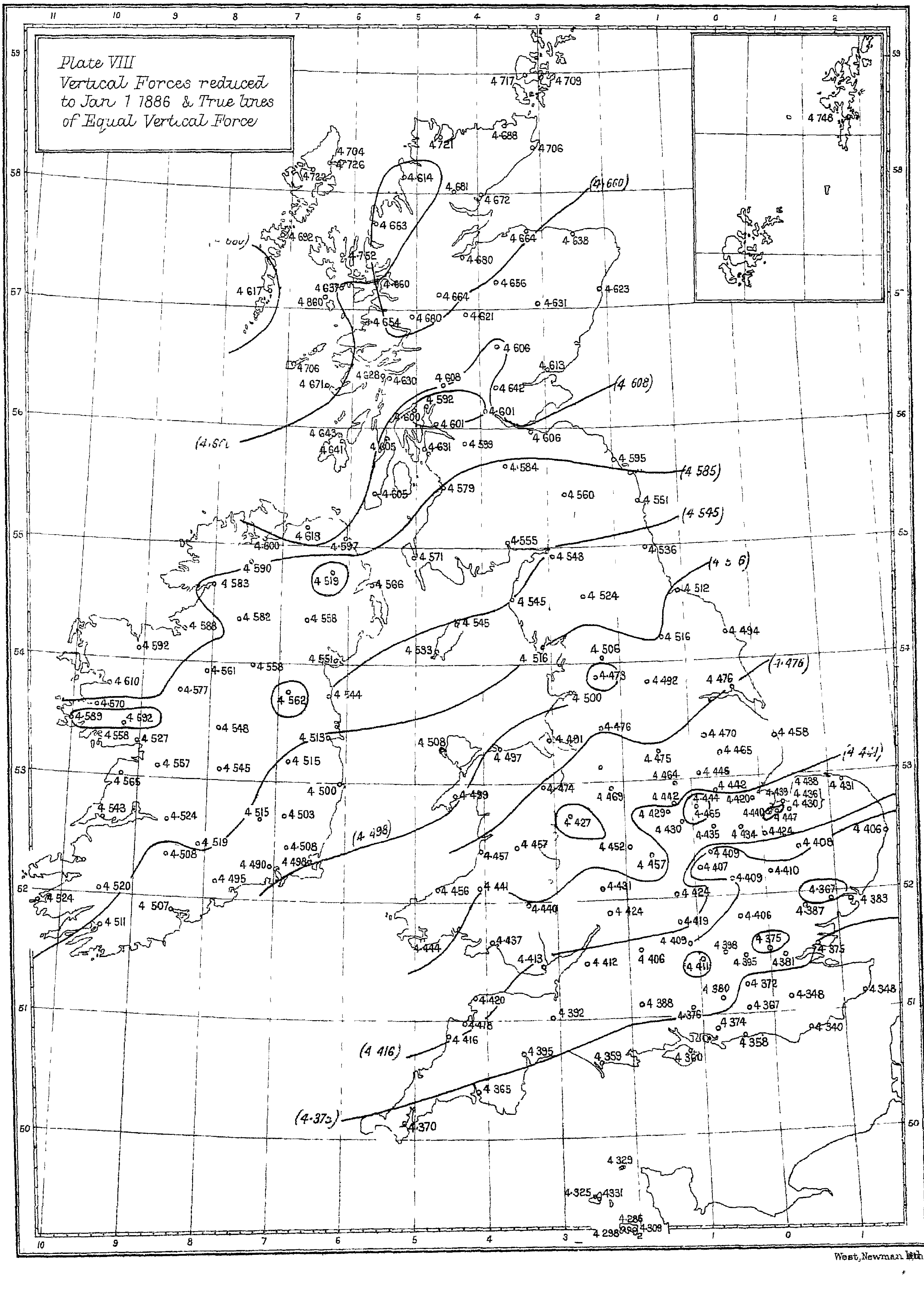


West, Newman lith

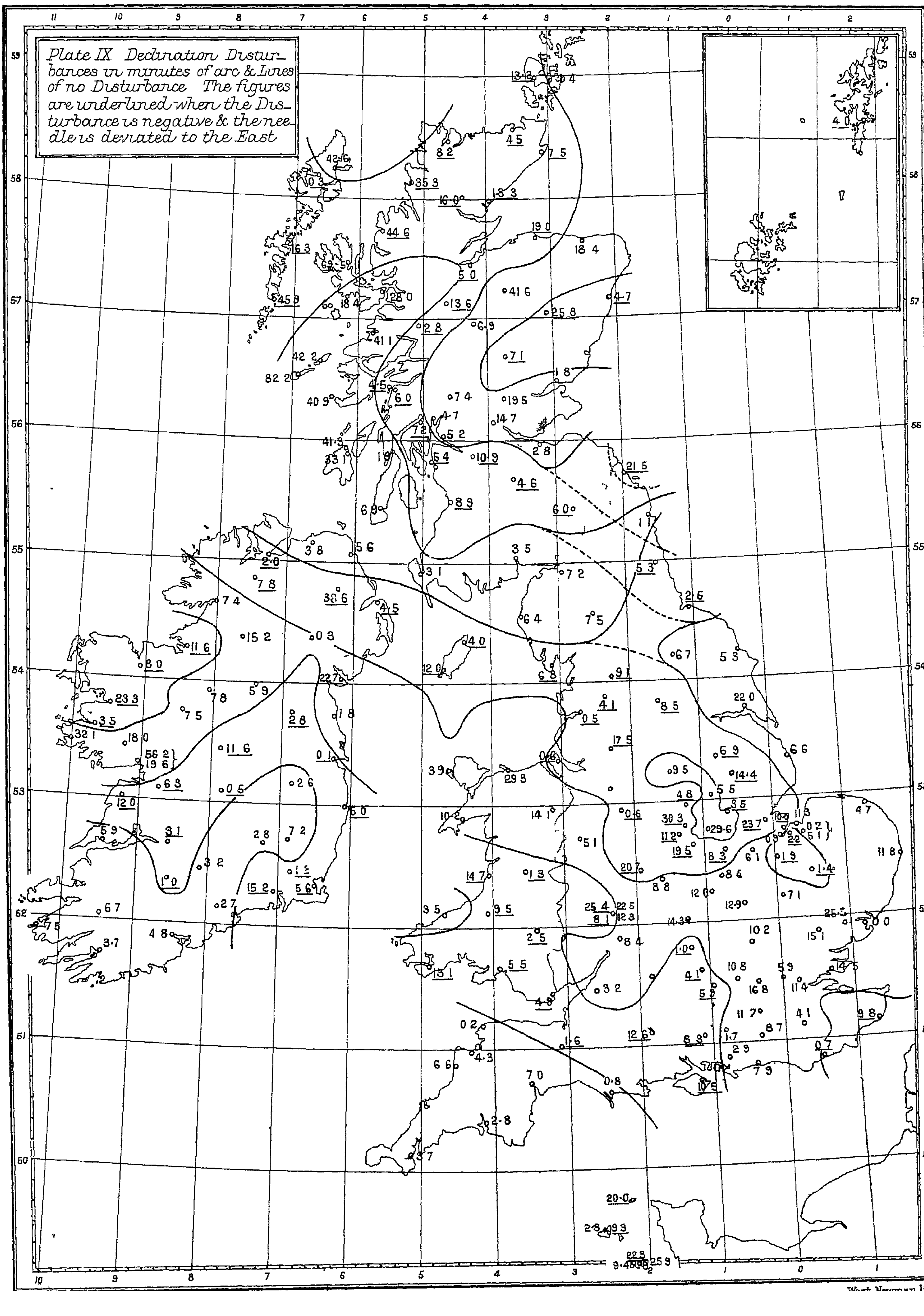
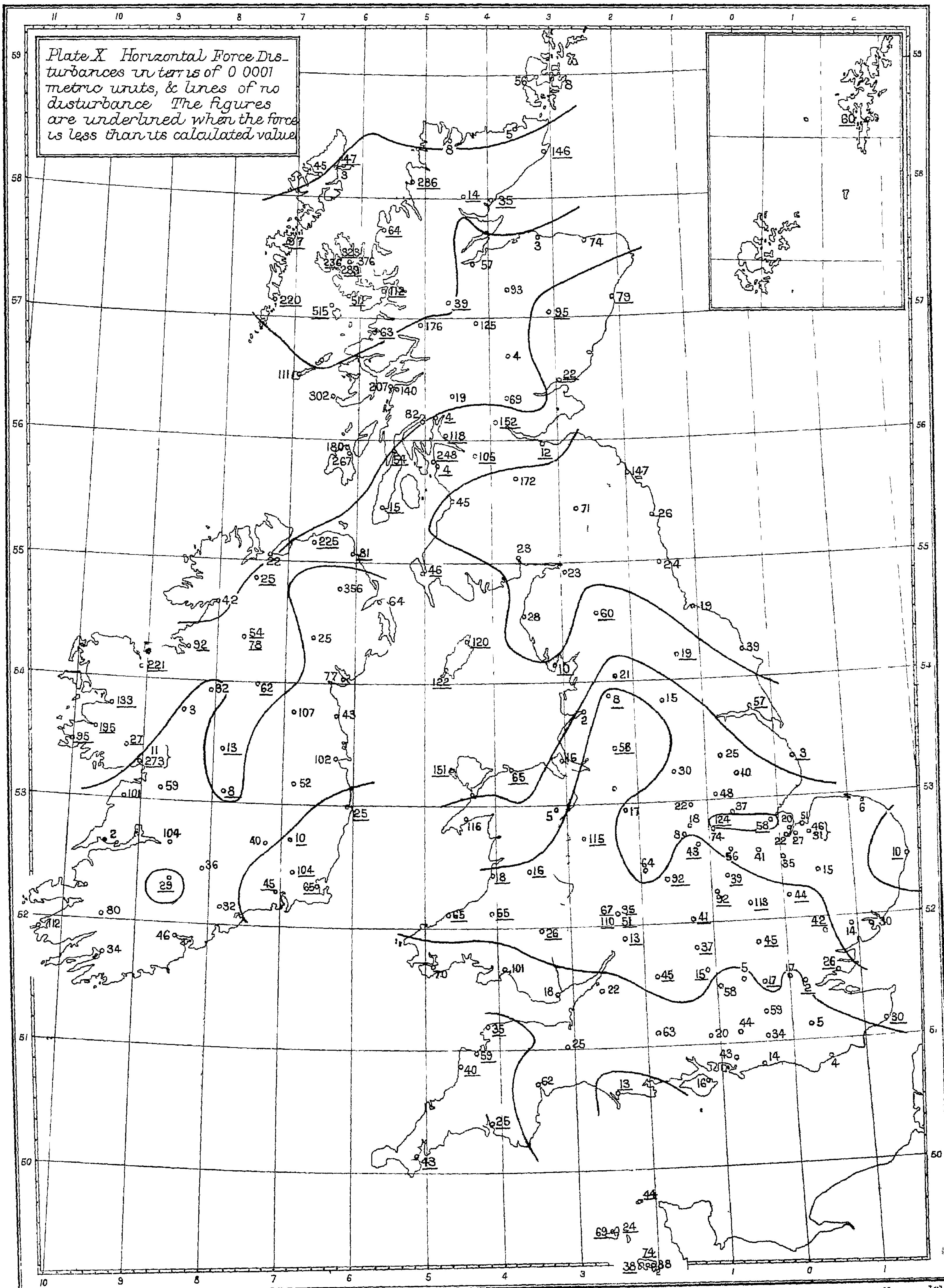
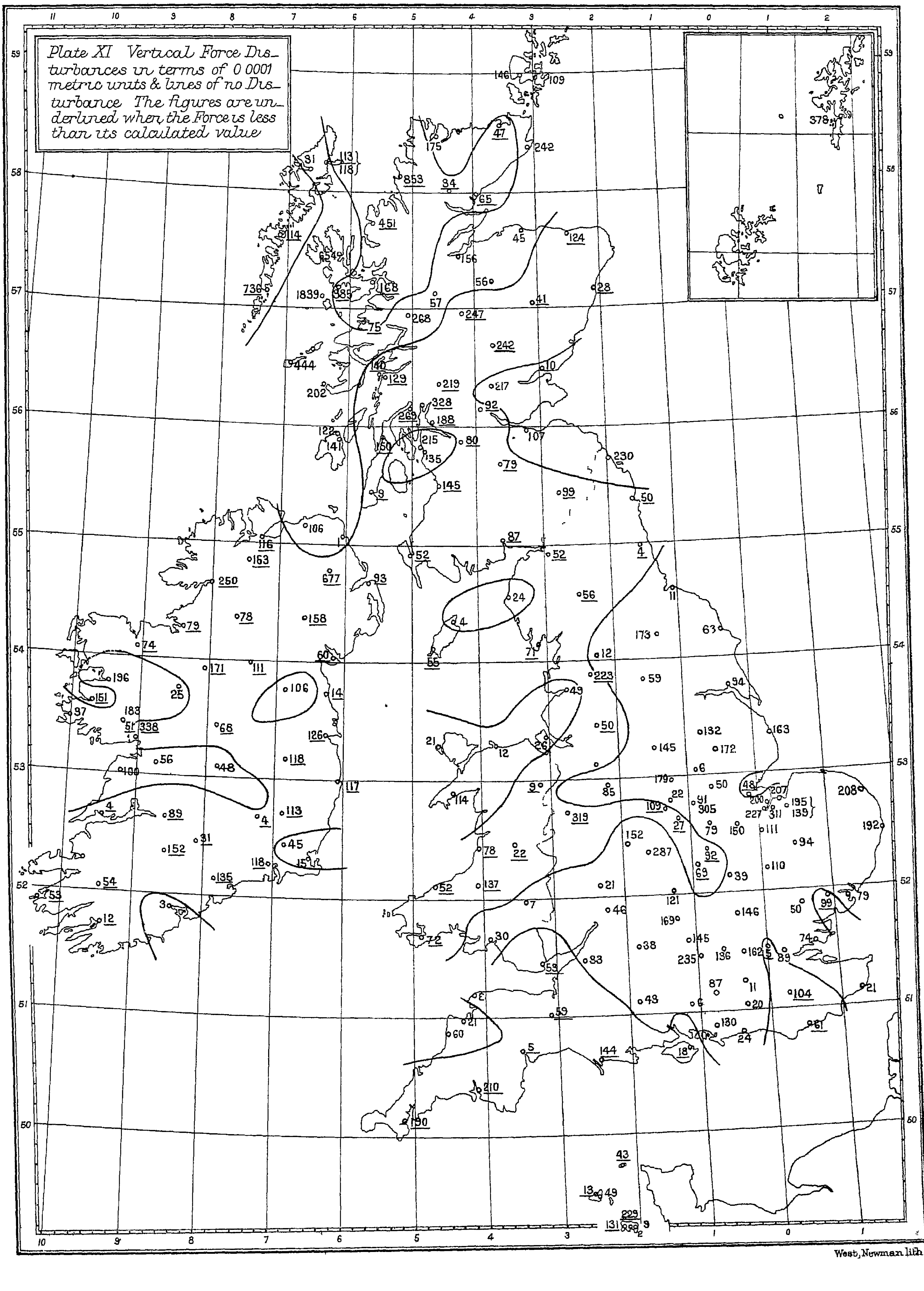


Plate IX Declination Disturbances in minutes of arc & Lines of no Disturbance The figures are underlined when the Disturbance is negative & the needle is deviated to the East
West, Newman lith.



West, Newman, lith.



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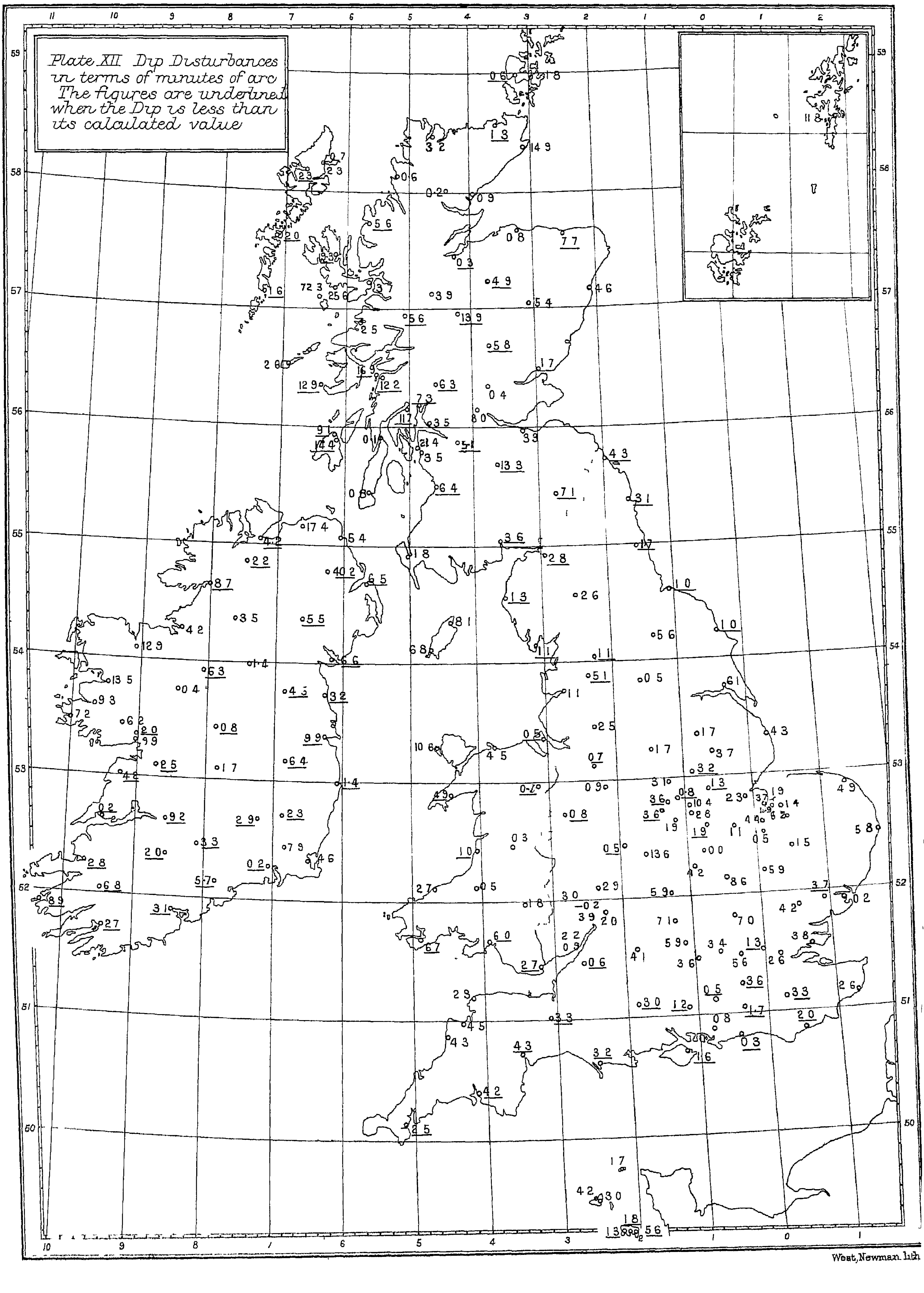

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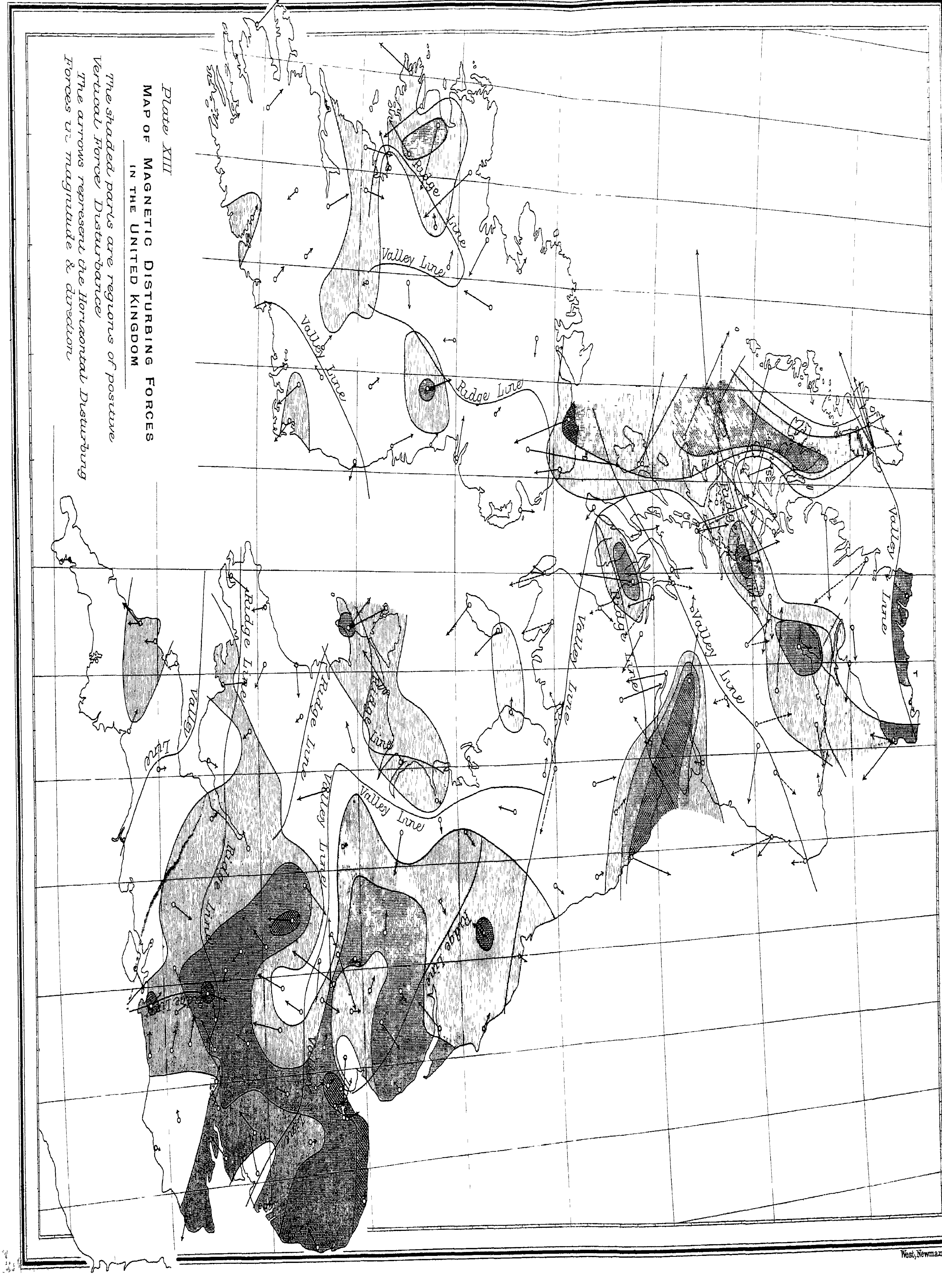

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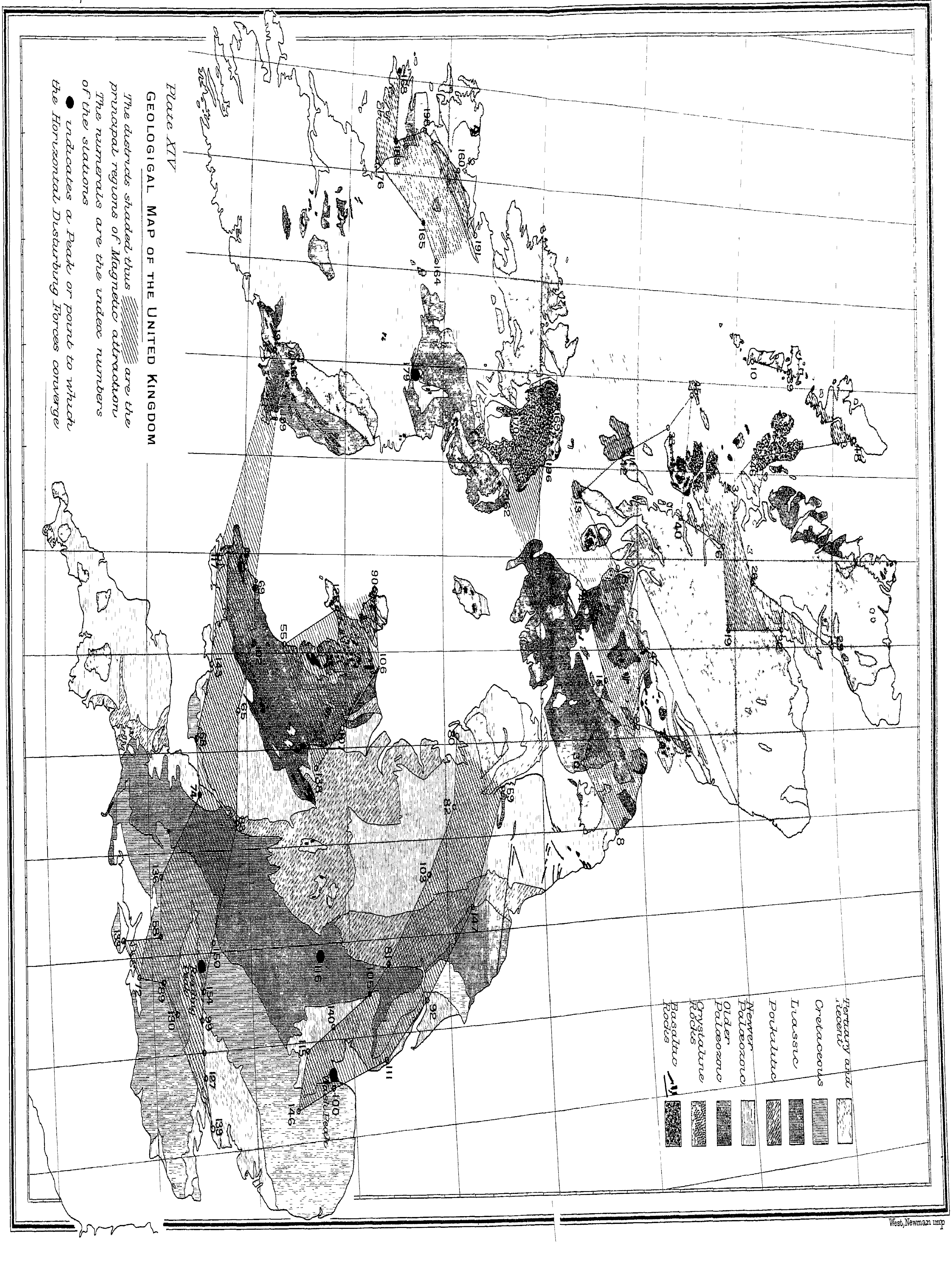

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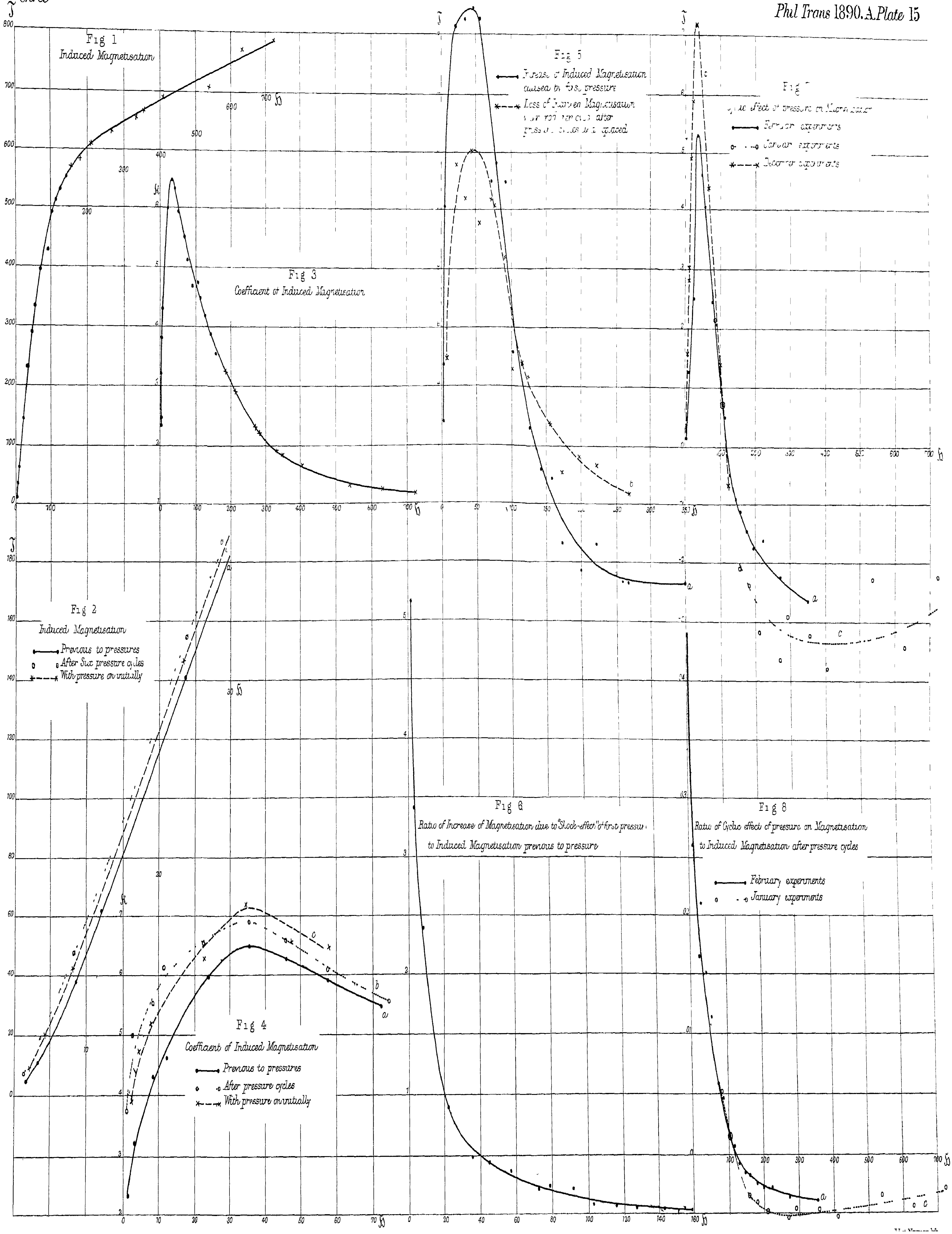
Fig 1
Induced Magnetisation
Fig 2
Induced Magnetisation
Previous to pressures
After Six pressure cycles
With pressure on initially
Fig 3
Coefficient of Induced Magnetisation
Fig 4
Coefficient of Induced Magnetisation
Previous to pressures
After pressure cycles
With pressure on initially
Fig 5
Increase of Induced Magnetisation caused by first pressure
Fig 6
Ratio of Increase of Magnetisation due to "Shock-effect" of first pressure to Induced Magnetisation previous to pressure
Fig 8
Ratio of Cyclic effect of pressure on Magnetisation to Induced Magnetisation after pressure cycles
February experiments
January experiments

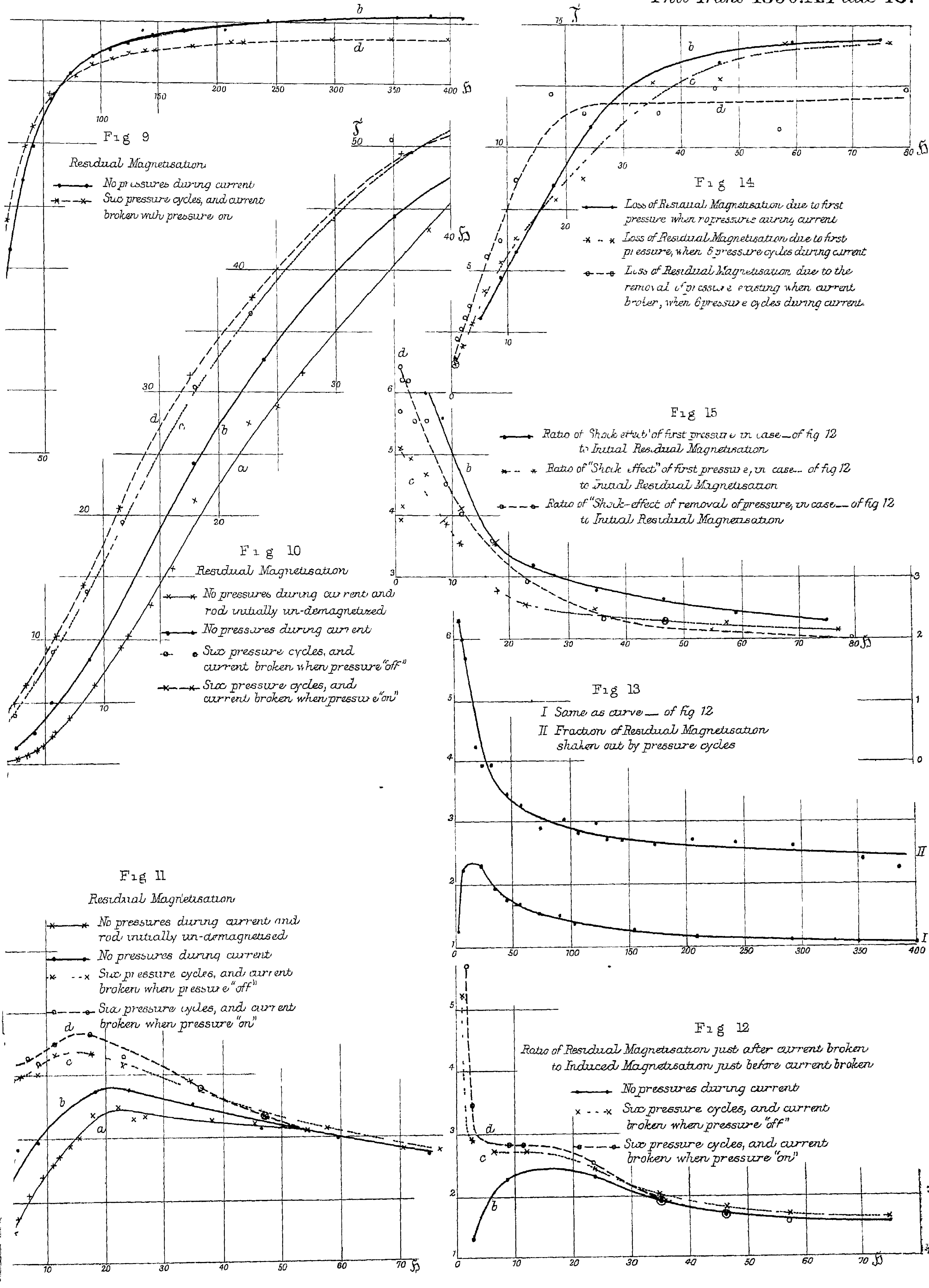
Fig 9
Residual Magnetisation
No pressures during current
Six pressure cycles, and current broken with pressure on
Fig 10
Residual Magnetisation
No pressures during current and rod initially un-demagnetized
No pressures during current
Six pressure cycles, and current broken when pressure "off"
Six pressure cycles, and current broken when pressure "on"
Fig 11
Residual Magnetisation
No pressures during current and rod initially un-demagnetised
No pressures during current
Six pressure cycles, and current broken when pressure "off"
Six pressure cycles, and current broken when pressure "on"
Fig 12
Ratio of Residual Magnetisation just after current broken to Induced Magnetisation just before current broken
No pressures during current
Six pressure cycles, and current broken when pressure "off"
Six pressure cycles, and current broken when pressure "on"
Fig 13
I Same as curve — of fig 12
II Fraction of Residual Magnetisation shaken out by pressure cycles
Fig 14
Loss of Residual Magnetisation due to first pressure when no pressures during current
Loss of Residual Magnetisation due to first pressure, when 6 pressure cycles during current
Loss of Residual Magnetisation due to the removal of pressure existing when current broken, when 6 pressure cycles during current
Fig 15
Ratio of "Shock effect" of first pressure in case — of fig 12 to Initial Residual Magnetisation
Ratio of "Shock effect" of first pressure, in case — of fig 12 to Initial Residual Magnetisation
Ratio of "Shock-effect of removal of pressure, in case — of fig 12 to Initial Residual Magnetisation